CASE STUDY: Optimal Hedging of CDO Book (meanabs_err, polynom_abs, cardn)

Background

This case study demonstrates several optimization setups for a hedging problem. The objective is to build a portfolio of Credit Default Swaps (CDS) and Credit Indices that hedge against changes in a CDO (Collateralized Debt Obligation) book. Risk (hedging error) is defined by Mean Absolute Error of losses for the tracking portfolio. We minimize risk subject to budget and cardinality constraints.

- Optimization Problem 1 minimizes Mean Absolute Error of losses subject to budget constraint and cardinality constraint (defined with PSG cardinality function).
- Optimization Problem 2 minimizes Mean Absolute Error of losses subject to budget and cardinality constraints. Compared to Problem 1, the cardinality constraint is defined with a linear function on boolean variables.

CDSs are the most popular credit derivatives; they provide insurance against the risk of default (credit event) of a particular company. The buyer of the insurance has the right to sell bonds issued by the company for their face value when a credit event occurs to the seller of the insurance. The buyer of the CDS makes periodic payments to the seller until the end of the life of the CDS or until a credit event occurs. The total amount paid per year, as a percent of the notional principal, to buy protection is known as the *CDS spread*. Credit Indexes track CDS spreads. CDO is a credit derivative based on defaults of a pool of assets. CDO makes available credit risk exposure to various investors. A common structure of CDO involves tranching or slicing the credit risk of the reference pool into different risk levels. The risk of loss on the reference portfolio is divided into tranches of increasing seniority. The losses first affect the equity (first loss) tranche, then the mezzanine tranche, and finally the senior and super senior tranches.

References

- Hull, J., *Risk Management and Financial Institutions*. Prentice Hall, 2006.

Notations

$\boldsymbol{I}$=number of CDS in the portfolio; $i=\{1,\ldots,I\}$ is the CDS number;

$\boldsymbol{M}$=number of indexes in the portfolio; $m=\{1,\ldots,M\}$ is the index number;

$\boldsymbol{J}$=number of scenarios; $j=\{1,\ldots,J\}$;

$\boldsymbol{h}=(h_1,\ldots,h_I)$ = decision vector of hedge positions in CDSs;

$\bar{\boldsymbol{h}}=(\bar{h}_1,\ldots,\bar{h}_M)$ = decision vector of hedge positions in indices;

$\boldsymbol{y}=(y_1,\ldots,y_I)$ = decision vector with Boolean components (defining positions in CDSs);

$\boldsymbol{z}=(z_1,\ldots,z_M)$ = decision vector with Boolean components (defining positions in indices);

P_i = change of the book value upon 1bp change in the credit spread of the i-th name (the book PVBP);

$\boldsymbol{D}=(D_1^c,\ldots,D_I^c)$ = vector of durations of CDSs;

$\bar{\boldsymbol{D}}=(\bar{D}_1,\ldots,\bar{D}_M)$ = vector of durations of indexes;

k_i = upper bound on position in CDS *i;*

$\bar{k}_m$ = upper bound on position in index *m;*

BO_i = bid-offer spread for CDS *i;*

$\bar{B}\bar{O}_m$ = bid-offer spread for index *m;*

C = available capital for hedging;

$S_{i,j}$ = spread for *CDS i* on scenario *j;*

$\bar{S}_{m,j}$ = spread for index *m* on scenario *j;*

$\Delta S_{i,j}$ = change in CDS *i* spread on scenario *j;*

$$\Delta S_{i,j} = S_{i,j} - S_{i\ ,j-1}$$

$\Delta \bar{S}_{m,j}$ = change in index *m* spread on scenario *j;*

$$\Delta \bar{S}_{m,j} = \bar{S}_{m,j} - \bar{S}_{m\ ,j-1}$$

$\Delta \Pi = \sum_i S_{i,j}(P_i + h_i D_i)\frac{\Delta S_{i,j}}{S_{i,j}} + \sum_m \bar{S}_{m,j}\bar{h}_m \bar{D}_m \frac{\Delta \bar{S}_{m,j}}{\bar{S}_{m,j}}$ = change in portfolio value on scenario *j*;

