Optimal Allocation of Retirement Portfolios

Giorgi Pertaia *[1], Morton Lane[2], Matthew Murphy[2], and Stan Uryasev[1]

[1]Industrial and Systems Engineering, University of Florida
[2]Department of Finance, University of Illinois at Champaign-Urbana

Abstract

A retiree with a savings account balance, but without a pension is confronted with an important investment decision that has to satisfy two conflicting objectives. Without a pension the function of the savings is to provide post-employment income to the retiree. At the same time, most retirees want to leave an estate to their heirs. Guaranteed income can be acquired by investing in an annuity. However, that decision takes funds away from investment alternatives that might grow the estate. The decision is made even more complicated because one does not know how long one will live. A long life expectancy may suggest more annuities, and short life expectancy could promote more risky investments. However there are very mixed opinions about either strategy. A framework has been developed to assess consequences and the trade-offs of alternative investment strategies. We propose a stochastic programming model to frame this complicated problem. The objective is to maximize expected terminal net worth (the estate), subject to cash outflow constraints. The cash outflow shortages are penalized in the objective function of the problem. We use kernel method to build position adjustment functions that control how much is invested in each asset. These adjustments nonlinearly depend upon asset returns in previous years. Case study was conducted using two variations of the model. The parameters used in this case study correspond to typical retirement situation. The case study shows that if the market forecasts are pessimistic, it is optimal to invest in annuity. The case study results, codes, and data are posted at the website.

1 Introduction

The problem of selecting optimal portfolios for retirement has unique features that are not addressed by more commonly used portfolio selection models used in trading. One distinct feature of a retirement portfolio is that it should incorporate the life span of an investor. The planning horizon depends on the age of investor, or more specifically, on a conditional life expectancy. Another important feature is to guarantee, in some sense, that the individual will be able to withdraw some amount of money every year from a portfolio by selling some predefined amount of assets without injecting external funds. Finally, one of the questions that the models tries to answer is, in what situation is it beneficial to invest in annuity instead of more risky assets.

Most of portfolio optimization literature considers portfolios focusing on risk minimization with some budget and expected profit constraints. The famous mean-variance (or Markowitz) portfolio Markowitz [1952] minimizes portfolio variance with constraints on the expected return. There are

*Corresponding author.
Email addresses: **gpertaia@ufl.edu** (G. Pertaia), **mnlane@illinois.edu** (M. Lane), **mdmurph@illinois.edu** (M. Murphy), **uryasev@ufl.edu** (S. Uryasev)

many directions that extend the original mean-variance portfolio and deal with its shortcomings. One direction is to substitute variance with some other risk measures. Variance does not distinguish positive and negative portfolio returns, however investors are mostly concerned only with negative returns. Rockafellar and Uryasev [2000, 2002] and Krokhmal et al. [2002] used Conditional Value-at Risk (CVaR) instead of the variance. CVaR is a convex function of its random variable and therefore problems involving CVaR can be solved efficiently in many cases. Another important risk measure, which is frequently used in trading, is drawdown. Drawdown can be optimized with convex and linear programming, see Chekhlov et al. [2003, 2005] and Zabarankin et al. [2014]. Other extension of the portfolio theory focuses on dynamic models. In dynamic models the decision to invest is made over time. The dynamic models can be of two types, continuous-time and discrete-time (multistage) models. In continuous time, the decision to invest is made continuously and in discrete-time, the investment decisions take place on specific time moments. For the continuous-time portfolio selection see Merton [1969, 1971]. For discrete-time stochastic control model see Samuelson [1969]. A comprehensive literature review on dynamic models is given in Rizal and Wiryono [2015]. Multistage models can be formulated as stochastic optimization problems. Mulvey and Shetty [2004] and Mulvey and Vladimirou [1992] developed a general multistage approach for modeling financial planning problems. Shang et al. [2016] and Bogentoft et al. [2001] use stochastic programming to solve dynamic cash flow matching and asset/liability management problems, respectively. In general, it is very hard to solve multistage stochastic optimization problems formulated with scenario trees, due to the size of the problem (number of variables) growing beyond tractable bounds. It should be mentioned that calibration of such trees is a difficult non-convex optimization problem.

In order to avoid the dimensionality problems, Calafiore [2008] models the investment decisions as linear functions that remain same across all scenarios and produce the investment decision based on previous performance of the asset.

Takano and Gotoh [2014] model the investment decisions with kernel method, resulting in the nonlinear control functions depending upon returns of instruments.

We follow ideas of Takano and Gotoh [2014] and model multistage portfolio decision process using the kernel method. The investment horizon is 35 years, starting from the retirement of the investor at the age of 65. The objective is to maximize the discounted expected terminal wealth subject to constraints on cash outflows from the portfolio. On every scenario, the discounted weighted portfolio value is calculated, where the probabilities of death are used as weights. The probability of death is calculated from the U.S. mortality tables. The investor wants to have predetermined cash outflows obtained by selling a portion of the portfolio. Risk of shortage of this cash outflows is managed by penalizing the cash outflow shortage in the objective function. Furthermore, the monotonicity constraints are imposed on the cash outflows from the portfolio. Without the monotonicity constraint the model might not provide the necessary cash outflow on certain periods and instead, reinvest that amount to increase the portfolio terminal value.

We conducted a case study corresponding to a typical investment decision upon retirement, in order to reveal the conditions leading to investments in annuities. Two types of asset return scenarios are considered. First type assumes that the asset returns will be similar to the historically observed rates of the asset. The second type of scenarios assumes the future asset returns will be significantly lower. These scenarios are created by subtracting 12% from the historical returns of all assets. The case study shows that for the first type of scenarios, where rates are similar to the ones observed in the past, investment in the annuities is not optimal. However, in the case when the asset growth rates are significantly lower, the model invests only in the annuities.

2 Notations

We start with introduction of.

