A New Approach to Credit Ratings

Giorgi Pertaia
Industrial and Systems Engineering
University of Florida

Stan Uryasev
Industrial and Systems Engineering
University of Florida

Artem Prokhorov
Business School
University of Sydney

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Abstract

Credit ratings are fundamental in assessing the credit risk of a security or debtor. Traditional credit ratings fail to provide a reliable risk assessments, which is evidenced, for example, by the failure of CDO ratings during the financial crisis of 2007-2008 and by the massive undervaluation of corporate risk leading up to the crisis. This paper reconsider the roots of the problems and present a new rating methodology based on the Buffered Probability of Exceedance (bPOE), which offers an improved and more conservative risk assessment.

1 Introduction

At the height of the financial crisis of 2008, American International Group, Inc. (AIG), once the largest insurance company in the US, was rescued from bankruptcy by a US government bailout worth $85 bn [see, e.g., Sjostrum, 2009]. This was part of the Troubled Asset Relief Program (TARP) that cost the US taxpayer in excess of $245 bn. What caused the companies that enjoyed stable AAA credit ratings to fail abruptly and what role did credit ratings play in the failure?

Early post-crisis literature focused on issues of risk mispricing caused by using dependence models that fail to accommodate realistic tail behavior of joint defaults and on issues around structured finance where loan securitization obscured the true riskiness of the collateral. For example, Coval et al. [2009] and Zimmer [2012] look at how securitized risky debt was repackaged as virtually risk-free. They show that the rating agencies were simply unfamiliar with assessing creditworthiness of financial instruments that cannot be ascribed to a single company and instead involve pooling loans, bonds and mortgages from various sources. The subsequent issue of claims, known as synthetic instruments, against those assets, prioritized using risk categories, known as tranches, were difficult to price.

Coval et al. [2009] make the point that the new developments in structured finance amplified errors in risk assessment, while Zimmer [2012] show that the commonly used dependence assumption known as the Gaussian copula was inappropriate. As a result, imprecision in credit risk estimation led to variations in the default risk of the synthetic securities that were large enough to cause an AAA-rated security to default with a high probability.

Ashcraft et al. [2011] looked at a large number of mortgage-backed securities (MBS), collaterized debt obligations (CDO) and other structured finance securities and found empirical evidence that higher credit ratings were closely associated with higher MBS prices after controlling for a large set of security fundamentals. They report that, in terms of value, 80 to 90 percent

of sub-prime MBS initially received AAA ratings but were in effect 6-10 rating notches lower [1]. This offers support for the widely held belief that more conservative credit ratings would have muted the crisis by making credit more expensive and providing reliable information about synthetic instruments to less informed investor. Bolton et al. [2012] describe the various conflicts of interest that may have added to the inability or unwillingness of credit ranking agencies to do that.

Moody's, Standard and Poor's and Fitch Group – the three major credit rating agencies known as the Big Three – have evolved since then. They are now more mindful of joint tail risk and synthetic instruments are hardly new any more. More recent papers focus on how credit rating inflation is affected by competition between agencies, by regulation of the industry and by business cycle [see, e.g., Dilly and Mahlmann, 2016, Alp, 2013, Altman and Rijken, 2004, Amato and Furfine, 2004, Baghai et al., 2014, Opp et al., 2013, He et al., 2012, Bar-Isaac and Shapiro, 2013, Rabanal and Rud, 2018]. For example, Dilly and Mahlmann [2016] find recent evidence credit ratings are inflated during the boom periods and Bar-Isaac and Shapiro [2013] present a model where ratings quality is countercyclical. Yet, the "boom bias" does not result from changes in rating regulations or competitive pressure but rather from rating agencies incentive conflicts.

This paper shows that the fundamental underlying risk measure used in credit ratings remains the same as before the crisis. The basic observation is that credit rating agencies still assign ratings based on a default risk measure known as probability of exceedance (POE), which measures the chance of a default-level loss, not the exposure associated with the default. We argue that POE is an overly optimistic risk measure.

We offer an alternative, more conservative way of credit rating, which is based on what we call *Buffered Probability of Exceedance* (bPOE). Like POE, bPOE is tied to a threshold. Unlike POE, bPOE takes into account the magnitude of outcomes in the tail exceeding the threshold. It is possible to stretch the tail of the loss distribution and increase the exposure without increasing of POE, but not bPOE.

More formally, bPOE is the probability of a tail event with the mean value equal to a specified threshold. Therefore, by definition bPOE controls both the average magnitude of the tail and its probability, adding a "buffer" to POE. In the engineering literature this concept has been introduced by Mafusalov and Uryasev [2018a] as an extension of the buffered failure probability suggested by Rockafellar [2009] and explored by Rockafellar and Royset [2010].

Given the above definition, it is not surprising that bPOE is closely related to Conditional Value-at-Risk (CVaR). CVaR, also known under such names as Expected Shortfall (EF), Superquartile, Average VaR and Tail VaR, is defined as the mean value of the tail with a given probability. In response to the 2007-2009 crisis, the Basel Committee on Banking Supervision, among other measures, moved from using an unconditional Value-at-Risk (VaR) to ES in order to provide an additional buffer to capital reserve requirements of financial institutions. We explore the relationship between CVaR, bPOE and POE and argue that bPOE is particularly useful for extremely heavy tailed distributions of losses, when conditional expectations are not informative.

From the optimization perspective, the pair bPOE and CVaR has several advantages compared to the pair POE and VaR. First, bPOE is quasi-convex [see, e.g., Mafusalov and Uryasev, 2018a] and CVaR is convex [see, e.g., Rockafellar and Uryasev, 2002b]. This means that there are efficient algorithms for solving optimization problems involving these measures. Second, bPOE is a monotonic function of the underlying random variable and a strictly decreasing function of the threshold on the interval between the mean value and the essential supremum. Third, it is well known that CVaR is coherent in the sense of Artzner et al. [1999] [see, e.g., Pflug, 2000, Acerbi and Tasche, 2002, Rockafellar and Uryasev, 2002a].

[1] Rating agencies commonly use 21-22 notch scales, from AAA to C or AAA to D.

Rating \Time	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
AAA	0	0.03	0.13	0.24	0.35	0.46	0.52	0.61	0.66	0.72
AA	0.02	0.06	0.13	0.23	0.34	0.45	0.55	0.63	0.71	0.79
A	0.06	0.15	0.26	0.4	0.55	0.72	0.92	1.1	1.28	1.48
BBB	0.19	0.53	0.91	1.37	1.84	2.3	2.71	3.11	3.5	3.89
BB	0.73	2.25	4.07	5.86	7.51	9.03	10.34	11.49	12.53	13.45
B	3.77	8.56	12.66	15.82	18.27	20.26	21.89	23.19	24.32	25.37
CCC/C	26.36	35.54	40.83	44.05	46.43	47.28	48.24	49.05	49.95	50.6
Investment grade	0.1	0.28	0.48	0.73	0.98	1.24	1.49	1.72	1.94	2.17
Speculative grade	3.8	7.44	10.6	13.15	15.24	16.94	18.38	19.58	20.65	21.61
All rated	1.49	2.94	4.21	5.27	6.17	6.92	7.57	8.12	8.62	9.09

Table 1: PD vs S&P's rating.

From the statistical perspective, bPOE is not much harder to estimate than POE [see, e.g., Mafusalov et al., 2018]. Yet, the new measure provides a more realistic approach to credit ratings. Our example in Section ??? shows that using a "naked options investment strategy" it is possible to achieve an arbitrary high rating based on POE, while the actual risk as measured by bPOE tends to one.

We use bPOE to develop a simple improvement to credit rating, allowing for a buffer to the conventional POE-based ratings. We study the disparity between the two measures under the most popular statistical distributions used in structured finance. We also show how to assess statistical precision of the new measure using real-world data. As an illustration of the new approach we consider what happens to the creditworthiness of an insurance company as it accumulates exposure in a way AIG did in the early 2000s.

