Dynamic Maximization of the Sharpe Ratio and bPOE

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The Monotone Sharpe ratio

Portfolio selection

Let X be the benchmark-adjusted future return of some stock portfolio.

The Sharpe ratio (W. F. Sharpe, 1966):

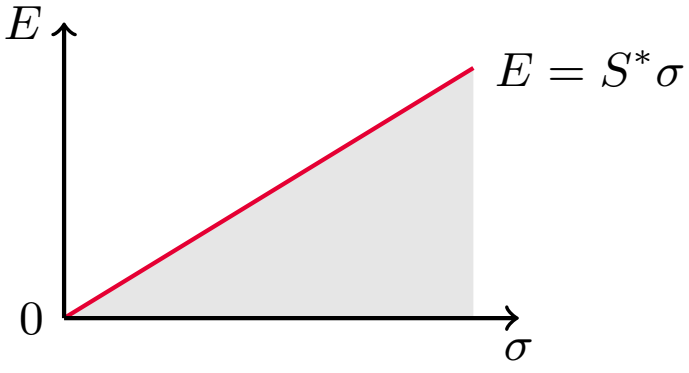
$$S(X) = \frac{EX}{\sigma(X)}.$$

Markowitz portfolio selection: let R_i, $i = 0, \ldots, n$, be asset returns; for given $\delta \geqslant 0$ find $x \in \mathbb{R}^{n+1}$ which

$$\text{minimizes } \sigma(x \cdot R) \quad \text{subject to} \quad \sum_i x_i = 1,\ E(x \cdot R) \geqslant \delta.$$

Efficient frontier

Assume $R_0 = 0$ (a risk-free asset). Then the solution in the (σ, E) coordinates:



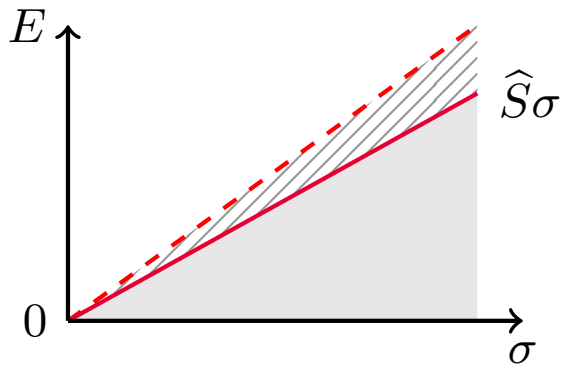
The slope of the efficient frontier is the maximal Sharpe ratio S^*:

$$S^* = \max_x S(x \cdot R).$$

The solution is not monotone

It may be possible to obtain a higher Sharpe ratio by disposing of a part of the return:

$$\mathrm{S}(Y) \geqslant \mathrm{S}(X) \text{ for some } Y \leqslant X.$$



Monotone Sharpe ratio

Define the monotone Sharpe ratio

$$\overline{S}(X) = \sup_{Y \leqslant X} \frac{E(Y)}{\sigma(Y)},$$

where $\sup$ is over random variables Y such that $P(Y \leqslant X) = 1$.

The following formula can be proved for $X \in L^2$ (Z., 2015):

$$\frac{1}{1 + (\overline{S}(X))^2} = \inf_{c \geqslant 0} E(1 - cX)_+^2,$$

where $\inf$ is over real numbers $c \geqslant 0$ and $(\ldots)_+ = \max(0, \ \ldots)$.

Connection between MSR and bPOE

Conditional value at risk (CVaR)

Let $Q(X, \lambda)$ denote the quantile function of a random variable X:

$$Q(X, \lambda) = x \text{ such that } P(X \leqslant x) = \lambda, \qquad \lambda \in [0, 1],$$

and $\overline{Q}(X, \lambda)$ denote the superquantile function (CVaR) for $X \in L^1$:

$$\overline{Q}(X, \lambda) = E(X \mid X \geqslant Q(X, \lambda)).$$

The well-known representation (Rockafellar, Uryasev, 2000):

$$\overline{Q}(X, \lambda) = \inf_{c \in \mathbb{R}} \left(\tfrac{1}{1-\lambda} E(X - c)_+ + c \right).$$

CVaR in L^p

In a similar way, define for $X \in L^p$, $p \in [1, \infty)$ (Krokhmal, 2007):

$$\overline{Q}_p(X, \lambda) = \inf_{c \in \mathbb{R}} \left(\tfrac{1}{1-\lambda} \|(X - c)_+\|_p + c \right).$$