- N := number of assets available for investments,
- S := number of scenarios,
- T := portfolio investment horizon,
- $r_{i,t}^s$:= rate of return of asset $i = 1, \ldots, N$ during period $t = 1, \ldots, T$ in scenario $s = 1, \ldots, S$; we will call rate of return by just return and denote the vector of returns by $\boldsymbol{r}_t^s = (r_{1,t}^s, \ldots, r_{N,t}^s)$,
- $\boldsymbol{v}_{m,t}^s = \{\boldsymbol{r}_m^s, \ldots, \boldsymbol{r}_{t-1}^s\}$:= set of returns observed from period m, until the end of period $t-1$ (not including the returns $\boldsymbol{r}_t^s$) in scenario s,
- d_t^s := discount factor at time t in scenario s; discounting is done using inflation rate ρ_t^s, $d_t^s = 1/(1+\rho_t^s)^t$,
- p_t := probability that a person will die at the age $65+t$ (conditional that he is alive at the age of 65),
- $\boldsymbol{y}_i$:= vector of control variables for investment adjustment function,
- $f(\boldsymbol{v}_t^s, \boldsymbol{y}_i)$:= investment adjustment function defining how much investment is made in each scenario s in asset i at the end of period t,
- $G(\boldsymbol{y}_i)$:= regularization function of control parameters,
- $K(\boldsymbol{v}_{m,t}^s, \boldsymbol{v}_{m,t}^k)$:= kernel function, $k = 1, \ldots, S$,
- $x_{i,t}^s$:= investment amount to i-th asset at the end of time period t in scenario s,
- x_i := investments to i-th asset at time $t = 0$,
- $u_{i,t}^s$:= adjustment (change in position) of asset i at the beginning of period t in scenario s,
- $R_{i,t}^s$:= cash outflow resulting from selling an asset i at the end of time t in scenario s,
- V_0 := portfolio value at time $t = 0$ (initial investment),
- V_t^s := portfolio value at time t in scenario s,
- z := investment in annuity at time $t = 0$(in dollars),
- A_t^s := yield of annuity at the end of time period t in scenario s,
- L := amount of money that the investor is planing to withdraw as each time t,
- λ := regularization coefficient, $\lambda > 0$,
- κ_t := penalty for the cash flow shortage at time t,
- α := upper bound on sum of absolute adjustments each year, expressed as a fraction of the portfolio.

3 Model Formulation

This section develops a model for optimization of a retirement portfolio. We consider a portfolio including stock indices, bond indices, and an annuity. The annuity pays amount $A_t^s z$ at the end of each period t and does not contribute funds to the terminal wealth. Annuity is bought at time $t = 0$ and can not be bought or sold after that moment. Given initial investments in assets x_i, the dynamics of investments in stocks and bonds are as follows

$$\begin{aligned} x_{i,1}^s &= (1 + r_{i,1}^s)x_i, \\ x_{i,t}^s &= (1 + r_{i,t}^s)(x_{i,t-1}^s + u_{i,t-1}^s - R_{i,t-1}^s) \qquad t = 2, \dots, T. \end{aligned} \tag{1}$$

Variables $u_{i,t}^s$ and $R_{i,t}^s$ control how much is invested at the end of each period in each asset. $u_{i,t}^s$ is a position adjustments for asset i at the end of time t in scenario s. $R_{i,t}^s$ is cash outflow from the portfolio, generated from selling asset i at time t in scenario s. The variable $u_{i,t}^s$ is defined as

$$u_{i,t}^s = f(\boldsymbol{v}_t^s, \boldsymbol{y}_i), \tag{2}$$

where $\boldsymbol{v}_t^s$ is a set of returns for all assets, up to time t, in scenario s, and $\boldsymbol{y}_i$ are some parameters defining the function f. Therefore, $u_{i,t}^s$, are some nonlinear functions of previous returns of assets. The explicit form of function f is not specified in this section. The only requirement on function f is that it should be linear in $\boldsymbol{y}_i$, i.e.

$$f(\boldsymbol{v}_t^s, \gamma_1 \boldsymbol{y}_i^1 + \gamma_2 \boldsymbol{y}_i^2) = \gamma_1 f(\boldsymbol{v}_t^s, \boldsymbol{y}_i^1) + \gamma_2 f(\boldsymbol{v}_t^s, \boldsymbol{y}_i^2),$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$. Also, it should be noted that f does not change with t. The linearity of f with respect to $\boldsymbol{y}_i$ is introduced to formulate the portfolio optimization problem as a convex programming problem.

The total asset adjustments must sum to 0, this is expressed as a constraint,

$$\sum_{i=1}^{N} u_{i,t}^s = 0\,. \tag{3}$$

In addition to (3) the sum of absolute adjustments (over each asset i) in each period t and scenario s is constrained to be less than or equal to some fraction α of the portfolio value in the previous year of the same scenario,

$$\sum_{i=1}^{N} |u_{i,t}^s| \leq \alpha V_{t-1}^s. \tag{4}$$

Constraint (4) serves as additional regularization on the adjustments. Without constraint (4) the values of $u_{i,t}^s$ can potentially be very large in absolute value but cancel out due to opposite signs and still satisfy (3).

The value of the portfolio at the end of time period t in scenario s equals,

$$V_t^s = \sum_{i=1}^{N} x_{i,t}^s\;. \tag{5}$$

The objective is to maximize terminal wealth of the portfolio. The terminal wealth is the weighted average of the discounted expected portfolio values in each scenario, where the probabilities of death p_t are used as weights. For every scenario s the portfolio value V_t^s, at the end of time period t, is discounted to time 0 using discounting coefficients d_t^s, defined by inflation, therefore,

$$\text{discounted terminal wealth in scenario } s = \sum_{t=1}^{T} p_t d_t^s V_t^s \ . \tag{6}$$

By averaging over scenarios we obtain the expected terminal wealth,

$$\frac{1}{S} \sum_{s=1}^{S} \sum_{t=1}^{T} p_t d_t^s V_t^s \ . \tag{7}$$

In order to avoid over-fitting the data, we included the regularization term $G(\boldsymbol{y}_i)$, defined for every instrument i. The total regularization term is

$$\sum_{i=1}^{N} G(\boldsymbol{y}_i) \ . \tag{8}$$

The total cash outflow from selling the assets in the portfolio equals

$$\text{cash flow from portfolio} = \sum_{i=1}^{N} R_{i,t}^s \ .$$

The amount of money that the investor receives from the portfolio and annuity at the end of time period t in scenario s equals $A_t^s z + \sum_{i=1}^N R_{i,t}^s$. If $A_t^s z + \sum_{i=1}^N R_{i,t}^s < L$ then there is a shortage of cash outflow and the resulting amount is penalized in the objective. Let $\{\kappa_t\}_{t=1}^T$ be some decreasing sequence of positive numbers, the following function is a penalty term of cash outflow shortages in the objective

$$\sum_{t=1}^{T} \kappa_t \left[L - A_t^s z - \sum_{i=1}^{N} R_{i,t}^s \right]^+ , \tag{9}$$

where $[*]^+ = \max\{*, 0\}$. To illustrate why it is important that $\{\kappa_t\}_{t=1}^T$ is a decreasing sequence, consider the case where all κ_t are equal. Also, lets assume that there is a shortage of cash outflow, equal to the amount w, at some year $t > 0$. Because, κ_t are all equal in (9), it does not make a difference for that penalty term if there is a shortage equal to w/t during every year until t, or just a single shortage of w at time t. However, if the amount of w/t is reinvested before time t in the portfolio, it will(probably) increase in value by the time t and therefore, it will increase the terminal wealth of the portfolio. So, if $\{\kappa_t\}_{t=1}^T$ is not a decreasing sequence, the model will try to incur penalty as soon as possible, even if there are enough funds in the portfolio at that earlier date, and reinvest that shortage amount in the portfolio. Therefore the parameters κ_t should be selected in a way that will outweigh any possible benefits from reinvesting at earlier dates. A simple possible formula for κ_t is $\kappa_t = \kappa(1+\bar{r})^{T-t}$, where $\kappa > 1$ is some constant and $\bar{r}$ is some percentage, such that it is significantly greater than the average growth rate of any asset considered in the portfolio. With this simple formula for κ_t, any benefits of reinvesting early will be outweighed by the corresponding penalty.