2 Probability of Exceedance, Loss Magnitude and Credit Ratings

As a risk measure, bPOE is gaining popularity in areas where tail events are important. For example, in engineering it has been used to assess tropical storm damages [see, e.g., Davis and Uryasev, 2016], to optimize network infrastructure [see, e.g., Norton and Uryasev, 2018c], and other areas. In machine learning, it has been used to improve on data mining algorithms [see, e.g., Norton and Uryasev, 2018b, Norton et al., 2017]. The measure remains very new to finance and has not been used in credit ratings.

Traditionally, credit ratings are driven by historical default rates. This measure gives the likelihood of a financial loss exceeding the default threshold for a given security or debtor. Based on historical data, a rating is assigned if POE is inside a range of default probability characterizing that specific rating. Agencies publish tables of default probabilities for each rating class over a given time horizon. Table 1 contains the conversion between Standard and Poor's rating and Probabilities of Default (PD). For example, the BBB rating is assigned to an entity with one-year PD in the range $0.08\% < PD \leq 0.23\%$.

When rating synthetic instruments, the agencies use complicated models reflecting the various assumptions and approaches involved in constructing the instruments. However, at the most basic level, the traditional approach to issuing a rating is quite simple. We will explain it with an example based on the Merton model.

Suppose that a firm finances its operation by issuing a single zero-coupon bond with face value B_T payable at time T. Assume that at every time $t \in [0, T]$ the company has total assets A_t. It is standard to assume that A_t follows a Geometric Brownian motion, in the Merton

model. Further, the Merton model assumes that the default of the company occurs when the firm has no capital (equity) to pay back the debt holders. Because the zero-coupon pays only at time T, default can occur only at T.

The probability of default at time T equals $\mathbb{P}(\text{default}) = \mathbb{P}(A_T < B_T)$. This formula can be rewritten in terms of POE by changing the sign of assets and liabilities,

$$\mathbb{P}(\text{default}) = \mathbb{P}(A_T < B_T) = \mathbb{P}(-A_T > -B_T).$$

Thus, PD is a POE of the random variable $-A_T$ with the threshold $-B_T$ and the probabilities in Table 1 can be used to convert the PD into a rating.

We illustrate POE on Figure 1 as the shaded area $1 - \alpha$. If we define Value-at-Risk (VaR) as the loss that is exceeded no more than a given (small) proportion of time $1 - \alpha$, then it is clear from the figure that POE is simply one minus the inverse of VaR.

The POE-VaR pair has been criticized on a number of conceptual and computational grounds. First, VaR is not a coherent risk measure because it fails the sub-additivity condition, which implies in essence that a diversified portfolio may have a higher, rather than lower, VaR [see, e.g., Ibragimov, 2009, Ibragimov and Prokhorov, 2016]. Second, VaR is non-continuous, non-differential and non-convex for empirical distributions – a major numerical difficulty for optimization algorithms [see, e.g., Uryasev, 2000].

Most importantly, the POE-VaR pair does not account for the magnitude of the default-level loss. A $(1 - \alpha)100\%$ probability default with a hundred billion dollar loss and with one billion dollar loss will have the same VaR as long as both losses are higher than the VaR for the distribution of these losses. Thus massive losses can be hidden behind a credit rating based on POE. This shortcoming of VaR has been recognized by the Basel Committee which replaced it by CVaR in capital reserve requirement calculation as part of Basel III.

Similarly, our approach to credit rating is to use bPOE instead of POE. It is not difficult to see that bPOE is related to CVaR in the same way as POE is related to VaR: bPOE is simply one minus the inverse of CVaR. We illustrate bPOE on Figure 2 as the shaded area. It is clear from the figure that bPOE provides the probability of a tail event with expected loss equal to CVaR, which captures the magnitude of the possible losses.

In these settings, VaR is an overly optimistic measure of risk due the insensitivity to the loss given default. In particular, if the loss distribution of one instrument is heavy-tailed while the other is light-tailed, POE-based ratings can be identical (in some cases, the instrument with heavy-tailed losses might even have a higher rating). Meanwhile, as we illustrate in Figure 3, the bPOE-based ratings are necessarily lower for instruments with heavy-tailed losses. Therefore we have a more realistic basis for credit rating and we remove the incentive to accumulate low default probabilities with high exposure.

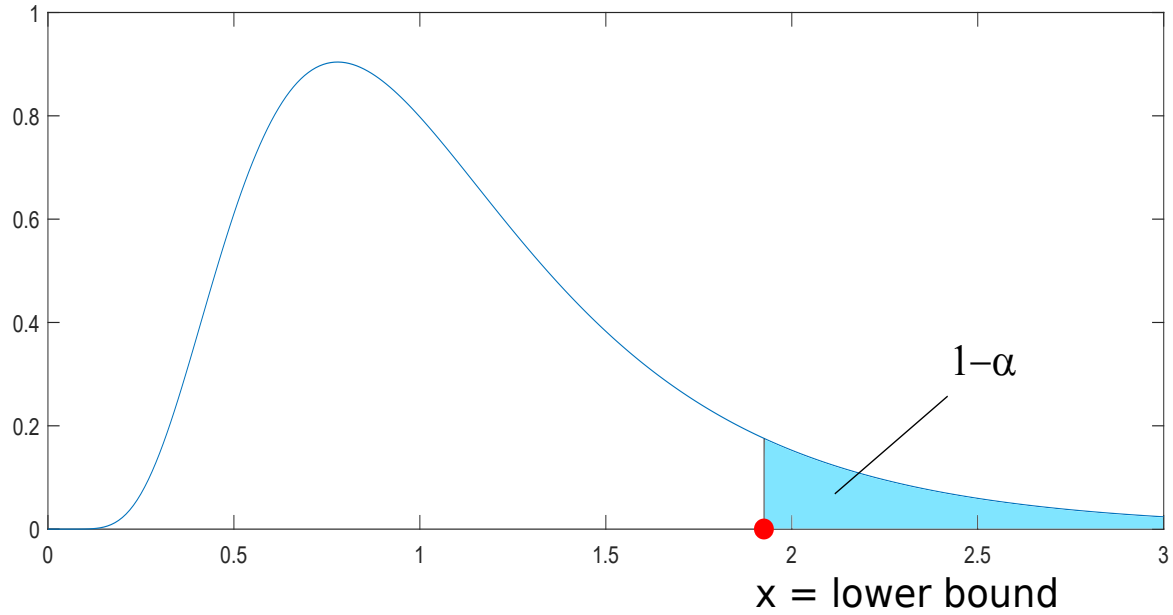

Figure 1: Relationship between POE and VaR: $\mathrm{VaR}(\alpha) \leq x \iff \mathrm{POE}(x) \leq 1 - \alpha$.

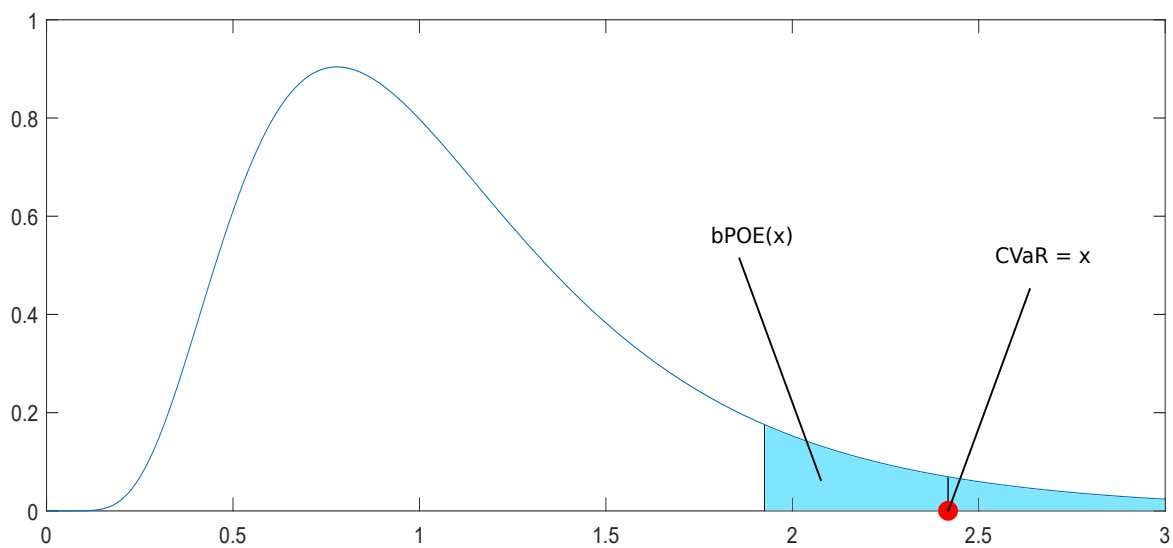

Figure 2: Relationship between bPOE and CVaR, $\mathrm{CVaR}(\alpha) \leq x \iff \mathrm{bPOE}(x) \leq 1 - \alpha$. The area of the shaded region is equal to the $\mathrm{bPOE}(x)$.

