CASE STUDY: Portfolio Replication with Cardinality and Buyin Constraints

(max risk, meanabs err, polynom abs, cardn pos, card neg, buyin pos, linear, variable)

Background

Index tracking is a portfolio construction problem that builds a portfolio with returns close to the returns of an index. The objective is to generate appropriate returns without purchasing all instruments in the index. A number of approaches and papers relative to the index tracking have been discussed in Canakgoz and Beasley (2009) and Beasley et al.(2003). This case study is motivated by the paper by Mezali and Beasley (2012).

Typical requirements to the tracking portfolio:
- Objective: minimize the tracking error.
- The number of assets in the tracking portfolio should not exceed a given number.
- Positions of chosen assets in the portfolio should not be small.
- Variable and fixed transaction costs are defined by changes in asset positions.
- Budget constraint, i.e., re-balanced portfolio value plus transaction costs $\leq$ current portfolio value.

References

- Beasley, J.E. (1990): OR-Library: distributing test problems by electronic mail, Journal of the Operational Research Society, 41 (11), 1069–1072 (http://people.brunel.ac.uk/ mastjjb/jeb/jeb.html)

- Beasley, J.E., Meade, N. and Chang, T-J. (2003): An evolutionary heuristic for the index tracking problem, European Journal of Operational Research, 148 (3), 621–643.

- Canakgoz, N.A. and Beasley, J. E. (2009): Mixed-integer programming approaches for index tracking and enhanced indexation, European Journal of Operational Research, 196 (1), 384–399.

- Mezali, H. and Beasley, J.E. (2014): Index Tracking with Fixed and Variable Transaction Costs. Optimization Letters, 8 (1), 61–80.

Notations

T = number of observation periods;

$t = 0, \ldots, T$ = index of time period;

N = number assets in the tracking portfolio;

$i = 1, \ldots, N$ = index of assets;

K = maximum number of assets to be held in tracking portfolio ($K < N$);

x_i = number of units (shares) of an asset in the rebalanced tracking portfolio;

x_i^0 = value if asset i in the current tracking portfolio;

$C = \sum_{i=1}^{N} x_i^0$ = total value of the current tracking portfolio;

Y_t = price of the index at time $t = 0, \ldots, T$;

V_{it} = price of i-th asset at time $t = 0, \ldots, T$;

$\vec{x} = (x_1, \ldots, x_n)$ = vector of values of assets in the rebalanced tracking portfolio;

$\theta_{t0} = \ln(Y_t) - \ln(Y_{t-1}) = \ln(Y_t/Y_{t-1})$ = log-returns of the index;

$\theta_{ti} = \frac{\ln(V_{it}) - \ln(V_{it-1})}{C(1-\gamma)}$, $i = 1, \ldots, N$ = normalized log-returns of instruments;

γ = limit on the fraction of value of the current portfolio that can be spent on transaction costs, $0 < \gamma < 1$;

$L_t(\vec{x}) = \theta_{t0} - \sum_{i=1}^{N} \theta_{ti} x_i$ = underperformance of the tracking portfolio;

$\delta(x)$ = indicator function: $\delta(x) = 0$ for $x = 0$, and $\delta(x) = 1$ for $x \neq 0$;

$\sum_{i=0}^{I} \delta(x_i) \leqslant K$ = cardinality constraint;

$$\beta_\sigma^+(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x \in (0, \sigma] \\ 0, & \text{if } x > \sigma \end{cases} = \textit{buy-in positive function}; \text{ it is used to assure that positions of assets} \geq \sigma;$$

$\sum_{i=1}^{N} \beta_\sigma^+(x_i) \leqslant 0$ = constraint assuring that all non-zero positions $\geq \sigma$;

$vcost + fcost \leqslant \gamma C$ = constraint assuring that the total transaction cost $\leq$ some fraction of the current value of the tracking portfolio;

$\sum_{i=1}^{N} x_i + vcost + fcost \leqslant C$ = constraint assuring that (rebalance portfolio value) + (transaction costs) $\leq$ (current portfolio value);

$x_i \geqslant 0$, $i = 1, \ldots, N$ = constraints assuring long positions in the rebalanced tracking portfolio;

$\mathcal{E}_{MAX}(\vec{x}) = \max_{1 \leqslant t \leqslant T} |L_t(\vec{x})|$ = Maximum Absolute Error (MAX);

$\mathcal{E}_{MAE}(\vec{x}) = \frac{1}{T} \sum_{t=1}^{T} |L_t(\vec{x})|$ = Mean Absolute Error (MAE);

$a_i > 0$ = variable cost coefficient for buying/selling asset i;

$vcost = \sum_{i=1}^{N} a_i |x_i - x_i^0|$ = variable transaction cost;

A = fixed cost of buying/selling asset i;