The model includes the constraint on monotonicity of the cash outflows from the portfolio

$$\sum_{i=1}^{N} R_{i,t-1}^s \geq \sum_{i=1}^{N} R_{i,t}^s \ . \tag{10}$$

Without the monotonicity constraint, the model might not provide necessary cash outflows at the end of certain years and instead, reinvest that amount to increase the terminal wealth of the portfolio. The monotonicty constraint ensures that the cash outflow shortage occurs only in years where the portfolio value drops below the cash outflow amount at the end of the previous year.

We minimize the objective function, containing expected costs with minus sign, regularization term with penalty coefficient $\lambda > 0$ and cash outflow shortage with penalty κ_t

$$-\frac{1}{S}\sum_{s=1}^{S}\sum_{t=1}^{T} p_t d_t^s V_t^s + \lambda \sum_{i=1}^{N} G(\boldsymbol{y}_i) + \sum_{t=1}^{T} \kappa_t \left[L - A_t^s z - \sum_{i=1}^{N} R_{i,t}^s \right]^+ . \tag{11}$$

The explicit form of function G is not defined in this section, however, it is assumed that the function $G(\boldsymbol{y})$ is a convex function in $\boldsymbol{y}$. This is important to formulate the problem as a convex optimization. The resulting objective function (11) is a convex function in $\boldsymbol{y}_i$ and linear in V_t^s.

Further we provide the general model formulation.

$$\min_{\substack{u_{i,t}^s, R_{i,t}^s,\\ V_0, V_t^s, \boldsymbol{y}_i,\\ x_i^s, x_{i,t}^s, z}} \quad \frac{1}{S}\sum_{s=1}^{S}\sum_{t=1}^{T} p_t d_t^s V_t^s + \lambda \sum_{i=1}^{N} G(\boldsymbol{y}_i) + \sum_{t=1}^{T} \kappa_t \left[L - A_t^s z - \sum_{i=1}^{N} R_{i,t}^s \right]^+ \tag{12}$$

s.t.

$$x_{i,1}^s = (1 + r_{i,1}^s) x_i \qquad i = 1,\dots,N; \quad s = 1,\dots,S$$

$$x_{i,t}^s = (1 + r_{i,t}^s)(x_{i,t-1}^s + u_{i,t-1}^s - R_{i,t-1}^s) \qquad i = 1,\dots,N; \quad t = 2,\dots,T; \quad s = 1,\dots,S$$

$$\sum_{i=1}^{N} x_i = V_0 - z$$

$$V_t^s = \sum_{i=1}^{N} x_{i,t}^s \qquad t = 1,\dots,T; \quad s = 1,\dots,S$$

$$\sum_{i=1}^{N} u_{i,t}^s = 0 \qquad t = 1,\dots,T; \quad s = 1,\dots,S$$

$$\sum_{i=1}^{N} R_{i,t}^s \le \sum_{i=1}^{N} R_{i,t-1}^s \qquad t = 2,\dots,T; \quad s = 1,\dots,S$$

$$u_{i,t}^s = f(\boldsymbol{v}_{m,t}^s, \boldsymbol{y}_i) \qquad i = 1,\dots,N; \quad t = 1,\dots,T; \quad s = 1,\dots,S$$

$$\sum_{i=1}^{N} |u_{i,t}^s| \le \alpha V_{t-1}^s \qquad t = 2,\dots,N; \quad s = 1,\dots,S$$

$$\sum_{i=1}^{N} |u_{i,1}^s| \le \alpha (V_0 - z)$$

$$z \ge 0$$

$$R_{i,t}^s \ge 0$$

$$x_i \ge 0 \qquad i = 1,\dots,N$$

$$x_{i,t}^s \ge 0 \qquad i = 1,\dots,N; \quad t = 1,\dots,T; \quad s = 1,\dots,S$$

4 Special Case of General Formulation

This section presents a special case of the general problem formulation. We picked functions $G(\boldsymbol{y}_i)$ and $f(\boldsymbol{r}_t^s, \boldsymbol{y}_i)$ similar to the model developed in Takano and Gotoh [2014].

Let $m > 0$ be some integer and $K_m(\boldsymbol{v}_t^s, \boldsymbol{v}_t^k)$ be the kernel function defined as follows

$$K(\boldsymbol{v}_{m,t}^s, \boldsymbol{v}_{m,t}^k) = \exp\left(-\frac{\sigma}{m} \sum_{i=1}^{N} \sum_{l=t-m-1}^{t-1} (r_{i,l}^k - r_{i,l}^s)^2 \right), \tag{13}$$

where $\sigma > 0$ is some constant. The parameter m controls how many previous years of information is used by the kernel function to calculate the portfolio adjustments. Given (13), the control function $f(\boldsymbol{v}_t^s, \boldsymbol{y}_i)$ is defined as

$$f(\boldsymbol{v}_t^s, \boldsymbol{y}_i) = \sum_{j=1}^{S} y_i^j K(\boldsymbol{v}_{m,t}^s, \boldsymbol{v}_{m,t}^j), \text{ where } \boldsymbol{y}_i = (y_i^1, \ldots, y_i^S). \tag{14}$$

Function (14) is linear in $\boldsymbol{y}_i$. By substituting (14) in constraint (2), we get the following adjustment functions

$$u_{i,t}^s = \sum_{j=1}^{S} y_i^j K(\boldsymbol{v}_{m,t}^s, \boldsymbol{v}_{m,t}^j) \qquad i = 1, \ldots, N; \quad t = 1, \ldots, T; \quad s = 1, \ldots, S. \tag{15}$$

We use L2 norm as the regularization function $G(\boldsymbol{y}_i)$,

$$G(\boldsymbol{y}_i) = ||\boldsymbol{y}_i||_2^2 = \sum_{s=1}^{S} (y_i^s)^2. \tag{16}$$

Substituting (16) in the objective, gives

$$-\frac{1}{S} \sum_{t=1}^{T} \sum_{s=1}^{S} p_t d_t^s V_t^s + \lambda \sum_{i=1}^{N} ||\boldsymbol{y}_i||_2^2 + \sum_{t=1}^{T} \kappa_t \left[L - A_t^s z - \sum_{i=1}^{N} R_{i,t}^s \right]^+ . \tag{17}$$

This model can be reduced to convex quadratic problem by linearizing (9). Other formulations are also possible. For example using L1 norm instead of L2 norm in (16) leads to a linear programming formulation after linearization of (9). Another variation of this model could be linear (with respect to rates $\boldsymbol{r}_{i,t}^s$) adjustment functions instead of the nonlinear kernel adjustment functions. Linear investment adjustments will lead to a lower terminal wealth. However the dimensionality of the problem will be reduced significantly, because the problem size (the number of parameters to be optimized) will increase linearly with the number of scenarios, instead of quadratically, with kernel functions.