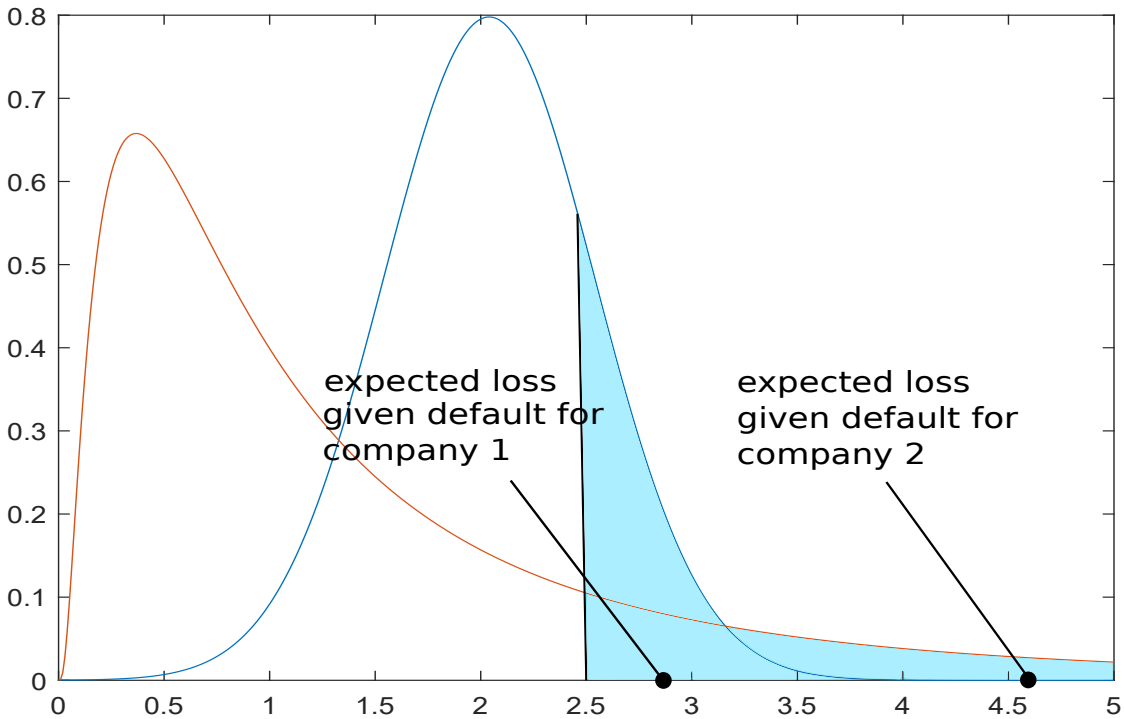

Figure 3: Loss distribution for two different companies. The POEs for both companies are equal however they have very different expected losses given default. The difference in expected loss given default comes from the fact that Company 2 has a heavy tailed loss distribution, while Company 1 does not.

3 bPOE vs POE: Mathematical Properties

We now turn to the mathematical properties of the bPOE-CVaR pair and compares them to the POE-VaR counterparts.

We start by noting that non-sub-additivity makes VaR a non-convex function. Non-convexity means that, as a rule, optimization problems with VaR constraints or with VaR as the objective function are intractable. At the same time, optimization problems involving CVaR constraints or with CVaR as the objective function are usually solvable in polynomial time, using convex or even linear programming methods.

Despite the wide adoption of CVaR in financial industry, there has not been an analogous substitution of POE-based methodologies with bPOE-based, even though bPOE inherits similar mathematical properties from CVaR. For example it is possible to efficiently solve a large dimensional portfolio optimization problem with a constraint on bPOE for the loss of the portfolio returns. Yet, in some critical areas, including credit rating and nuclear engineering, POE still serves as the valid regulation tool.

Given a random variable X, its cumulative distribution function F_X and some confidence level $\alpha \in (0, 1)$, VaR (or quantile) is defined as follows

$$VaR_\alpha(X) = \inf \left\{ v \in R \mid F_X(v) \geq \alpha \right\},$$

and CVaR is defined as follows

$$CVaR_\alpha(X) = \min_C \left(C + \frac{1}{1-\alpha}\mathbb{E}[X-C]^+ \right),$$

where $[x]^+ = \max\{x, 0\}$. For more detail on CVaR function and its properties see Rockafellar and Uryasev [2002a]. CVaR is a coherent risk measure as it satisfies the translation invariance, sub-additivity, positive homogeneity and monotonicity properties whenever it is well defined. The convexity of CVaR follows from the sub-additivity and positive homogeneity properties.

The relationship between bPOE and POE is similar to that between CVaR and VaR. The value of bPOE with threshold v of a random variable X equals to the probability mass in the right tail of the distribution of X such that the average value of X in this tail is equal to the threshold v. It is convenient to define bPOE formally as follows

$$bPOE_v(X) = \min_{a \geq 0} \mathbb{E}[\, a(X-v)+1]^+ \,. \tag{1}$$

bPOE is equal to one minus inverse of CVaR, where CVaR is the average of the tail having probability $1-\alpha$. For more detailed discussion of bPOE functions see Mafusalov and Uryasev [2018b]. Similarly, bPOE equals POE for the right tail with CVaR equal to v, see Figure 2.

The function $bPOE_v(X)$ is a quasi-convex function of the random variable X and the quasi-convexity makes bPOE well-suited for optimization, while POE-based problems are, again, intractable in general [see Rockafellar and Uryasev, 2002a, Mafusalov and Uryasev, 2018b, for more details]. Similar to CVaR, the quasi-convexity makes it possible to efficiently solve problems that have constraints on bPOE. For example, efficient solvers of portfolio optimization problems with constraints on the bPOE of the portfolio loss are available.

4 bPOE Ratings

Our proposed methodology for assigning credit ratings consists in using bPOE, in place of POE, calculated for the same threshold. As bPOE, by construction, is always greater than the POE calculated for the same threshold, this means the new ratings are lower. For example, if the losses are distributed according to the standard normal distribution, bPOE is roughly 2.4 times higher than POE calculated for the same threshold; if they are distributed as log-normal with parameters $\mu = 0$ and $\sigma = 1$ then bPOE is roughly 3.2 time higher than POE. Figure 4 demonstrates the ratio bPOE/POE for standard normal losses as a function of POE (left panel) and as a function of quantile v (right panel).

Correspondingly, we propose to rescale the default probabilities in the rating table by multiplying them by the bPOE/POE ratio calculated for the exponential distribution. bPOE ratings will be calculated using resulting intervals from the new table. There are two reasons why exponential distribution is a good candidate for rescaling:

1. Exponential distribution is the "demarcation line" between heavy-tailed an light-tailed distributions. The distribution is called heavy-tailed if

   $$\lim_{v\to\infty} e^{\lambda v}\mathbb{P}(X \geq v) = \infty \quad \forall \lambda > 0.$$

   I.e., heavy-tailed distribution has heavier tails than the exponential distribution with arbitrary parameter λ.