$fcost = A \sum_{i=1}^{N} \delta(|x_i - x_i^0|)$ = fixed transaction cost defined by the # of assets with changed positions;

Optimization Problem 1
minimizing the maximum absolute difference between returns of tracking portfolio and index

$$\min_{\vec{x}} \quad \mathcal{E}_{MAX}(\vec{x})$$

subject to

$$\sum_{i=1}^{N} \delta(x_i) \leqslant K$$

$$\sum_{i=1}^{N} \beta_{\sigma}^{+}(x_i) \leqslant 0$$

$$\sum_{i=1}^{N} x_i + \sum_{i=1}^{N} a_i |x_i - x_i^0| + A \sum_{i=1}^{N} \delta(|x_i - x_i^0|) \leqslant C$$

$$\sum_{i=1}^{N} a_i |x_i - x_i^0| + A \sum_{i=1}^{N} \delta(|x_i - x_i^0|) \leqslant \gamma C$$

$$x_i \geqslant 0, \quad i = 1, \ldots, N$$

Optimization Problem 2
minimizing the average of the absolute differences between returns of tracking portfolio and index

$$\min_{\vec{x}} \quad \mathcal{E}_{MAE}(\vec{x})$$

subject to

$$\sum_{i=1}^{N} \delta(x_i) \leqslant K$$

$$\sum_{i=1}^{N} \beta_{\sigma}^{+}(x_i) \leqslant 0$$

$$\sum_{i=1}^{N} x_i + \sum_{i=1}^{N} a_i |x_i - x_i^0| + A \sum_{i=1}^{N} \delta(|x_i - x_i^0|) \leqslant C$$

$$\sum_{i=1}^{N} a_i |x_i - x_i^0| + A \sum_{i=1}^{N} \delta(|x_i - x_i^0|) \leqslant \gamma C$$

$$x_i \geqslant 0, \quad i = 1, \ldots, N$$

Optimization Problem 3
MIP version of Problem 1 (formulated with PSG nonlinear functions and automatically converted to MIP)

$$\min_{y,\vec{x},\vec{v},\vec{w},\vec{z}} \quad y$$

subject to

$$y \geqslant \sum_{i=1}^{N} \theta_{ti} x_i - \theta_{t0}, \quad t = 1, \ldots, T$$

$$y \geqslant -\left(\sum_{i=1}^{N} \theta_{ti} x_i - \theta_{t0}\right), \quad t = 1, \ldots, T$$

$$\sum_{i=1}^{N} v_i \leqslant K$$

$$l_i v_i \leqslant x_i \leqslant u_i v_i, \quad i = 1, \ldots, N$$

$$z_i \geqslant x_i - x_i^0, \quad z_i \geqslant -(x_i - x_i^0), \quad i = 1, \ldots, N$$

$$z_i \leqslant U w_i, \quad i = 1, \ldots, N$$

$$\sum_{i=1}^{N} x_i + \sum_{i=1}^{N} a_i z_i + A \sum_{i=1}^{N} w_i \leqslant C$$

$$\sum_{i=1}^{N} a_i z_i + A \sum_{i=1}^{N} w_i \leqslant \gamma C$$

$$v_i \in \{0,1\}, \ w_i \in \{0,1\}, \ x_i \geqslant 0, \ i = 1, \ldots, N$$

Optimization Problem 4
MIP version of Problem 2 (formulated with PSG nonlinear functions and automatically converted to MIP)

$$\min_{y_1,\ldots,y_T,\vec{x},\vec{v},\vec{w},\vec{z}} \quad \sum_{t=1}^{T} y_t$$

subject to

$$y_t \geqslant \sum_{i=1}^{N} \theta_{ti} x_i - \theta_{t0}, \quad t = 1, \ldots, T$$

$$y_t \geqslant -\left(\sum_{i=1}^{N} \theta_{ti} x_i - \theta_{t0}\right), \quad t = 1, \ldots, T$$

$$\sum_{i=1}^{N} v_i \leqslant K$$

$$l_i v_i \leqslant x_i \leqslant u_i v_i, \quad i = 1, \ldots, N$$

$$z_i \geqslant x_i - x_i^0, \quad z_i \geqslant -(x_i - x_i^0), \quad i = 1, \ldots, N$$

$$z_i \leqslant Uw_i, \quad i = 1, \ldots, N$$

$$\sum_{i=1}^{N} x_i + \sum_{i=1}^{N} a_i z_i + A \sum_{i=1}^{N} w_i \leqslant C$$

$$\sum_{i=1}^{N} a_i z_i + A \sum_{i=1}^{N} w_i \leqslant \gamma C$$

$$v_i \in \{0, 1\}, \ w_i \in \{0, 1\}, \ x_i \geqslant 0, \ i = 1, \ldots, N$$