5 Simulation of Return Scenarios and Mortality Probabilities

5.1 Historical Simulations

We simulate return scenarios of considered investment instruments for T years in the future. The simulations are based on end-of-year data of N assets over $\bar{T}$ years. Let $\bar{t} \in \{1, \ldots, \bar{T}\}$ be a year

	$\hat{p}$(age)	
Age	Male	Female
65	0.0158	0.0098
66	0.0170	0.0107
...	...	...
119	0.8820	0.8820

Table 1: USA Mortality table for the year 2017 with probabilities of death for male and female USA citizens. This table give a conditional probability of death at some age, given that person is alive at year earlier of that age.

index for a historical dataset and $\bar{r}_{i,\bar{t}}$ be a historical return of asset i. The returns of the indices are represented as the $N \times \bar{T}$ matrix,

$$\begin{bmatrix} \bar{r}_{1,1} & \bar{r}_{2,1}, & \dots, & \bar{r}_{N,1} \\ \bar{r}_{1,2} & \bar{r}_{2,2}, & \dots, & \bar{r}_{N,2} \\ \dots & \dots & \dots & \dots \\ \bar{r}_{1,\bar{T}} & \bar{r}_{2,\bar{T}}, & \dots, & \bar{r}_{N,\bar{T}} \end{bmatrix} \tag{18}$$

We generate return sample paths (scenarios) with the historical simulation method, also known as the “Bootstrap” method. The historical simulation method samples a random row from the matrix (18) and uses this row as a possible future realization of returns of instruments. Therefore the future simulation of returns is just sampling of the matrix (18) with replacement. Each such sample represents a future dynamics of return of the assets. Note that the simulation method samples entire row from matrix (18), therefore the correlations among assets are maintained in the random sample.

5.2 mortality probabilities p_t

Let τ be a random variable that denotes an age of death of the investor. The probability that an investor dies in time interval $[t-1, t)$ since retirement at the age 65 is defined as follows

$$p_t = \mathbb{P}(t + 64 < \tau \le t + 65 \mid \tau > 65), \qquad t = 1, \dots, T.$$

It is possible to calculate p_t using the mortality table of USA. We use the mortality Table 1, which gives probability $\hat{p}_t$ that $t + 64 < \tau \le t + 65$, conditional that $\tau > t + 64$,

$$\hat{p}_t = \mathbb{P}(t + 64 < \tau \le t + 65 | \tau > t + 64), \qquad t = 1, \dots, T.$$

It can be shown that

$$p_t = \begin{cases} \hat{p}_t, & \text{if } t = 1 \\ \hat{p}_t \prod_{j=1}^{t-1} (1 - \hat{p}_j), & \text{if } t = 2, \dots, T \end{cases}$$

Figure 1 shows the function p_t.

6 Case Study

6.1 Case Study Parameters

The case study results, codes, and data are posted at web, see Pertaia and Uryasev [2019].

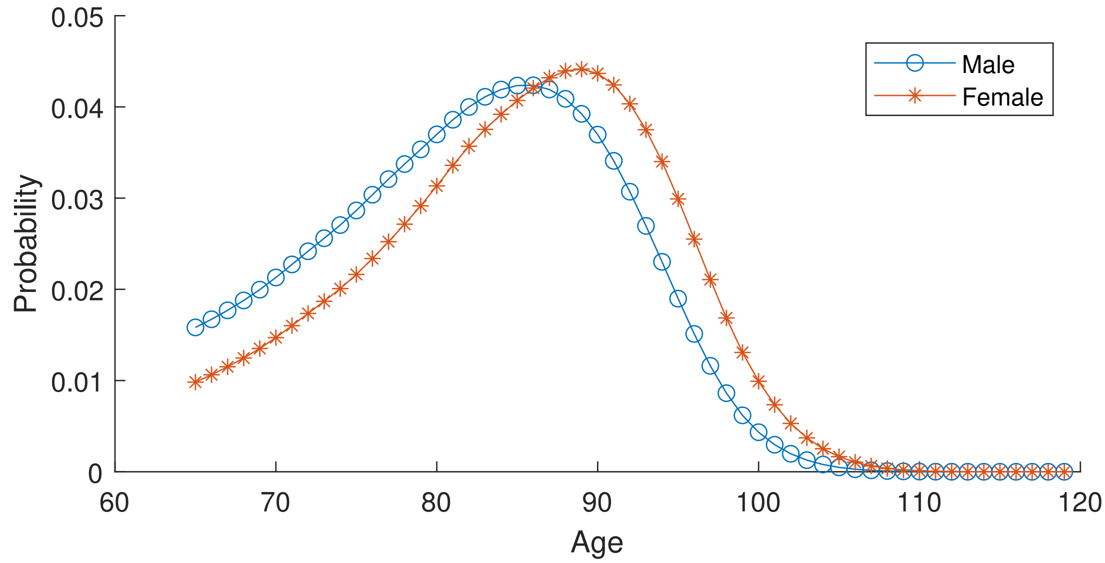

Figure 1: Probabilities that person dies while he/she is $t + 64$ years old ($t = 1, \ldots, T$), conditional that he/she is alive at the age of 65.

This case study considers a typical retirement situation in USA. Two variants of future asset return scenarios are considered. These two variants correspond to an optimistic and pessimistic views regarding the future market dynamics. In the optimistic case, the future returns over 35 years, for all instruments, are sampled from the historical returns over the recent 30 years. In the pessimistic case, the market is assumed to enter into a stagnation, similar to the Japaneses market, which has approximately zero cumulative return for the recent 30 years. In the pessimistic case, 12% is subtracted from each asset return, every year in every scenario.

Here are parameters of the model, which correspond to a typical retirement conditions in USA.