2. The $bPOE(v)/POE(v)$ ratio for the exponential distribution with arbitrary parameter $\lambda > 0$ and arbitrary threshold value $v > \mathbb{E}X$ is constant and equal to $e = 2.718....$ Indeed, bPOE for the exponential distribution (see Mafusalov et al. [2018]) equals $e^{1-\lambda v}$. The POE for the exponential distribution equals $e^{-\lambda v}$, thus, the ratio of bPOE to POE is equal to e.

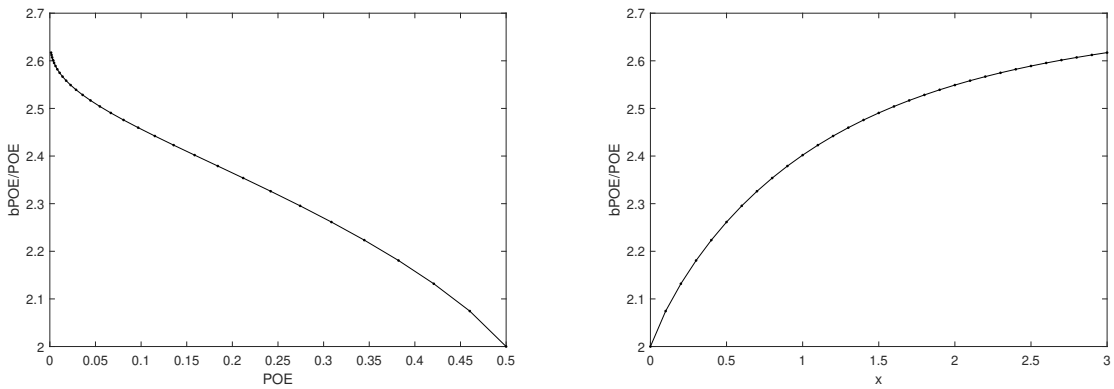

Figure 4: Left graph shows the relationship between $POE_v(X)$ and $bPOE_v(X)/POE_v(X)$ for the standard normal distribution. Right graph shows relationship between quantile v and $bPOE_v(X)/POE_v(X)$ for the standard normal distribution.

Rating \Time	**1**	**2**	**3**	**4**	**5**	**...**	**10**
AAA	0.00	0.08	0.35	0.65	0.95	...	1.96
AA	0.05	0.16	0.35	0.63	0.92	...	2.15
A	0.16	0.41	0.71	1.09	1.50	...	4.02
BBB	0.52	1.44	2.47	3.72	5.00	...	10.57
BB	1.98	6.12	11.06	15.93	20.41	...	36.56
B	10.25	23.27	34.41	43.00	49.66	...	68.96
CCC/C	71.65	96.61	100.00	100.00	100.00	...	100.00
Investment grade	0.27	0.76	1.30	1.98	2.66	...	5.90
Speculative grade	10.33	20.22	28.81	35.75	41.43	...	58.74

Table 2: The table of rating intervals for bPOEs calculated for the same threshold as the POEs. Each element of this table is simply a multiplication of Table 1 elements by e.

5 bPOE Rating: Uncovered Call Options Investment Strategy

This section contains two examples illustrating bPOE ratings for the investment strategy with uncovered call options. The second example serves as a conceptual explanation of the AIG operation, where the AIG loaded the book with upper tranches of CDOs without appropriate hedging. Tremendous exposure to uncovered upper tranches did not affect the rating of AIG, until it almost went bankrupt in the credit crisis of 2008. The problem is that the rating system, at the time, has not adequately captured the risk of AIG's portfolio. The following examples show the cases when the POE based rating does not measure the risk appropriately, versus bPOE capturing the risks.

Case I. Conceptual Comparison of POE and bPOE. Suppose that a portfolio manager sells a number of uncovered call options with same underlying asset and strike price K. Without loss of generality we can assume that interest rates are 0. Let $P(K)$ be the price of the option with strike price K. We assume that the portfolio has no capital, except proceeds from selling the call options. The portfolio manager sells $n_K = \frac{1}{P(K)}$ number of options so that the proceeds from the sale are equal to \$1. Given an underlying asset price S_T at maturity time T, the call option payoff at time T equals $f_K = \max\{S_T - K, 0\}$. The portfolio will have a negative

balance, when $n_K f_K - 1 > 0$, which is considered to be a default. Thus, the probability of portfolio default is $\mathbb{P}(n_K f_K - 1 > 0)$, which is POE of random variable $n_K f_K - 1$ with threshold 0. We assume that S_t evolves over time according to geometric Brownian motion, so

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t},$$

where W_t is the Wiener process. Price S_t is log-normally distributed with cumulative distribution function

$$F_t(x) = \frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{ln(x) - m_t}{\sqrt{2}s_t}\right)$$

with parameters $m_t = ln(S_0) + (\mu - 0.5\sigma^2)t$ and $s_t = \sigma\sqrt{t}$, for each time moment $t \in (0, T]$, (see chapter 14 in Hull [2009]). The probability of default equals

$$\mathbb{P}(n_K f_K - 1 > 0) = \mathbb{P}(f_K > \frac{1}{n_K}) = \mathbb{P}(\max\{S_T - K, 0\} > \frac{1}{n_K}) = \mathbb{P}(\max\{S_T - K - \frac{1}{n_K}, -\frac{1}{n_K}\} > 0)$$

$$= \mathbb{P}(S_T - K - \frac{1}{n_K} > 0) = \mathbb{P}(S_T > K + \frac{1}{n_K}) = 1 - \mathbb{P}(S_T < K + \frac{1}{n_K}) = 1 - F_T\big(K + \frac{1}{n_K}\big).$$

Since the distribution function tends to one,

$$F_T\big(K + \frac{1}{n_K}\big) \to 1 \quad \text{for} \quad K \to \infty,$$

then, the last chain of equalities implies

$$POE_0(n_K f_K - 1) = \mathbb{P}(n_K f_K - 1 > 0) \to 0 \quad \text{for} \quad K \to \infty. \tag{2}$$

Therefore, if POE is used for evaluation of rating, then the portfolio manager can set AAA portfolio rating for sufficiently large K. Further we will show that bPOE, for $n_K f_K - 1$ at the threshold value 0 always equals 1.

Proposition:

$$bPOE_0(n_K f_K - 1) = 1 \quad \forall K \in R.$$

Proof:

The probability density function of random variable $X_K = n_K f_K - 1$ has a single atom located at point $X_K = -1$ with probability $\mathbb{P}(n_K f_K - 1 = -1) = \mathbb{P}(S_T \leq K)$. The CDF of $X_K \geq x$ where $x \in (-1, \infty)$ is

$$\mathbb{P}(X_K \geq x) = \mathbb{P}(n_K f_K - 1 \geq x) = \mathbb{P}\left(f_K \geq \frac{x-1}{n_K}\right) = \mathbb{P}\left(S_T \geq \frac{x-1}{n_K} + K\right).$$

Thus, for values grater than -1, the distribution of X_K is the same as log-normal distribution corresponding to S_T, however it is shifted left by $K - 1/n_K$ and is scaled by n_K. Let $I_{\{*\}}$ be an indicator function. In order to calculate $bPOE_0(X_K)$ we need to consider two possibilities. In the firs case, when $\mathbb{E}\left[X_K I_{\{X_K > -1\}}\right] \leq 0$, the value $bPOE_0(X_K)$ can be calculated using only the log-normal part of the the distribution of X_K. However, in the second case, when $\mathbb{E}\left[X_K I_{\{X_K > -1\}}\right] > 0$, it is necessary to take some fraction the probability of the atom, in order to bring the conditional expectation down to value 0. Thus, in second case, $bPOE_0(X_K)$ is calculated as

$$bPOE_0(X_K) = \mathbb{P}(X_K > -1) + p_{-1},$$

where p_{-1} is the fraction of the probability of the atom, such that

$$0 = \mathbb{E}\left[X_K I_{\{X_K > -1\}}\right] + (-1)p_{-1}. \tag{3}$$

Because n_K is always positive

$$bPOE_0(X_K) = bPOE_0(n_K f_K - 1) = \mathbb{P}(f_K > 0) + p_{-1} = \mathbb{P}(S_T > K) + p_{-1}$$

