Case Study: Data Envelopment Analysis, Stochastic Case with Buffered-Ranking (pr pen, pr pen g, bpoe, bpoe g, linearmulti)

Background

This case study implements a new approach for evaluation of efficiency score for a decision making unit (DMU) in Data Envelopment Analysis (DEA). Charnes-Cooper-Rhodes (CCR) model evaluates the efficiency score of DMU as its highest achievable efficiency ratio (see, Charnes et all (1978)). Salo and Punkka (2011) studies the best and worst rankings that one DMU can attain over feasible input/output weights. The best efficiency ranking of this DMU is obtained by minimizing the number of DMUs with efficiency measure higher than this DMU, while the worst ranking of this DMU is obtained by minimizing the number of DMUs with efficiency measure lower than this DMU. Both the best and worst rankings should be obtained by solving a mixed-integer linear program (MILP). It is difficult to solve these problems, because the objective function includes the cardinality of a set that depends on the decision variables. Wang and Uryasev (2019) proposed to evaluate the efficiency of DMU with the best and the worst buffered-ranking. The buffered-rankings has a great computational advantage over rankings based on MILP. Finding the best and the worst buffered-rankings can be reduced to convex and linear programming.

References

- Charnes A, Cooper WW, Rhodes E. (1978): Measuring the efficiency of decision-making units. Eur J Oper Res.3:429444.
- Salo, A., Punkka, A. (2011): Ranking intervals and dominance relations for ratio-based efficiency analysis. Management Science, 57, 200-214.
- Wang, Y., Uryasev, S. (2019): Buffered ranking in efficiency analysis, Working paper.

Notations

J = number of DMUs;

N = number of types of inputs, which consumes each DMU;

M = number of types of outputs, which produces each DMU;

x_{nj} = number of units of the n-th input type, which consumes the j-th DMU, $x_{nj} \geqslant 0$;

y_{mj} = number of units of the m-th output type, which produces the j-th DMU, $y_{mj} \geqslant 0$;

$\vec{x}_j = (x_{1j}, \dots, x_{Nj})^T$ = vector of input consumptions for DMU_j, $j = 1, \dots, J$;

$\vec{y}_j = (y_{1j}, \dots, y_{Mj})^T$ = vector of output productions for DMU_j $j = 1, \dots, J$;

$\vec{\nu} = (\nu_1, \dots, \nu_N)^T$ = vector of non-negative weights for inputs;

$\vec{\mu} = (\mu_1, \dots, \mu_N)^T$ = vector of non-negative weights for outputs;

$E_j(\vec{v},\vec{\mu}) = \vec{\mu}^T\vec{y}_j/\vec{v}^T\vec{x}_j$ = effiiency of DMU_j;

A_v, A_μ = coefficient matrices derived from the preference statements about value of inputs and outputs. If no preference information is imposed on the inputs and outputs, then A_v, A_μ are null matrices;

$S = \{(\vec{v},\vec{\mu}) \neq 0 | \vec{v} \geqslant 0, A_v\vec{v} \geqslant 0, \vec{\mu} \geqslant 0, A_\mu\vec{\mu} \geqslant 0\}$ = feasible set of weight vectors $\vec{v},\vec{\mu}$;

DMU_o = evaluated DMU, $o \in \mathcal{J} = \{0,\ldots,J\}$;

$\phi = \max_{(\vec{v},\vec{\mu})\in\mathcal{S}} \quad E_j(\vec{v},\vec{\mu}) \quad s.t. \quad E_j(\vec{v},\vec{\mu}) \leq 1, \forall j \in \mathcal{J}$ = Charnes-Cooper-Rhodes (CCR) model, which evaluates DMU_o. Under the constraint that all efficiency ratios do not exceed 1, CCR evaluates the efficiency score of DMU_o as its highest achievable efficiency ratio by choosing the optimal $(\vec{v},\vec{\mu}) \in S$;

$|\mathcal{A}|$ = cardinality of set $\mathcal{A}$;

X = random variable;

z = threshold;

$p_z(X) = P(X \geq z)$ = probability of exceedance (POE);

$\bar{p}_z(X) = \min_{a\geqslant 0} E[a(X-z)+1]^+ =$ buffered probability of exceedance (bPOE);

$\text{Rank}_o^+(\vec{v},\vec{\mu}) = 1 + \left|\{j \in \mathcal{J} | E_j(\vec{v},\vec{\mu}) > E_o(\vec{v},\vec{\mu})\}\right|$ = upper ranking for DMU_o under one setting of input/output weights $(\vec{v},\vec{\mu})$;