- The retiree is 65 years old.
- Investment horizon is 35 years.
- Portfolio is re-balanced at the end of each year.
- Retiree is a male (mortality probabilities for males are used in objective function).
- \$500,000 is available for investment at time $t = 0$.
- Yearly inflation rate is 3% during the entire investment horizon.
- Yearly rate of return of annuity is 5%.
- Adjustment rules use kernel functions with parameter $\sigma = 1$.
- $\lambda = 100$
- $\kappa_t = 2 \cdot 1.2^{(35-t)}$
- $\alpha = 20\%$

- $m = 5$

There are 10 stock and bond indexes available for investment, see Table 2.

Index Name	Index Abbreviation
Barclays Muni	FI-MUNI
Barclays Agg	FI-INVGRD
Russell 2000	USEQ-SM
Russell 2000 Value	USEQ-SMVAL
Russell 2000 Growth	USEQ-SMGRTH
S&P 500	USEQ-LG
S&P 400 Mid Cap	USEQ-MID
S&P Citi 500 Value	USEQ-LGVAL
S&P Citi 500 Growth	USEQ-LGGRTH
MSCI EAFE	NUSEQ

Table 2: The list of assets in the retirement portfolio.

For each index, 30 years of yearly returns (from 1985, to 2015) are used to create future scenarios (return sample-paths). Each scenario includes 35 yearly returns, sampled from the 30 year historical dataset (see the Historical Simulation method in Section 5). 200 scenarios are generated for both, optimistic and pessimistic cases. 100 scenarios out of 200, for both optimistic and pessimistic scenario datasets, are used to find optimal investment rules in the model. The remaining 100 scenarios, not included in the optimization, are used for evaluating the out-of-sample performance of the model.

6.2 Optimal Portfolio

The considered optimization problems are reduces to Quadratic Programming, by linearizing function (9) in the objective. Gurobi version 8.1.0 and Pyomo version 5.5.0 are used for solving th resulting quadratic programming problem. The case study link (Pertaia and Uryasev [2019]) contains the corresponding code.

The coefficients of the adjustment functions $\boldsymbol{y}_i$, are obtained by solving the quadratic optimization problem corresponding to the optimal portfolio problem (12). Next, the adjustment values for the out-of-sample dataset are evaluated, according to the formula (14). The adjustment functions, for end of the time moment t, take previous m rates of returns of all assets in the portfolio, observed in time interval $[t-m, t-1]$ and produce an asset adjustment for that time moment. Note that returns that go into these functions are different on each scenario, therefore the adjustment values will be different on each scenario, as well.

In order to calculate the portfolio values on the out-of-sample data, the cash outflows $R^s_{i,t}$ are required. The model does not provide the cash outflow $R^s_{i,t}$ for the out-of-sample scenarios, as those values are calculated for the in-sample scenarios. Therefore, it is unclear what values of $R^s_{i,t}$ should be use in the out-of-sample scenarios. Additionally, despite the constraint on positivity of asset positions in the in-sample optimization problems, a small portion of the assets may be allocated to short positions in out-of-sample runs. Usually, the retirement portfolios do not have short positions, since it is considered a risky strategy and therefore not suitable for a risk averse retiree investors. Next, we show how to circumvent these problems for the out of sample datasets.

Let $P^{s,t}_+$ and $P^{s,t}_-$ be the total dollar investment in long and short positions, in a portfolio at the end of time period t in scenario s,

$$P_+^{s,t} = \sum_{i=1}^{N} [x_{i,t}^s]^+ ,$$
$$P_-^{s,t} = \sum_{i=1}^{N} [-x_{i,t}^s]^+.$$

The cash outflows are calculated as follows

$$R_{i,t}^s = L \frac{[x_{i,t-1}^s]^+}{P_+^{s,t-1}}. \tag{19}$$

So the cash outflows originate only from the long positions and are proportional to $P_+^{s,t-1}$.

All short positions, at the end of time period t in scenario s, are set to 0. As a result, the amount of money equal to $P_-^{s,t}$ has to be subtracted from the remaining (long) part of the portfolio. To shrink the portfolio by $P_-^{s,t}$, each long asset position is reduced in a proportion to $P_+^{s,t}$. Thus, the new positions $\bar{x}_{i,t}^s$ are

$$\bar{x}_{i,t}^s = \begin{cases} 0, & \text{if } x_{i,t}^s \leq 0 \\ x_{i,t}^s - \frac{x_{i,t}^s}{P_+^{s,t}} P_-^{s,t}, & \text{otherwise.} \end{cases}$$

Tables 3 through 7 show the average (over scenarios) investments in assets over time for optimistic out-of-sample scenarios, corresponding to the model (12), with the minimum cash flows requirements $L \in \{\$10,000; \$30,000; \ldots, \$90,000\}$. Tables 8,10 and 8 show the average (over scenarios) investments in assets over time for pessimistic out-of-sample scenarios, corresponding to the model (12), with the minimum cash flows requirements $L \in \{\$10,000; \$25,000; \$30,000; \$50,000\}$. Tables 8,10 and 11, show that, in the pessimistic case, for $L = \$10,000$, the model invests 30% of funds in the annuity and for $L = \$25,000$, 100% of investment goes into the annuities. However for $L = \$30,000$ the model decreases the annuity investment to 56%. As for $L = \$50,000$ (and higher) nothing is invested in the annuities and the model selects the stock/bond indexes. The graph 2 shows the average (taken over scenarios) portfolio values through time, constructed using the adjustment functions, corresponding to the model (12) with the minimum cash flows requirements of $L \in \{\$10,000; \$30,000; \ldots, \$90,000\}$. However in the optimistic scenarios, the model does not invest in annuities at any minimum cash outflow requirement L.