Note that $I_{\{X_K > -1\}} = I_{\{n_K f_K - 1 > -1\}} = I_{\{S_T > K\}}$, then, from (3) we have

$$\begin{aligned} bPOE_0(X_K) &= \mathbb{P}(S_T > K) + \mathbb{E}\left[X_K I_{\{X_K > -1\}}\right] \\ &= \mathbb{P}(S_T > K) + \mathbb{E}\left[(n_K f_K - 1) I_{\{S_T > K\}}\right] \\ &= \mathbb{P}(S_T > K) + n_K \mathbb{E}\left[f_K I_{\{S_T > K\}}\right] - \mathbb{P}(S_T > K) = n_K \mathbb{E}\left[f_K\right] \end{aligned} \tag{4}$$

From the fundamental theorem of asset pricing (see chapter 14 in Hull [2009]) and our assumption that interest rates (including risk free) are 0, we know that

$$n_K = \frac{1}{P(K)} = \frac{1}{\mathbb{E}\left[f_K\right]} \tag{5}$$

Substituting the right-hand side of (5) in (4) we get

$$bPOE_0(X_K) = n_K \mathbb{E}\left[f_K\right] = \frac{1}{\mathbb{E}\left[f_K\right]} \mathbb{E}\left[f_K\right] = 1 \tag{6}$$

Finally, note that

$$\mathbb{E}\left[X_K I_{\{X_K > -1\}}\right] = n_K \mathbb{E}\left[f_K\right] - \mathbb{P}(S_T > K) = 1 - \mathbb{P}(S_T > K) > 0$$

holds for any K, thus the proposition is proven.

6 bPOE Rating Used in Risk Management

Before the crisis of 2008 CDOs were the major credit derivative instruments traded on the market. The CDO consists of the pool of the assets that generate cash flow. This asset pool is repackaged in number of tranches with ordered priority on the collateral in the event of default. Each of these tranches has a separate rating assigned to it. Usually the Senior tranche has the first priority on collateral payment in the event of default, this is why Senior tranche has the highest rating compared to other tranches in the CDO (Mezzanine and Equity tranche). Each tranche has an attachment and a detachment point that controls the amount of loss absorbed by a tranche in the event of default. A CDO tranche defaults when the cumulative loss reaches the attachment point. The tranche rating is calculated based on the POE of the loss distribution given the attachment point of the tranche as a threshold. These tranches are sold to investors as the separate assets that pays some cash flow. The payoff of the tranche depends on the assigned rating.

The synthetic CDO consists of the pool of credit default swaps (CDS).The CDS buyer pays some amount of money over time. The difference between the amount paid by the CDO originator and the CDO buyer, for the assets in that tranche, is called the tranche spread.

In this section illustrates how the bPOE ratings can be used in risk management. Given the fixed pool of assets, the objective is to select a CDO with attachment and detachment points that change over time. The CDO tranches have constraints on rating and the sum of discounted spread payments is minimized. This case study extends the models in Veremyev et al. [2012].

For each time moment t there is a set of attachment points that determine the width of a tranche, see figure 5.

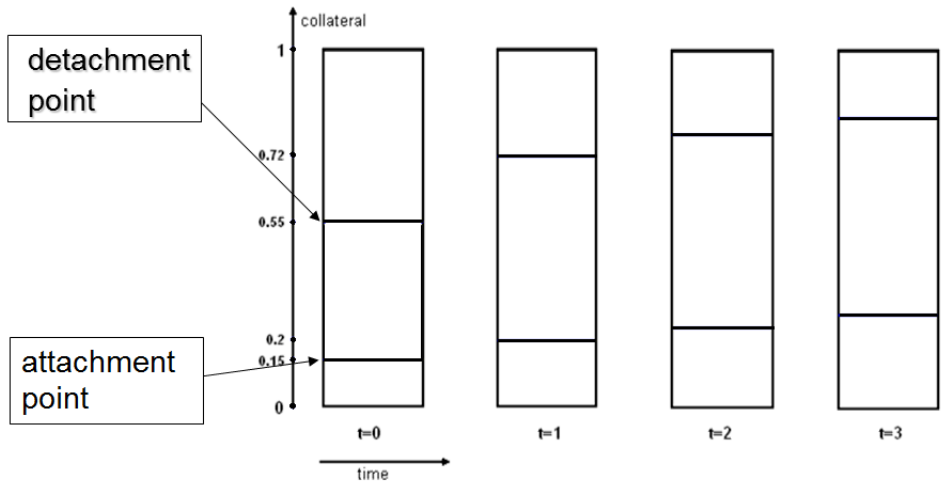

Figure 5: CDO attachment and detachment points structure

6.1 Finding the optimal attachment points of the CDO with fixed pool of CDSs, POE case.

Let M be the number of tranches, T is the number of periods, $L_t, \quad t \in 1, \ldots, T$ is loss at each time t and $s_m, \ m \in \{1, \ldots, M\}$ are the spread payments for each tranche m. The total payment for a tranche with given attachment point x_m^t time t is

$$\sum_{m=1}^{M} (x_{m+1}^t - \max(x_m^t, L_t))^+ s_m. \tag{7}$$

Given the discount rate r we want to minimize the expected present value of payments with respect to tranche attachment points $\{x_1^t, \ldots, x_m^t\}$

$$\sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=1}^{M} \mathbb{E}(x_{m+1}^t - \max(x_m^t, L_t))^+ s_m. \tag{8}$$

In paper Veremyev et al. [2012] it was shown that there exists a function equivalent to (8), that is convex with respect to the attachment points x_m^t. Let $\Delta s_m = s_m - s_{m+1}$, then the following equation holds

$$\sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=1}^{M} \mathbb{E}[(x_{m+1}^t - \max(x_m^t, L_t))^+ s_m] = \sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=1}^{M} \Delta s_m \mathbb{E}[(x_{m+1}^t - L_t)^+]. \tag{9}$$

The function on the right hand side of (9) is a convex function with respect to x_m^t (its the sum of expectations of convex functions). Convexity makes right hand side of (9) a desirable objective function for the minimization problem.

Each tranche in the CDO has a predefined rating. Let $\hat{p}_m$ be the appropriate PD for the given rating, then the rating constraints are

$$1 - \mathbb{P}(L_1 \leq x_m^1, \ldots, L_T \leq x_m^T) \leq \hat{p}_m \quad m = 1, \ldots, M. \tag{10}$$

Also the attachment points should satisfy monotonicity constraints

$$x_m^t \geq x_{m-1}^t \quad m = 3, \ldots, M; \ t = 1, \ldots, T. \tag{11}$$

Let $\boldsymbol{x} = \{x_1^t, \ldots, x_m^t\}$, combining (9), (10) and (11) the optimization problem is

$$\min_{\boldsymbol{x}} \sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=1}^{M} \Delta s_m \mathbb{E}[(x_{m+1}^t - L_t)^+] \tag{12}$$

$$\begin{aligned}
&s.t. \\
&1 - \mathbb{P}(L_1 \leq x_m^1, \ldots, L_T \leq x_m^T) \leq \hat{p}_m && m = 1, \ldots, M \\
&x_m^t \geq x_{m-1}^t && m = 3, \ldots, M;\ t = 1, \ldots, T. \\
&0 \leq x_m^t \leq 1 && m = 1, \ldots, M;\ t = 1, \ldots, T
\end{aligned}$$