$\text{Rank}_o^-(\vec{v},\vec{\mu}) = \left|\{j \in \mathcal{J} | E_j(\vec{v},\vec{\mu}) \geq E_o(\vec{v},\vec{\mu})\}\right|$ = lower ranking for DMU_o under one setting of input/output weights $(\vec{v},\vec{\mu})$;

$\text{Rank}_o^{+*} = \min_{(\vec{v},\vec{\mu})\in\mathcal{S}} \text{Rank}_o^+(\vec{v},\vec{\mu})$ = best (highest) ranking that DMU_o can achieve by choosing the weights $(\vec{v},\vec{\mu}) \in \mathcal{S}$ most favorable to DMU_o;

$\text{Rank}_o^{-*} = \max_{(\vec{v},\vec{\mu})\in\mathcal{S}} \text{Rank}_o^-(\vec{v},\vec{\mu})$ = worst (lowest) ranking that DMU_o can achieve by choosing the weights $(\vec{v},\vec{\mu}) \in \mathcal{S}$ least favorable to DMU_o;

$\mathbf{E}(\vec{v},\vec{\mu}) = [E_1(\vec{v},\vec{\mu}),\cdots,E_J(\vec{v},\vec{\mu})]^T$ = efficiency measures for J DMUs with input/output weights $(\vec{v},\vec{\mu})$;

$E_{[1]}(\vec{v},\vec{\mu}) \geq E_{[2]}(\vec{v},\vec{\mu}) \geq E_{[3]}(\vec{v},\vec{\mu}) \geq \cdots \geq E_{[J]}(\vec{v},\vec{\mu})$ = non-increasing order of efficiency measures for J DMUs with input/output weights $(\vec{v},\vec{\mu})$;

$\text{UA}_k[\mathbf{E}(\vec{v},\vec{\mu})] = \frac{1}{k}\left(\sum_{j=1}^{\lfloor k\rfloor} E_{[j]}(\vec{v},\vec{\mu}) + (k-\lfloor k\rfloor)E_{\lceil k\rceil}(\vec{v},\vec{\mu})\right)$ = k-upper average, $k \in [1,J]$, $\lfloor k\rfloor$ denotes the largest integer less than or equal to k, and $\lceil k\rceil$ denotes the smallest integer greater than or equal to k;

$\text{bRank}_o^+(\vec{v},\vec{\mu}) = \max\{k \in [1,J] | \text{UA}_k[\mathbf{E}(\vec{v},\vec{\mu})] \geq E_o(\vec{v},\vec{\mu})\}$ = upper buffered-ranking for DMU_o with input/output weights $(\vec{v},\vec{\mu})$;

$\text{bRank}_o^{+*} = \min_{(\vec{v},\vec{\mu})\in\mathcal{S}} \text{bRank}_o^+(\vec{v},\vec{\mu})$ = the best buffered-ranking (effiency score for DMU_o);

$\mathrm{LA}_k[\mathbf{E}(\vec{v},\vec{\mu})] = \frac{1}{k}\left(\sum_{j=1}^{\lfloor k \rfloor} E_{[J+1-j]}(\vec{v},\vec{\mu}) + (k - \lfloor k \rfloor) E_{[J+1-\lceil k \rceil]}(\vec{v},\vec{\mu})\right)$ = k-lower average;

$\mathrm{bRank}_o^-(\vec{v},\vec{\mu}) = J + 1 - \max\{k \in [1,J] | \mathrm{LA}_k[\mathbf{E}(\vec{v},\vec{\mu})] \leq E_o(\vec{v},\vec{\mu})\}$ = lower buffered-ranking for DMU_o with weights $(\vec{v},\vec{\mu})$;

$\mathrm{bRank}_o^{-*} = \max_{(\vec{v},\vec{\mu}) \in \mathcal{S}}\{J + 1 - \mathrm{bRank}_o^-(\vec{v},\vec{\mu})\}$ = the worst buffered-ranking for DMU_o;

$(\tilde{\vec{x}},\tilde{\vec{y}})$ =random vectors which follow the uniform multivariate distribution with support $\{(\tilde{\vec{x}}_1,\tilde{\vec{y}}_1),\ldots,(\tilde{\vec{x}}_J,\tilde{\vec{y}}_J)\}$, i.e. $\forall j \in \mathcal{J}, \mathbb{P}\{(\tilde{\vec{x}},\tilde{\vec{y}}) = (\tilde{\vec{x}}_j,\tilde{\vec{y}}_j)\} = 1/J$;