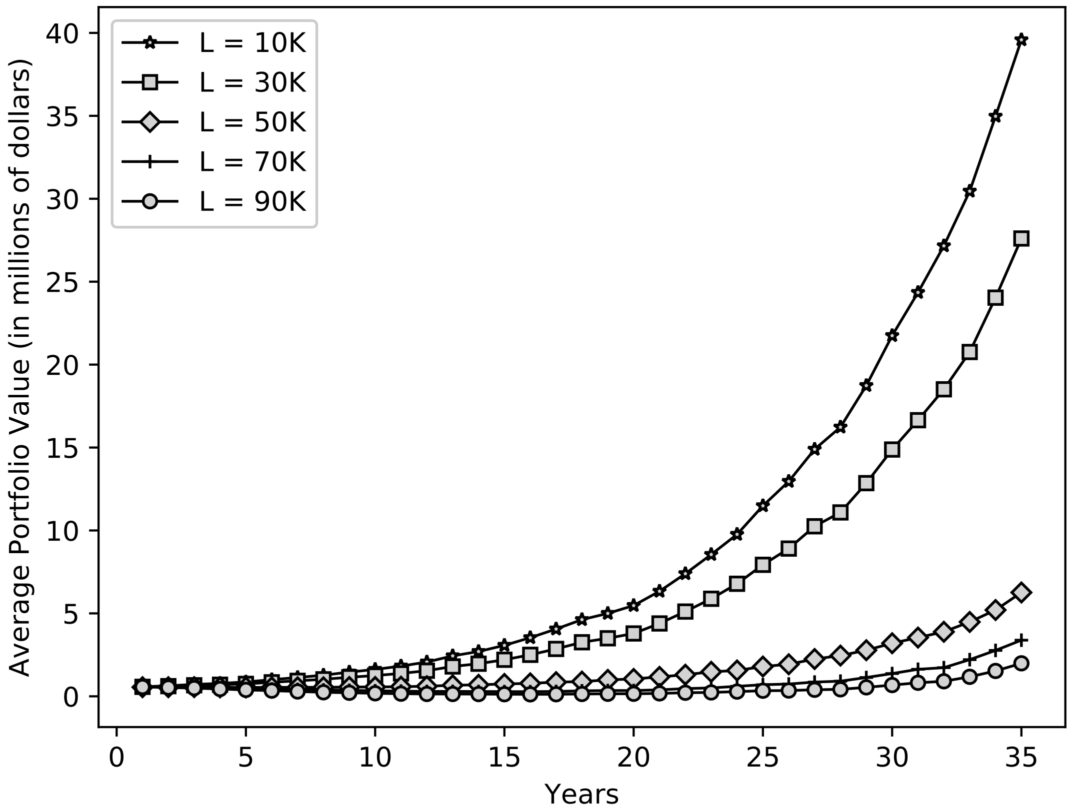

Figure 2: The average(over scenarios) portfolio value for the optimistic out-of-sample dataset, constructed using adjustment functions, corresponding to model (12) wit minimum cash outflow requirements $L \in \{\$10,000; \$30,000; \ldots, \$90,000\}$

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
Annuity	0	0	0	0	0	0	0	0
FI-MUNI	0	0	0	0	0	0	0	0
FI-INVGRD	0	3	4	6	7	11	16	25
USEQ-SM	0	0	0	0	0	0	0	0
USEQ-SMVAL	0	28	54	104	171	360	635	1,177
USEQ-SMGRTH	0	1	1	2	4	7	13	19
USEQ-LG	0	0	0	0	0	0	0	0
USEQ-MID	500	779	1,475	2,791	4,993	10,593	20,183	36,797
USEQ-LGVAL	0	0	0	0	0	0	0	0
USEQ-LGGRTH	0	4	8	15	30	72	139	380
NUSEQ	0	50	80	153	268	444	762	1,186

Table 3: Average investment (in thousand dollars) in assets over time for the optimistic out-of-sample scenarios with $L = \$10,000$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
Annuity	0	0	0	0	0	0	0	0
FI-MUNI	0	0	0	0	0	0	0	0
FI-INVGRD	3	34	44	51	64	95	138	194
USEQ-SM	0	0	0	0	0	0	0	0
USEQ-SMVAL	69	28	57	105	192	366	592	1,121
USEQ-SMGRTH	0	1	2	4	7	14	27	42
USEQ-LG	0	0	0	0	0	0	0	0
USEQ-MID	402	612	1,025	1,818	3,136	6,642	12,574	22,657
USEQ-LGVAL	0	0	0	0	0	0	0	0
USEQ-LGGRTH	25	30	54	102	181	448	920	2,594
NUSEQ	0	45	69	124	209	365	628	996

Table 4: Average investment (in thousand dollars) in assets over time for the optimistic out-of-sample scenarios with $L = \$30,000$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
Annuity	0	0	0	0	0	0	0	0
FI-MUNI	0	7	7	7	7	10	14	21
FI-INVGRD	330	244	202	206	246	328	492	680
USEQ-SM	0	0	0	0	0	0	0	0
USEQ-SMVAL	57	137	194	281	424	693	1,163	1,875
USEQ-SMGRTH	0	0	0	0	0	0	0	0
USEQ-LG	0	0	0	0	0	0	0	0
USEQ-MID	36	47	58	84	108	224	416	820
USEQ-LGVAL	0	0	0	0	0	0	0	0
USEQ-LGGRTH	77	65	66	92	157	386	857	2,515
NUSEQ	0	33	35	74	104	154	255	349

Table 5: Average investment (in thousand dollars) in assets over time for the optimistic out-of-sample scenarios with $L = \$50,000$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
Annuity	0	0	0	0	0	0	0	0
FI-MUNI	0	0	0	0	0	0	0	0
FI-INVGRD	195	117	67	40	32	35	44	56
USEQ-SM	0	0	0	0	0	0	0	0
USEQ-SMVAL	46	66	67	48	43	65	99	132
USEQ-SMGRTH	0	0	0	0	0	0	0	0
USEQ-LG	0	0	0	0	0	0	0	0
USEQ-MID	107	118	73	69	88	170	320	596
USEQ-LGVAL	0	0	0	0	0	0	0	0
USEQ-LGGRTH	136	92	77	90	142	350	748	2,300
NUSEQ	16	67	48	35	42	78	162	300

Table 6: Average investment (in thousand dollars) in assets over time for the optimistic out-of-sample scenarios with $L = \$70,000$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
Annuity	0	0	0	0	0	0	0	0
FI-MUNI	0	0	0	0	0	0	0	0
FI-INVGRD	65	54	17	6	5	5	6	7
USEQ-SM	0	0	0	0	0	0	0	0
USEQ-SMVAL	70	83	51	30	29	46	76	115
USEQ-SMGRTH	0	0	0	0	0	0	0	0
USEQ-LG	0	0	0	0	0	0	0	0
USEQ-MID	164	85	30	28	33	67	133	302
USEQ-LGVAL	0	0	0	0	0	0	0	0
USEQ-LGGRTH	140	107	56	48	76	204	439	1,522
NUSEQ	61	58	26	12	9	14	23	42

Table 7: Average investment (in thousand dollars) in assets over time for the optimistic out-of-sample scenarios with $L = \$90,000$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
Annuity	147	147	147	147	147	147	147	147
FI-MUNI	0	0	0	0	0	0	0	0
FI-INVGRD	1	3	3	2	2	2	1	1
USEQ-SM	0	0	0	0	0	0	0	0
USEQ-SMVAL	2	3	2	3	2	1	1	0
USEQ-SMGRTH	0	0	0	1	0	0	0	0
USEQ-LG	0	0	0	0	0	0	0	0
USEQ-MID	350	350	360	378	384	355	311	303
USEQ-LGVAL	0	0	0	0	0	0	0	0
USEQ-LGGRTH	0	4	7	7	7	5	5	4
NUSEQ	0	4	4	3	3	3	2	2