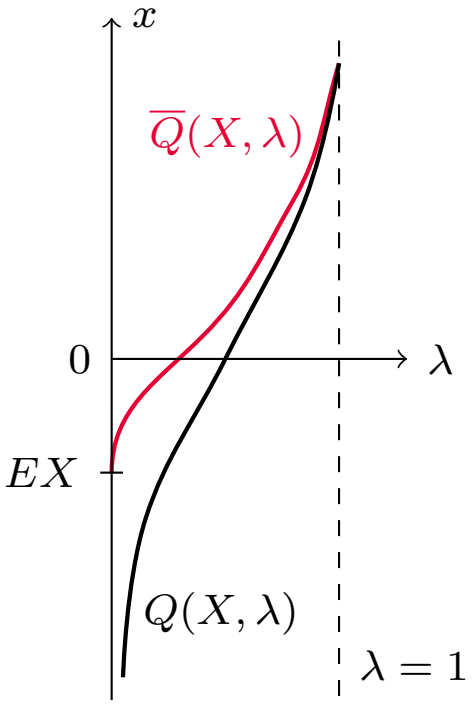
Define the inverse of $\overline{Q}_p(X, \lambda)$ in λ (buffered probability that $X > x$):

$$\overline{P}_p(X, x) = \begin{cases} 1 - \mathbb{Q}_p^{-1}(X, x), & \text{if } x > EX, \\ 1, & \text{if } x \leqslant EX \end{cases}$$

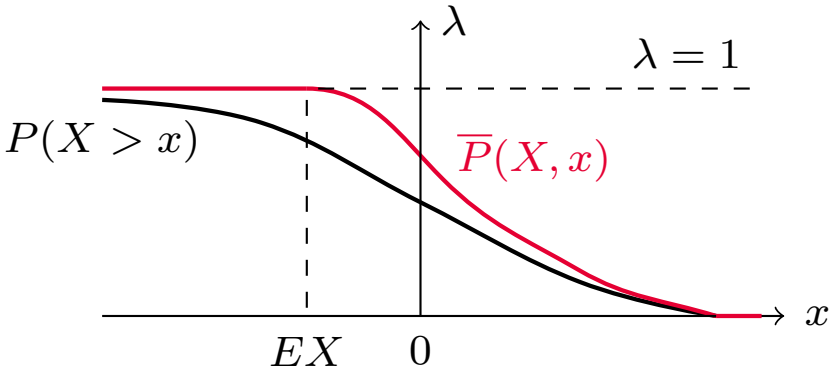
(for simplicity, it is assumed here that X is unbounded from above).

For L^1, bPOE was studied in Rockafellar, Royset (2010), Mafusalov, Uryasev (2014).

Quantile and superquantile



POE and bPOE



It can be proved similarly to Mafusalov, Norton, Uryasev (2014):

$$\overline{P}_p(X,x) = \inf_{c \geqslant 0} \|(c(X-x)+1)_+\|_p.$$

MSR and bPOE

For $X \in L^2$:

$$\overline{P}_2(-X, 0) = (1 + (\overline{S}(X))^2)^{-1/2}.$$

For $X \in L^1$:

$$\overline{P}_1(-X, 0) = (1 + \overline{S}_1(X))^{-1},$$

where

$$\overline{S}_1(X) = \inf_{Y \leqslant X} \frac{EY}{E|Y - \mathrm{med}(Y)|}.$$

There is no simple formula for $p \neq 1, 2$ but a dual representation can be proved for MSR in L^p, $p \in [1, \infty)$.

If $E(X) > 0$, the following representation is true for $p > 1$:

$$(\overline{S}_p(X))^q = \max_{a,b\in\mathbb{R}} \Big\{ b - E\Big(\tfrac{q-1}{q^p} \big|(aX+b)_+ - q\big|^p + (aX+b)_+ \Big) \Big\}.$$

A dynamic maximization problem for the Sharpe ratio

A market model

- One riskless asset with price $B_t = e^{rt}$, where $r \geqslant 0$.
- One risky asset, a geometric Brownian motion:

$$dS_t = S_t(\mu dt + \sigma dW_t), \qquad S_0 = 1.$$

- A trading strategy: (v_t, u_t), the amount of money invested in the riskless and risky assets.
- The capital $X_t^{u,v} = v_t + u_t$ of a strategy (v_t, u_t) satisfies the equation

$$dX_t^{u,v} = v_t \frac{dB_t}{B_t} + u_t \frac{dS_t}{S_t}, \qquad X_0^{u,v} = x_0 > 0.$$

Equivalently: $dX_t^{u,v} = (rv_t + \mu u_t)dt + \sigma u_t dW_t$.