6.2 Finding the optimal attachment points of the CDO and optimal CDS pool, POE case.

The model (12) finds the optimal attachment points but not the optimal allocation of the CDS pool. The following problem solves the optimal pool allocation problem as well as the optimal attachment point search. Let there be K number available CDSs, $y_k,\ k \in \{1, \ldots, K\}$ be the weight of th k-th asset in the asset pool and c_k be the annual income spread payments of the CDS. The CDS portfolio should have annual spread of at least ζ, where ζ is a free parameter. θ_k^t denotes the random cumulative loss of the CDS k at time t. The total loss of the CDS pool at time t is $L(\boldsymbol{\theta}^t, \boldsymbol{y}) = \sum_{k=1}^{K} \theta_k^t y_k$, where $\boldsymbol{\theta}^t = (\theta_1^t, \ldots, \theta_K^t)$ and $\boldsymbol{y} = (y_1, \ldots, y_K)$. The objective is to minimize the total spread payments minus the total income spread payments of th CDS pool.

$$\min_{\boldsymbol{x},\boldsymbol{y}} \sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=1}^{M} \Delta s_m \mathbb{E}[(x_{m+1}^t - L(\boldsymbol{\theta}^t, \boldsymbol{y}))^+] \tag{13}$$

$$\begin{aligned}
&s.t. \\
&1 - \mathbb{P}(L(\boldsymbol{\theta}^t, \boldsymbol{y}) \leq x_m^1, \ldots, L(\boldsymbol{\theta}^t, \boldsymbol{y}) \leq x_m^T) \leq \hat{p}_m && m = 1, \ldots, M \\
&x_m^t \geq x_{m-1}^t && m = 3, \ldots, M;\ t = 1, \ldots, T. \\
&\sum_{k=1}^{K} y_k = 1 \\
&\sum_{k=1}^{K} c_k y_k \geq \zeta \\
&y_k \geq 0 && k = 1, \ldots, K \\
&0 \leq x_m^t \leq 1 && m = 1, \ldots, M;\ t = 1, \ldots, T
\end{aligned}$$

6.3 Using bPOE ratings for CDO attachment point and CDS pool selection.

Both models (12) and (13) are non-convex optimization problems because of the probability constraints. Substituting POE with bPOE will make problems (12) and (13) convex optimization problems. The probability in the constraint (10)

$$1 - \mathbb{P}(L_1 \leq x_m^1, \ldots, L_T \leq x_m^T)$$

is equivalent to the following POE

$$POE_0(\max(L_1 - x_m^1, \ldots, L_T - x_m^T)). \tag{14}$$

Thus the bPOE analog of the constraint (10) is

$$bPOE_0(\max(L_1 - x_m^1, \dots, L_T - x_m^T)) \leq e\hat{p}_m. \tag{15}$$

Note that, the same change applies to the constraint $1 - \mathbb{P}(L_1(\boldsymbol{\theta}^t, \boldsymbol{y}) \leq x_m^1, \dots, L_T(\boldsymbol{\theta}^t, \boldsymbol{y}) \leq x_m^T) \leq \hat{p}_m$ in problem (13). Substituting constraint (15) in place of POE constraint (10) for problems (12) and (13) gives a convex optimization problems in both cases.

Also, the bPOE and the CVaR constraints are equivalent, so the constraint (15) is equivalent ot the following CVaR constraint

$$CVaR_{e\hat{p}_m}(\max(L_1 - x_m^1, \dots, L_T - x_m^T)) \leq 0 \qquad m = 1, \dots, M. \tag{16}$$

Using CVaR constraints is more desirable, because CVaR function is available in a number of widely used software languages and packages, such as MATLAB.

The Full optimization problems with bPOE constraints, corresponding to problems (12) and (13) respectively are given below.

$$\min_{\boldsymbol{x}} \sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=1}^{M} \Delta s_m \mathbb{E}[(x_{m+1}^t - L_t)^+] \tag{17}$$

$$\begin{aligned}
s.t.\quad & \\
& bPOE_0(\max(L_1 - x_m^1, \dots, L_T - x_m^T)) \leq e\hat{p}_m && m = 1, \dots, M \\
& x_m^t \geq x_{m-1}^t && m = 3, \dots, M;\ t = 1, \dots, T. \\
& 0 \leq x_m^t \leq 1 && m = 1, \dots, M;\ t = 1, \dots, T
\end{aligned}$$

Problem (17) is a reformulation of problem (12) with bPOE constraints.

$$\min_{\boldsymbol{x}, \boldsymbol{y}} \sum_{t=1}^{T} \frac{1}{(1+r)^t} \sum_{m=1}^{M} \Delta s_m \mathbb{E}[(x_{m+1}^t - L(\boldsymbol{\theta}^t, \boldsymbol{y}))^+] \tag{18}$$

$$\begin{aligned}
s.t.\quad & \\
& bPOE_0(\max(L(\boldsymbol{\theta}^t, \boldsymbol{y}) - x_m^1, \dots, L(\boldsymbol{\theta}^t, \boldsymbol{y}) - x_m^T)) \leq e\hat{p}_m && m = 1, \dots, M \\
& x_m^t \geq x_{m-1}^t && m = 3, \dots, M;\ t = 1, \dots, T. \\
& \sum_{k=1}^{K} y_k = 1 \\
& \sum_{k=1}^{K} c_k y_k \geq \zeta \\
& y_k \geq 0 && k = 1, \dots, K \\
& 0 \leq x_m^t \leq 1 && m = 1, \dots, M;\ t = 1, \dots, T
\end{aligned}$$

Problem (18) is a reformulation of problem (13) with bPOE constraints.

6.4 Numerical example using PSG.

We compare the POE and bPOE constraint based solutions for problems (12) and (13). The data comes from the research paper Veremyev et al. [2012] and can be downloaded from Veremyev et al..

Consider a CDO with 5 trenches (M=5) and the planning horizon of 5 years (T=5). The interest rate is assumed to be 7% (r = 7%) and the discounting is done in the middle of the

year. The default probabilities applied to the tranches are (ordered by the decreasing seniority) 0.12%, 0.36%, 0.71%, 0.281%. The corresponding bPOE probabilities are multiplied by the number e. The least senior tranche does not have a rating constraint. For more details see section 3 (Case study) in Veremyev et al. [2012].

The dataset represents a loss distribution of CDS pool that is an underlying asset for the CDO over the period of 5 years. The loss distribution for CDO pool is generated by Standard & Poor's CDO Evaluator. We use the Portfolio Safeguard (PSG) version 2.3, running on windows 10 machine, with Intel Core i7-8550U CPU.

Variables	Constraint type	**Solving time (seconds)**	**Objective value**
Attachment points	POE	134.18	0.592
Attachment points	bPOE	8.02	0.642
Attachment points & CDS pool	POE	—	—
Attachment points & CDS pool	bPOE	3694.08	537.56

Table 3: Comparison of numerical results for the problems (12) and (13), with POE constraints and problems (17) and (18) with bPOE constraints

From table 3 it is clear that the problems with bPOE constraints are solved much faster. Also, for problem with POE constraints we can not guarantee that the solution is a global, while for bPOE constrained problems we have a guarantee of global optimality.