Table 8: Average investment (in thousand dollar) in assets over time for the pessimistic out-of-sample scenarios $L = \$10,000$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
Annuity	500	500	500	500	500	500	500	500
FI-MUNI	0	0	0	0	0	0	0	0
FI-INVGRD	0	0	0	0	0	0	0	0
USEQ-SM	0	0	0	0	0	0	0	0
USEQ-SMVAL	0	0	0	0	0	0	0	0
USEQ-SMGRTH	0	0	0	0	0	0	0	0
USEQ-LG	0	0	0	0	0	0	0	0
USEQ-MID	0	0	0	0	0	0	0	0
USEQ-LGVAL	0	0	0	0	0	0	0	0
USEQ-LGGRTH	0	0	0	0	0	0	0	0
NUSEQ	0	0	0	0	0	0	0	0

Table 9: Average investment (in thousand dollar) in assets over time for the pessimistic out-of-sample scenarios $L = \$25,000$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
Annuity	282	282	282	282	282	282	282	282
FI-MUNI	0	0	0	0	0	0	0	0
FI-INVGRD	43	15	1	0	0	0	0	0
USEQ-SM	0	0	0	0	0	0	0	0
USEQ-SMVAL	35	18	2	0	0	0	0	0
USEQ-SMGRTH	0	0	0	0	0	0	0	0
USEQ-LG	0	0	0	0	0	0	0	0
USEQ-MID	61	33	4	1	0	0	0	0
USEQ-LGVAL	0	0	0	0	0	0	0	0
USEQ-LGGRTH	54	27	3	0	0	0	0	0
NUSEQ	24	11	1	0	0	0	0	0

Table 10: Average investment (in thousand dollar) in assets over time for the pessimistic out-of-sample scenarios $L = \$30,000$. Average is taken over scenarios.

Asset Investment	t=0	t=5	t=10	t=15	t=20	t=25	t=30	t=35
Annuity	0	0	0	0	0	0	0	0
FI-MUNI	0	0	0	0	0	0	0	0
FI-INVGRD	67	32	6	1	0	0	0	0
USEQ-SM	0	0	0	0	0	0	0	0
USEQ-SMVAL	95	64	24	6	3	1	0	0
USEQ-SMGRTH	0	0	0	0	0	0	0	0
USEQ-LG	0	0	0	0	0	0	0	0
USEQ-MID	148	105	42	13	5	1	0	0
USEQ-LGVAL	0	0	0	0	0	0	0	0
USEQ-LGGRTH	128	85	30	8	3	1	0	0
NUSEQ	62	37	13	3	1	0	0	0

Table 11: Average investment (in thousand dollar) in assets over time for the pessimistic out-of-sample scenarios $L = \$50,000$. Average is taken over scenarios.

6.3 Expected Shortage Time for Different Cash Outflows L

When the investor demands higher cash outflows from the portfolio, the terminal value of the portfolio should decrease. Also, with higher cash outflow demands, there are higher chances that there will not be enough money in the portfolio, at some point, to finance these outflows.

To measure the cash outflow shortage resulting from the different values of L, the following measure, named *Expected Shortage Time* (or ETS) is defined

$$ETS(L) = \frac{1}{S} \sum_{s=1}^{S} \sum_{t=1}^{T} p_t (T-t) \frac{\left(L - \sum_{t=1}^{T} R_{i,t}^{s}\right)^{+}}{L}$$

ETS is measured in years and calculates the amount of time the retiree will spend without the necessary cash outflow L.

The parameters of the case study are used to construct the ETS values for the optimistic and pessimistic cases. ETS is calculated on the in-sample data, for the cash outflow values of $L \in \{\$10,000; \$15,000; \ldots; \$100,000\}$. The resulting ETS values are shown on graphs 3 and 4 for optimistic and pessimistic scenarios respectively.

The graph 3 shows that, in the optimistic scenario, the retiree can have cash outflows up to \$50,000, without heaving any shortage at any time. For the values of L greater than \$50,000, the ETS grows roughly linearly. For $L = \$100,000$ the retiree will spend most of his expected life without necessary cash outflow, because the portfolio can not provide this much cash outflow, given the initial investment of \$500,000.

It should be noted that, in the pessimistic case, if $L \leq \$25,000$ the annuities can fully cover the cash flow requirements and therefore ETS = 0. However, if $L > \$25,000$ the investment in the annuities can no longer cover the cash outflow requirements. Even if the entire initial investment goes into the annuities, it will provide only $A \cdot z = 3\% \cdot \$500,000 = \$25,000$. Therefore, for L values higher than \$25,000, the model starts to invest in stock and bond indexes and the ETS is greater than 0.

For the pessimistic scenario, if the cash flow requirement is $L = \$100,000$ the ETS is almost equal to the life expectancy of the retiree. This happens because, on most pessimistic scenarios,

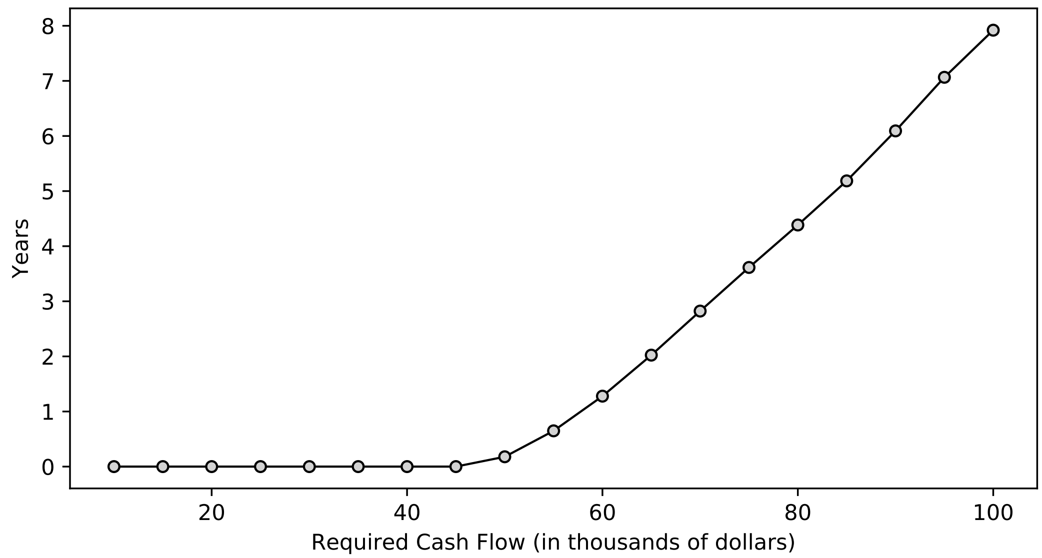

Figure 3: ETS values for required cash flows $L \in \{\$20,000; \$30,000; \ldots; \$100,000\}$, calculated for the optimistic scenario