– The model can be reduced to $B_t \equiv 1$, then

$$dX_t^u = \mu u_t dt + \sigma u_t dW_t, \qquad X_0^u = x_0.$$

– The class of admissible control processes:

$$\mathcal{U} = \{u_t : E \int_0^T u_t^2 dt\} < \infty.$$

The dynamic Sharpe ratio maximization problem on $[0, T]$:

$$\text{maximize } S(X_T^u) = \frac{EX_T^u - x_0}{\sigma(X_T^u)} \quad \text{over } u \in \mathcal{U}.$$

Literature review

The problem can be reduced to the constraint optimization problem

$$\begin{aligned} \text{minimize:} \quad & E(X_T^u)^2 \\ \text{subject to:} \quad & EX_T^u = \delta. \end{aligned}$$

1. Richardson (1989) solved this problem by martingale methods:

First, find the optimal terminal capital X_T^* in the set $\{X_T^u \mid u \in \mathcal{U}\}$ of capitals of all admissible strategies.

Then, find the process X_t^*, $t \in [0, T]$.

Finally, find the optimal u_t^*.

2. Pedersen, Peskir (2013) solved the problem by solving the Hamilton–Jacobi–Bellman equation for the Lagrangian:

$$\text{minimize } E(X_T^u)^2 - \lambda E X_T^u. \qquad (*)$$

The HJB equation:

$$\inf_{u \in \mathbb{R}} \left\{ V_t' + \mu u V_x' + \tfrac{\sigma^2}{2} u^2 V_{xx}'' \right\} = 0.$$
$$V(T, x) = x^2 - \lambda x,$$

where $V(t, x)$ is the minimal value of $(*)$ under condition $X_t = x$.

The key idea is to look for the solution in the form

$$V(t, x) = a(t)x^2 + b(t)x + c(t).$$

A simple (and more general) solution using bPOE and MSR

We'll first solve the bPOE minimization problem for $p > 1$:

$$V = \min_{u \in \mathcal{U}} \overline{P}_p(-X_T^u, 0).$$

From the representation of $\overline{P}_p(X, x)$:

$$V = \min_{c \geqslant 0} \min_{u \in \mathcal{U}} \|(1 - cX_T^u)_+\|_p = \min_{u \in \mathcal{U}} \|(1 - X_T^u)_+\|_p,$$

where we use that c can be included in the control by changing $\widetilde{u}_t = cu_t$.

Then

$$V^p = \min_{u \in \mathcal{U}} E|\widetilde{X}_T^u|^p$$

for

$$d\widetilde{X}_t^u = -\mu u_t dt - \sigma u_t dW_t, \qquad \widetilde{X}_0^u = 1.$$

We can remove $(\cdot)_+$ since it's never optimal to go below zero.

Let $w_t = -u_t/\widetilde{X}_t^u$. Then

$$E|\widetilde{X}_T^u|^p = E\Big\{Z_T \exp\Big(\int_0^T \big(\mu p w_s + \tfrac{1}{2}\sigma^2(p^2 - p)w_s^2\big)ds\Big)\Big\},$$

where $Z_t > 0$ is a martingale (cstochastic exponent of $\sigma p w_t$ w.r.t. W_t) and we can show that $EZ_T = 1$.

By changing the measure to $dQ = Z_T dP$ we obtain

$$E|\widetilde{X}_T^u|^p = E^Q\Big\{\exp\Big(\int_0^T \big(\mu p w_s + \tfrac{1}{2}\sigma^2(p^2-p)w_s^2\big)ds\Big)\Big\},$$

which is minimized by minimizing the integrand for each t:

$$w^* \equiv -\frac{\mu}{\sigma^2(p-1)}.$$

Then the optimal control in the original problem

$$u_t = \frac{\mu}{\sigma^2(p-1)}(1 - X_t^u).$$

Since any control $\widetilde{u}_t = Cu_t$ will be optimal as well, the family of optimal control functions is

$$u(t,x) = \frac{\mu}{\sigma^2(p-1)}(C - x), \qquad C > 0.$$

$\overline{S}$-optimality is the same as S-optimality

The previous result provides the solution to the problem:

$$\text{maximize } \overline{S}(X_T^u) = \sup_{Y \leqslant X_T^u} \frac{EY}{\sigma(Y)} \quad \text{over } u \in \mathcal{U}.$$

We show that the same strategy also maximizes $S(X_T^u)$.

