CASE STUDY: Portfolio Replication with Risk Constraint (meanabs_err, cvar_risk)

Background

The case study demonstrates three settings for a portfolio replication problem with the replication error measured by Mean Absolute Error. Underperformance of the portfolio compared to S&P100 index is measures by CVaR. Distribution of residuals is shaped with a CVaR constraint (several constraints can be specified, if of interest). We replicated S&P100 index using 30 stocks belonging to this index (tickers: GD, UIS, NSM, ORCL, CSCO, HET, BS, TXN, HM, INTC, RAL, NT, MER, KM, BHI, GEN, HAL, BDK, HWP, LTD, BAC, AVP, AXP, AA, BA, AGC, BAX, AIG, AN, AEP). Historical data on stock prices are used for building scenario matrices.

This case study was considered in Rockafellar and Uryasev (2002). For other references on portfolio replication, see, for instance, Andrews et al. (1986), Beasley and Meade (1999), Buckley and Korn (1998), Connor and Leland (1995), Dalh et al. (1993), Konno and Wijayanayake (2000), Rudd (1980), and Toy and Zurack (1989).

References

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Notations

I = number of instruments in the portfolio replicating of S&P100 index; $i=\{1,\dots,I\}$ index of instrument in the portfolio;

J = number of scenarios (days); $j=\{1,\dots,J\}$ index of scenarios;

U = budget available for investment to the replicating portfolio on day J ;

θ_{ij} = price of i-th instrument (i=1,…,I) in the portfolio on day j ;

θ_{0j} = price of S&P100 index on day j ;

$\boldsymbol{U}$ = amount of money to be on hand on the final day J ;

$\boldsymbol{\chi = \frac{U}{\theta_{0J}}}$ = the number of units of the S&P100 index at final day J ;

x_i = the number of shares of i-th instrument (i=1,…,I) in the replicating portfolio;

$\boldsymbol{\theta = (\theta_0, \theta_1, \ldots, \theta_I)}$ = random scenario vector having J equally probable outcomes $\boldsymbol{(\theta_{0j}, \theta_{1j}, \ldots, \theta_{Ij})}$ j={1,…,J};

$\boldsymbol{L(x,\theta_j) = \left(\theta_{0j}\chi - \sum_{i=1}^{I}\theta_{ij}x_i\right) / \theta_{0j}\chi = 1 - \sum_{i=1}^{I}\left(\frac{\theta_{ij}}{\theta_{0j}\chi}\right)x_i}$ = relative shortfall of the tracking portfolio on day j ;

$\boldsymbol{meanabs\_err(L(x,\theta)) = \frac{1}{J}\sum_{j=1}^{J}\left|L(x,\theta_j)\right| = \frac{1}{J}\sum_{j=1}^{J}\left|1 - \sum_{i=1}^{I}\left(\frac{\theta_{ij}}{\theta_{0j}\chi}\right)x_i\right|}$ = mean absolute error of the tracking portfolio;

w = bound on the underperformance of the portfolio compared to the index;
α = confidence level in CVaR.

Optimization Problem

minimizing replication error

$$\boldsymbol{\min_x meanabs\_err(L(x,\theta)) = \max_x\left[-meanabs\_err(L(x,\theta))\right]} \quad \text{(CS.1)}$$

subject to

CVaR constraint on the underperformance of the portfolio compared to the index

$$CVaR(L(x,\theta)) \le w \quad \text{(CS.2)}$$

budget constraint

$$\sum_{i=1}^{I}\theta_{iJ}x_i = U\ , \quad \text{(CS.3)}$$

no-short constraints on exposures

$$\boldsymbol{x_i \ge 0, \quad i = 1,\ldots,I}\ . \quad \text{(CS.4)}$$