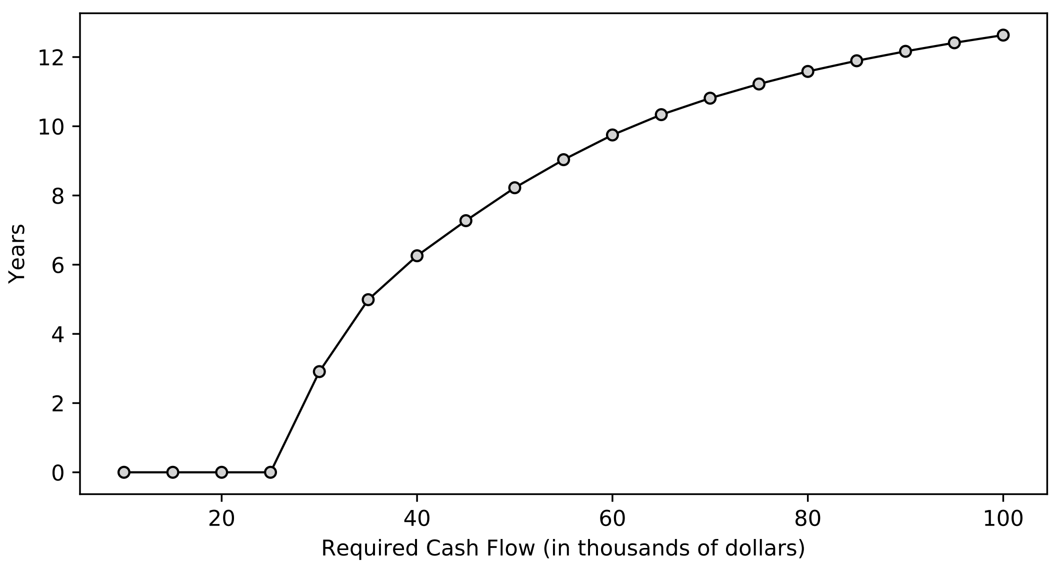

Figure 4: ETS values for required cash flows $L \in \{\$10,000; \$15,000; \ldots; \$100,000\}$, calculated for the pessimistic scenario

the portfolio shrinks to 0 in a 3 or 4 years for $L = \$100,000$. However, if $L = \$30,000$, in the pessimistic scenario, the retiree still has relatively small ETS values of around 3 year.

7 Summary

This paper developed a multi-period investment model for retirement portfolios. The parameters of the model represent a typical portfolio selection problem solved in the beginning of retirement. The model maximizes expected terminal wealth of an investor subject to constraints on minimum cash outflows from the portfolio. Investment decisions are based on adjustment rules implemented with kernel functions.

The case study showed performance of the model with pessimistic and optimistic asset return scenarios. In the pessimistic scenarios the market is assumed to enter a long term stagnation modeled by subtracting 12% from all rates of returns of the stock/bond indexes considered for investment. In this case it is optimal to invest a considerable portion of initial capital in annuities. In the optimistic case the returns of stock/bond indexes are expected to remain similar to past observations. In this case it is not beneficial to invest in the annuities, for the given model parameters.

We defined a new cash outflow shortage measure called Expected Shortage Time (ETS). The ETS calculates the number of years with shortage of cash outflows, given the retiree minimum cash outflow requirements. The case study shows that even in the pessimistic asset return scenarios a retiree can have zero ETS for some small cash outflows, due to a significant investment in the annuities.

References

E. Bogentoft, H. Romeijn, and S. Uryasev. Asset/liability management for pension funds using cvar constraints. *The Journal of Risk Finance*, 3(1):57–71, 2001.

G. C. Calafiore. Multi-period portfolio optimization with linear control pol-https://www.overleaf.com/project/5c2ceb14e2e3af05da32ac72icies. *Automatica*, 44:2463–2473, 2008.

A. Chekhlov, S. Uryasev, and M. Zabarankin. *Portfolio Optimization With Drawdown Constraints, B. Scherer (Ed.) Asset and Liability Management Tools, Risk Books, London.* 2003.

A. Chekhlov, S. Uryasev, and M. Zabarankin. Drawdown measure in portfolio optimization. *International Journal of Theoretical and Applied Finance*, 8(1):13–58, 2005.

P. Krokhmal, J. Palmquist, and S. Uryasev. Portfolio optimization with conditional value-at-risk objective and constraints. *The Journal of Risk*, 4:11–27, 2002.

H. Markowitz. Portfolio selection. *The Journal of Finance*, 7:77–91, 1952.

R. C. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51:247–257, 1969.

R. C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3:373–413, 1971.

J. M. Mulvey and B. Shetty. Financial planning via multi-stage stochastic optimization. *Computers and Operations Research*, 31(1):1–20, 2004.

J. M. Mulvey and H. Vladimirou. Stochastic network programming for financial planning problems. *Management Science*, 38(11):1642–1664, 1992.

G. Pertaia and S. Uryasev. Retirement portfolio selection case study, Jan. 2019. URL `https://www.ise.ufl.edu/uryasev/research/testproblems/financial_engineering/case-study-retirement-portfolio-selection/`.

N. A. Rizal and S. Wiryono. A literature review: Modelling dynamic portfolio strategy under defaultable assets with stochastic rate of return, rate of inflation and credit spread rate. *GSTF Journal on Business Review (GBR)*, 4(2), 2015.

R. Rockafellar and S. Uryasev. Optimization of conditional value-at-risk. *The Journal of Risk*, 2: 21–41, 2000.

R. Rockafellar and S. Uryasev. Conditional value-at-risk for general loss distributions. *Journal of Banking and Finance*, 26:1443–1471, 2002.

P. A. Samuelson. Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics*, 51:239–246, 1969.

D. Shang, V. Kuzmenko, and S. Uryasev. Cash flow matching with risks controlled by buffered probability of exceedance and conditional value-at-risk. *Annals of Operations Research*, pages 1–14, 2016.

Y. Takano and J. Gotoh. Multi-period portfolio selection using kernel-based control policy with dimensionality reduction. *Expert Systems with Applications*, 41:3901–3914, 2014.

M. Zabarankin, K. Pavlikov, and S. Uryasev. Capital asset pricing model (capm) with drawdown measure. *European Journal of Operational Research*, 234:508–517, 2014.
s