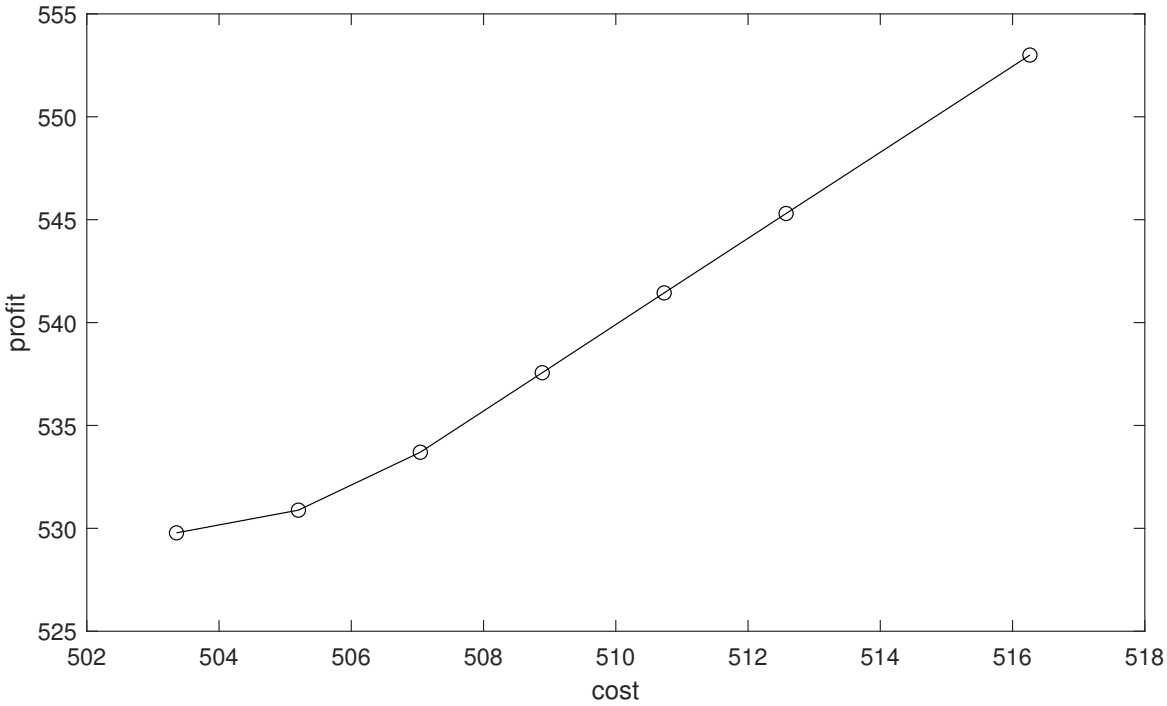

Figure 6: Pareto frontier for the income and loss of the CDO issuer in basis points. The X axis shows the discounted total cost of CDS pool. The Y axis shows the discounted total profit of the CDO

For the CDO issuer, the income is the sum of discounted spread payments received annually for the insurance of the CDS pool. The loss for the CDO issuer is the sum of discounted spread

payments made to the buyers of each tranche. Figure 6 shows Pareto frontier for the income and loss of the CDO issuer, the numbers are in basis points.

7 Confidence Intervals for bPOE Estimates

In this chapter we will give an insight on the size of estimation error for the bPOE ratings. We will use the asymptotic distributions for the bPOE estimates, developed in Mafusalov et al. [2018]. The results will be shown for the exponential and Pareto distributions. Pareto distribution is a heavy tailed distribution and thus it will provide an insight into to the estimation errors for the heavy tailed distributions.

On practice, the bPOE ratings will be applied to the empirical data. The data might come from some observations taken from the market or the result of a simulation. Given N the observations $\hat{X}_N = (X_1, X_2, \ldots, X_N)$ of some random variable X, the empirical bPOE is

$$bPOE_v(\hat{X}_N) = \min_{a \geq 0} \frac{1}{N} \sum_{i=1}^{N} \big[a(X_i - v) + 1\big]^+. \tag{19}$$

Formula (19) is an estimate of the true bPOE and is itself a random number.

Mafusalov, Shapiro and Uryasev, in their paper Mafusalov et al. [2018], give asymptotic distribution for the quantity $N^{1/2}\big(bPOE_v(\hat{X}_N) - bPOE_v(X)\big)$. By $Var(X)$ we denote the variance of X, then $N^{1/2}\big(bPOE_v(\hat{X}_N) - bPOE_v(X)\big)$ converges in distribution to normal with mean equal to 0 and variance

$$\sigma^2(v) = Var\Big(\big[a^*(X - v) + 1\big]^+\Big) = \mathbb{E}\Big[\big(\big[a^*(X - v) + 1\big]^+\big)^2\Big] - \big(bPOE_v(X)\big)^2 \tag{20}$$

where

$$a^* = \arg\min_{a \geq 0} \mathbb{E}\big[a(X - v) + 1\big]^+.$$

Let $\bar{q}_L^{\beta}(v)$ and $\bar{q}_U^{\beta}(v)$ denote the lower and upper bound of a confidence interval for bPOE estimate for a given quantile v, calculated for confidence level β. Given that $N^{1/2}\big(bPOE_v(\hat{X}_N) - bPOE_v(X)\big)$ is asymptotically normally distributed, the (asymptotic) confidence interval $[\bar{q}_L^{\beta}(v), \bar{q}_U^{\beta}(v)]$ is

$$\bar{q}_L^{\beta}(v) = \max\left(0, bPOE_v(X) - \Phi^{-1}(\beta)\frac{\sigma(v)}{\sqrt{N}}\right) \tag{21}$$

$$\bar{q}_U^{\beta}(v) = \min\left(1, bPOE_v(X) + \Phi^{-1}(\beta)\frac{\sigma(v)}{\sqrt{N}}\right) \tag{22}$$

Using formulas (21) and (22), it is also possible to calculate the necessary sample size $\hat{N}$ to achieve a given confidence level length. The case of exponential distribution is presented next.

7.1 Confidence Intervals for Exponential Distribution

When X is exponentially distributed with parameter λ, the variance has a closed form solution

$$\sigma^2(v) = e^{-\lambda v+1}(2 - e^{-\lambda v+1})$$

Given sample size of 10, 50 and 100 for an exponential distribution with parameter $\lambda = 1$, table 4 shows the values of the variance for different quantile values. Table 4 also shows the values of the necessary sample size in order to achieve 5% precision in estimating the bPOE with confidence level 90% (so that $(\bar{q}_U^{0.9}(v) - \bar{q}_L^{0.9}(v))/bPOE_v(X) < 5\%$), $\hat{N}_\beta^\epsilon = \min_N\{N \in \mathbb{N} \mid (\bar{q}_U^\beta(v) - \bar{q}_L^\beta(v))/bPOE_v(X) < \epsilon\}$. Figure 7 shows the confidence intervals for the 90% confidence level.

Quantile	**True bPOE**	σ^2	$\hat{N}_{90\%}^{5\%}$
6.2983	0.50%	0.0100	1048492
5.6052	1.00%	0.0199	522933
4.9120	2.00%	0.0396	260153
3.9957	5.00%	0.0975	102485
3.3026	10.00%	0.1900	49929
2.8971	15.00%	0.2775	32410
2.6094	20.00%	0.3600	23651
2.3863	25.00%	0.4375	18395
2.2040	30.00%	0.5100	14891
2.0498	35.00%	0.5775	12389
1.9163	40.00%	0.6400	10512
1.7985	45.00%	0.6975	9052
1.6931	50.00%	0.7500	7884
1.5978	55.00%	0.7975	6928
1.5108	60.00%	0.8400	6132
1.4308	65.00%	0.8775	5458
1.3567	70.00%	0.9100	4881
1.2877	75.00%	0.9375	4380
1.2231	80.00%	0.9600	3942
1.1625	85.00%	0.9775	3556
1.1054	90.00%	0.9900	3212
1.0513	95.00%	0.9975	2905
1.0000	100.00%	1.0000	657

Table 4: The results in this table are for the exponential distribution with parameter $\lambda = 1$. This table shows relationship between bPOE, variance of $N^{1/2}\big(bPOE_v(\hat{X}_N) - bPOE_v(X)\big)$ and the necessary sample size to insure that the 90% confidence interval length is at most 5% of the actual bPOE.