$\xi = \vec{\mu}^T\tilde{\vec{y}} - \vec{v}^T\tilde{\vec{x}} - \vec{\mu}^T\vec{y}_o + \vec{v}^T\vec{x}_o$ = random variable;

$$\mathrm{Rank}_o^{+*} = 1 + J \times \min_{(\vec{v},\vec{\mu}) \in \mathcal{S}} \mathrm{POE}_0(\vec{\mu}^T\tilde{\vec{y}} - \vec{v}^T\tilde{\vec{x}} - \vec{\mu}^T\vec{y}_o + \vec{v}^T\vec{x}_o);$$

$$\mathrm{Rank}_o^{-*} = J - J \times \min_{(\vec{v},\vec{\mu}) \in \mathcal{S}} \mathrm{POE}_0(\vec{v}^T\tilde{\vec{x}} - \vec{\mu}^T\tilde{\vec{y}} - \vec{v}^T\vec{x}_o + \vec{\mu}^T\vec{y}_o);$$

$$\mathrm{bRank}_o^{+*} = J \times \min_{(\vec{v},\vec{\mu}) \in \mathcal{S}} \mathrm{bPOE}_0(\vec{\mu}^T\tilde{\vec{y}} - \vec{v}^T\tilde{\vec{x}} - \vec{\mu}^T\vec{y}_o + \vec{v}^T\vec{x}_o);$$

$$\mathrm{bRank}_o^{-*} = J + 1 - J \times \min_{(\vec{v},\vec{\mu}) \in \mathcal{S}} \mathrm{bPOE}_0(\vec{v}^T\tilde{\vec{x}} - \vec{\mu}^T\tilde{\vec{y}} - \vec{v}^T\vec{x}_o + \vec{\mu}^T\vec{y}_o).$$

Optimization Problem 1 (best ranking)

minimizing probability that ξ exceeds zero

$$\min_{(\vec{v},\vec{\mu}) \in \mathcal{S}} \mathrm{POE}_0(\vec{\mu}^T\tilde{\vec{y}} - \vec{v}^T\tilde{\vec{x}} - \vec{\mu}^T\vec{y}_o + \vec{v}^T\vec{x}_o)$$

subject to

$$\vec{v}^T\vec{x}_o + \vec{\mu}^T\vec{y}_o = 1$$

$$\vec{v} \geq 0,\ \mathbf{A}_{\vec{v}}\vec{v} \geq 0,\ \vec{\mu} \geq 0,\ \mathbf{A}_{\vec{\mu}}\vec{\mu} \geq 0.$$

Optimization Problem 2 (worst ranking)

minimizing probability that $-\xi$ exceeds zero

$$\min_{(\vec{v},\vec{\mu}) \in \mathcal{S}} \mathrm{POE}_0(\vec{v}^T\tilde{\vec{x}} - \vec{\mu}^T\tilde{\vec{y}} - \vec{v}^T\vec{x}_o + \vec{\mu}^T\vec{y}_o)$$

subject to

$$\vec{v}^T\vec{x}_o + \vec{\mu}^T\vec{y}_o = 1$$

$$\vec{v} \geq 0,\ \mathbf{A}_{\vec{v}}\vec{v} \geq 0,\ \vec{\mu} \geq 0,\ \mathbf{A}_{\vec{\mu}}\vec{\mu} \geq 0.$$

Optimization Problem 3 (best ranking)

minimizing bPOE that ξ exceeds zero

$$\min_{(\vec{v},\vec{\mu})\in\mathcal{S}} \mathrm{bPOE}_0(\vec{\mu}^T\tilde{\vec{y}} - \vec{v}^T\tilde{\vec{x}} - \vec{\mu}^T\vec{y}_o + \vec{v}^T\vec{x}_o)$$

subject to

$$\vec{v} \geq 0,\ \mathbf{A}_{\vec{v}}\vec{v} \geq 0,\ \vec{\mu} \geq 0,\ \mathbf{A}_{\vec{\mu}}\vec{\mu} \geq 0.$$

Optimization Problem 4 (worst ranking)

minimizing bPOE that $-\xi$ exceeds zero

$$\min_{(\vec{v},\vec{\mu})\in\mathcal{S}} \mathrm{bPOE}_0(\vec{v}^T\tilde{\vec{x}} - \vec{\mu}^T\tilde{\vec{y}} - \vec{v}^T\vec{x}_o + \vec{\mu}^T\vec{y}_o)$$

subject to

$$\vec{v} \geq 0,\ \mathbf{A}_{\vec{v}}\vec{v} \geq 0,\ \vec{\mu} \geq 0,\ \mathbf{A}_{\vec{\mu}}\vec{\mu} \geq 0.$$