$$L(\mathrm{h}, \bar{\mathrm{h}}, \Delta \mathrm{S}_{\mathrm{j}}, \Delta \bar{\mathrm{S}}_{\mathrm{j}}) = \sum_i S_{i,j} h_i D_i \frac{\Delta S_{i,j}}{S_{i,j}} + \sum_m \bar{S}_{m,j}\bar{h}_m \bar{D}_m \frac{\Delta \bar{S}_{m,j}}{\bar{S}_{m,j}} - \sum_i S_{i,j} P_i \frac{\Delta S_{i,j}}{S_{i,j}}$$

= portfolio loss on scenario *j*;

$\mathbf{meanabs\_err}(L(\mathrm{h}, \bar{\mathrm{h}}, \Delta \mathrm{S}, \Delta \bar{\mathrm{S}})) = \frac{1}{J}\sum_{j=1}^{J}\left| L(\mathrm{h}, \bar{\mathrm{h}}, \Delta \mathrm{S}_{\mathrm{j}}, \Delta \bar{\mathrm{S}}_{\mathrm{j}}) \right|$ = mean absolute penalty;

$\mathbf{polynom\_abs}(\mathrm{h}, \bar{\mathrm{h}}) = \sum_{i=1}^{I}\frac{BO_i D_i}{2}\left| h_i \right| + \sum_{m=1}^{M}\frac{\bar{B}\bar{O}_m \bar{D}_m}{2}\left| \bar{h}_m \right|$ = polynomial absolute function;

$$\mathbf{linear}(\boldsymbol{y}, \boldsymbol{z}) = \sum_{i=1}^{I} y_i + \sum_{m=1}^{M} z_m \ ;$$

$$\mathbf{cardn}(\mathrm{h}, \bar{\mathrm{h}}, w) = \sum_{i=1}^{I} u\left(\left(\frac{BO_i D_i}{2}\cdot h_i\right), w\right) + \sum_{m=1}^{M} u\left(\left(\frac{\bar{B}\bar{O}_m \bar{D}_m}{2}\bar{h}_m\right), w\right),$$

where

$$u(y, w) = \begin{cases} 0, & \text{if } -w < \mathrm{y} < w \\ 1, & \text{otherwise} \end{cases};$$

w is threshold value;

$Card$ = upper bound on cardinality.

Optimization Problem 1

minimizing hedging risk

$$\textbf{min} \ \ \textbf{meanabs\_err}(L(\mathrm{h},\bar{\mathrm{h}},\Delta\mathrm{S},\Delta\bar{\mathrm{S}}))$$

subject to

budget constraint

$$\textbf{polynom\_abs}(\mathrm{h},\bar{\mathrm{h}}) \le C2$$

constraint on cardinality

$$\textbf{cardn}(\mathrm{h},\bar{\mathrm{h}},w) \le Card$$

bounds on positions

$$|h_i| \le k_i,\ \ i = 1,\ldots,I\ ,$$

$$|\bar{h}_m| \le \bar{k}_m,\ \ m = 1,\ldots,M$$

Optimization Problem 2

minimizing hedging risk

$$\textbf{min} \ \ \textbf{meanabs\_err}(L(\mathrm{h},\bar{\mathrm{h}},\Delta\mathrm{S},\Delta\bar{\mathrm{S}}))$$

subject to

budget constraint

$$\textbf{polynom\_abs}(\mathrm{h},\bar{\mathrm{h}}) \le C2$$

constraint on cardinality

$$\textbf{linear}(\boldsymbol{y},\boldsymbol{z}) \le Card$$

bounds on positions

$$|h_i| \le k_i y_i,\ \ i = 1,\ldots,I\ ,$$

$$|\bar{h}_m| \le \bar{k}_m z_m,\ \ m = 1,\ldots,M$$

definition of Boolean variables

$$y_i \in \{0,1\},\ \ i = 1,\ldots,I;\ \ z_m \in \{0,1\},\ \ m = 1,\ldots,M$$