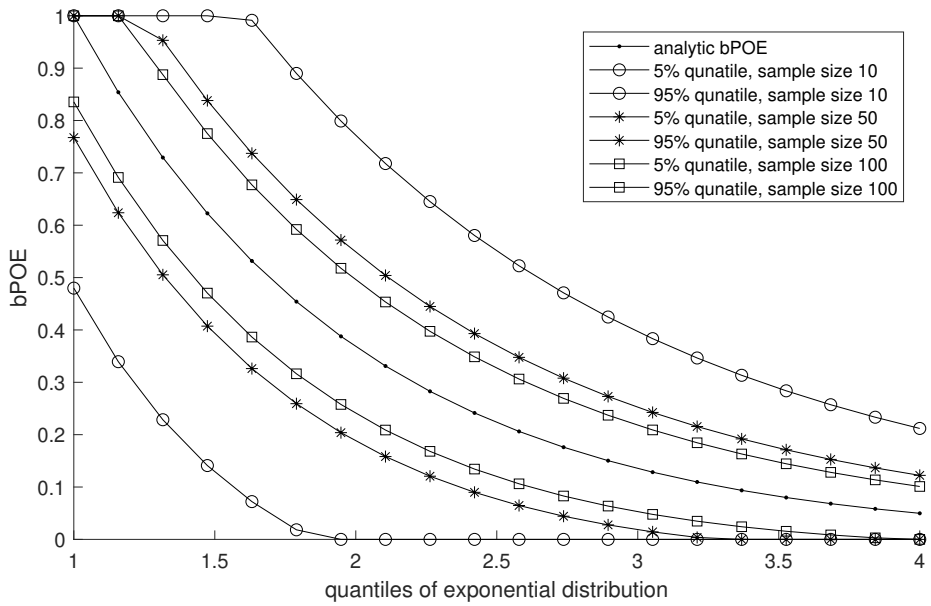

Figure 7: 90% confidence intervals for the bPOE estimates of the samples of exponential distribution with $\lambda = 1$, given sample sizes 10, 50 and 100.

7.2 Confidence Intervals for Pareto Distribution

Pareto distribution, with parameters α and x_m, is given by the following probability density function

$$f(x) = \begin{cases} \frac{\alpha x_m^\alpha}{x^{\alpha+1}} & x \geq x_m \\ 0 & \text{otherwise,} \end{cases}$$

and corresponding cumulative distribution function is

$$F(x) = \begin{cases} 1 - \left(\frac{x_m}{x}\right)^\alpha & x \geq x_m \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

bPOE for Pareto distributed random variable Y for parameter $\alpha > 1$ is,

$$bPOE_v(Y) = \left(\frac{x_m \alpha}{x(\alpha - 1)}\right)^\alpha. \tag{24}$$

The formula (24) was derived in Norton and Uryasev [2018a]. When $\alpha \in [0, 1]$, $bPOE_v(Y) = 1$ for all α. Using (23) and (24), the ratio of bPOE to POE is

$$bPOE_v(Y)/POE_v(Y) = \left(\frac{\alpha}{\alpha - 1}\right)^\alpha. \tag{25}$$

In this section we will consider Pareto distribution with parameters $\alpha = 2.5$ and $x_m = 1$. The ratio of bPOE to POE for $\alpha = 2.5$ is 3.586. Table 5 shows for the Pareto distribution, a relationship between quantile, bPOE evaluated for that quantile, variance of bPOE estimate and necessary sample size in order to estimate the bPOE with 5% precision with 90% confidence. Graph 8 shows the 90% confidence intervals for the sample sizes of 10, 50 and 100.

Quantile	**True bPOE**	σ^2	$\hat{N}^{5\%}_{90\%}$
13.8759	0.50%	0.0300	3150703
10.5160	1.00%	0.0599	1574069
7.9696	2.00%	0.1196	785697
5.5241	5.00%	0.2975	312721
4.1865	10.00%	0.5900	155041
3.5597	15.00%	0.8775	102488
3.1728	20.00%	1.1600	76206
2.9018	25.00%	1.4375	60441
2.6977	30.00%	1.7100	49929
2.5364	35.00%	1.9775	42421
2.4045	40.00%	2.2400	36790
2.2938	45.00%	2.4975	32409
2.1992	50.00%	2.7500	28906
2.1169	55.00%	2.9974	26039
2.0445	60.00%	3.2400	23650
1.9801	65.00%	3.4775	21629
1.9222	70.00%	3.7101	19897
1.8699	75.00%	3.9375	18395
1.8223	80.00%	4.1600	17081
1.7786	85.00%	4.3774	15922
1.7384	90.00%	4.5900	14891
1.7012	95.00%	4.7975	13969
1.6667	100.00%	2.7249	1791

Table 5: The results in this table are for the Pareto distribution with parameter $\alpha = 2.5$ and $x_m = 1$. This table shows relationship between bPOE, variance of $N^{1/2}\big(bPOE_v(\hat{X}_N) - bPOE_v(X)\big)$ and the necessary sample size to insure that the 90% confidence interval length is at most 5% of the actual bPOE.

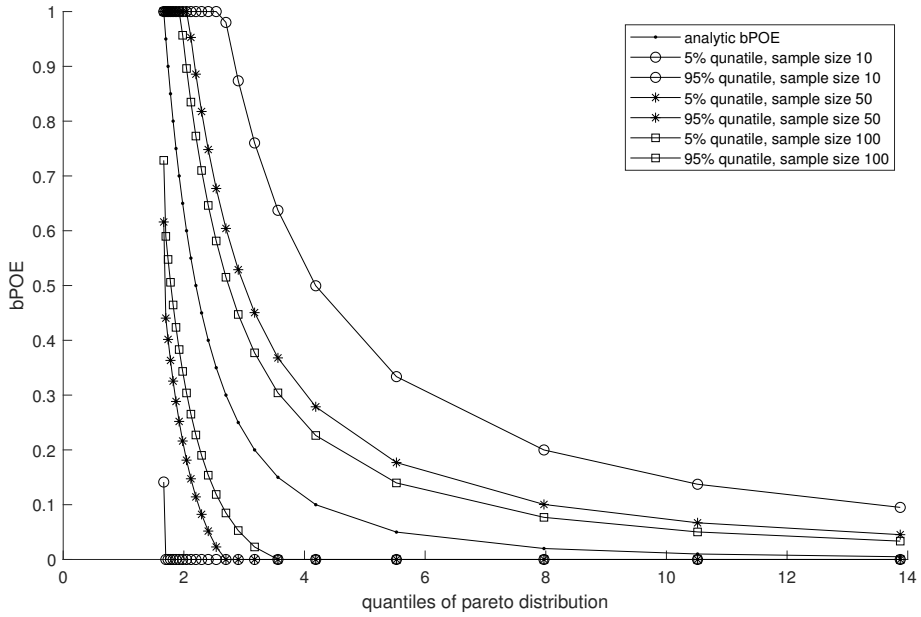

Figure 8: 90% confidence intervals for the bPOE estimates of the samples of Pareto distribution with $\alpha = 2.5$ and $x_m = 1$, given sample sizes 10, 50 and 100.

From tables 4 and 5 we see that we need roughly 3 times bigger sample, for Pareto distribution with $\alpha = 2.5$, compared to exponential distribution, to estimate the bPOE with given precision.

8 Summary

This paper presents a new approach to measuring the credit ratings based on the concept of bPOE. The bPOE ratings have a number of advantages over the POE based ratings. The first advantage is that the bPOE based ratings account for the heaviness of the loss distribution, while the POE based ratings completely ignore this characteristic. The second advantage is that bPOE is a quasi-convex function of its random variable, meaning that the level sets of the bPOE function are convex sets. The qusi-convexity implies that bPOE is a desirable function for optimization, compared to the POE based optimization problems that result in non-convex and discontinuous problems. This paper also presents a case study illustrating the efficiency of the bPOE ratings. In this case study a CDO structuring problem is considered. The goal is to find the optimal weights of a CDS pool that is the underlying of the CDO, as well as finding the optimal attachment points for the CDO tranches. The bPOE ratings are used as constraints in the optimization problem. The resulting problem is a convex optimization problem, due to the fact that the bPOE constraints define a convex set and the objective is a convex function. The results are compared to the nonconvex problem with POE rating constraints. The comparison clearly illustrates that with bPOE constraints the problem can be solved significantly faster.

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