Suppose for optimal u^* there is $Y \leqslant X_T^{u^*}$ such that $S(Y) > S(X_T^{u^*})$. Since the BS-market is complete, there is $y < 0$ and a control u_t so that $X_0^u = y_0 < 0$ and $X_T^u = Y$.

Then the capital process $\widetilde{X}_t = y_0 + X_t^u$ has a higher Sharpe ratio than Y and a higher monotone Sharpe ratio than X^{u*} – a contradiction.

A mean-variance optimal selling problem

As before, one risky asset:

$$dS_t = S_t(\mu dt + \sigma dW_t), \qquad S_0 = 1.$$

Consider the problem:

$$\text{maximize } \frac{ES_\tau - 1}{\sigma(S_\tau)} \text{ over stopping times } \tau.$$

The interesting case is only when $\mu \in (0, \frac{\sigma^2}{2})$:

- $\mu \leqslant 0$: the optimal $\tau = 0$,
- $\mu \geqslant \frac{\sigma^2}{2}$: the process $S_t = \exp(\sigma B_t + (\mu - \frac{\sigma^2}{2})t)$ reaches any level A; take $\tau = \tau_A$ and $A \to +\infty$.

A result of Pedersen and Peskir (2012)

Pedersen and Peskir considered an equivalent problem

$$\text{maximize } ES_\tau - c \operatorname{Var} S_\tau.$$

Their solution briefly:

1. Equivalence to a constrained problem: $\min_\tau \{ES_\tau^2 \mid ES_\tau = \delta\}$.
2. Lagrange multipliers: $\min_\tau \{ES_\tau^2 - \lambda ES_\tau\}$.
3. Markov formulation: the value function $V(s) = \min_\tau E_s(S_\tau^2 - \lambda S_\tau)$, where $E_s(\cdot) = E(\,\cdot \mid S_0 = s)$.
4. Apply methods for Markov optimal stopping problems.

A simple solution using the monotone Sharpe ratio

First, let's maximize the monotone Sharpe ratio, or, equivalently

$$\text{minimize } \overline{P}_2(-(S_\tau - x), 0) \text{ over Markov times } \tau.$$

From the representation of $\overline{P}_2(X, 0)$:

$$V = \min_{c \geqslant 0} \min_{\tau} E(1 - c(S_\tau - x))_+^2.$$

The inner problem $\min_\tau E(\ldots)$ is Markovian, and by a simple argument it can be shown that the solution is

$$\tau_c = \inf\{t \geqslant 0 : S_t \geqslant b_c\},$$

where $\tau_c = \infty$ if $S_t < b_c$ for all t.

Hence, for the optimal τ^* the distribution of S_{τ^*} is binomial:

$$S_{\tau^*} = \begin{cases} b^*, & \text{if } \tau^* < \infty, \\ 0, & \text{if } \tau^* = \infty. \end{cases}$$

Next observe that if τ^* maximizes $\overline{S}(S_\tau)$, then

$$\overline{S}(S_{\tau^*}) = \sup_{Y \leqslant S_{\tau^*}} \frac{EY - x}{\sigma(Y)} = \frac{ES_{\tau^*} - x}{\sigma(S_{\tau^*})},$$

i.e. $Y = S_{\tau^*}$, which follows from that only $\mathscr{F}^{S_{\tau^*}}$-measurable Y can be considered.

Hence, the same τ^* maximizes both $\overline{S}$ and S.

To find the optimal level b, use the well-known result that:

$$P(\tau_b < \infty) = b^{\gamma-1}, \qquad \text{where } \gamma = \frac{2\mu}{\sigma^2} < 1.$$

Then

$$\frac{ES_{\tau_b} - x}{\sigma(S_{\tau_b})} = \frac{bp_b - x}{b\sqrt{p_b(1-p_b)}} = \frac{b^\gamma - x}{b^{\frac{\gamma+1}{2}}(1-b^{\gamma-1})^{\frac{1}{2}}}.$$

From here, $b > 1$ which maximizes the right-hand side can be found numerically.

Thank you for your attention