Boundary Value Problems in Uncertain Environment: Applications and Software

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Formulation, investigation and solving of control problems for boundary value problems with stochastic parameters

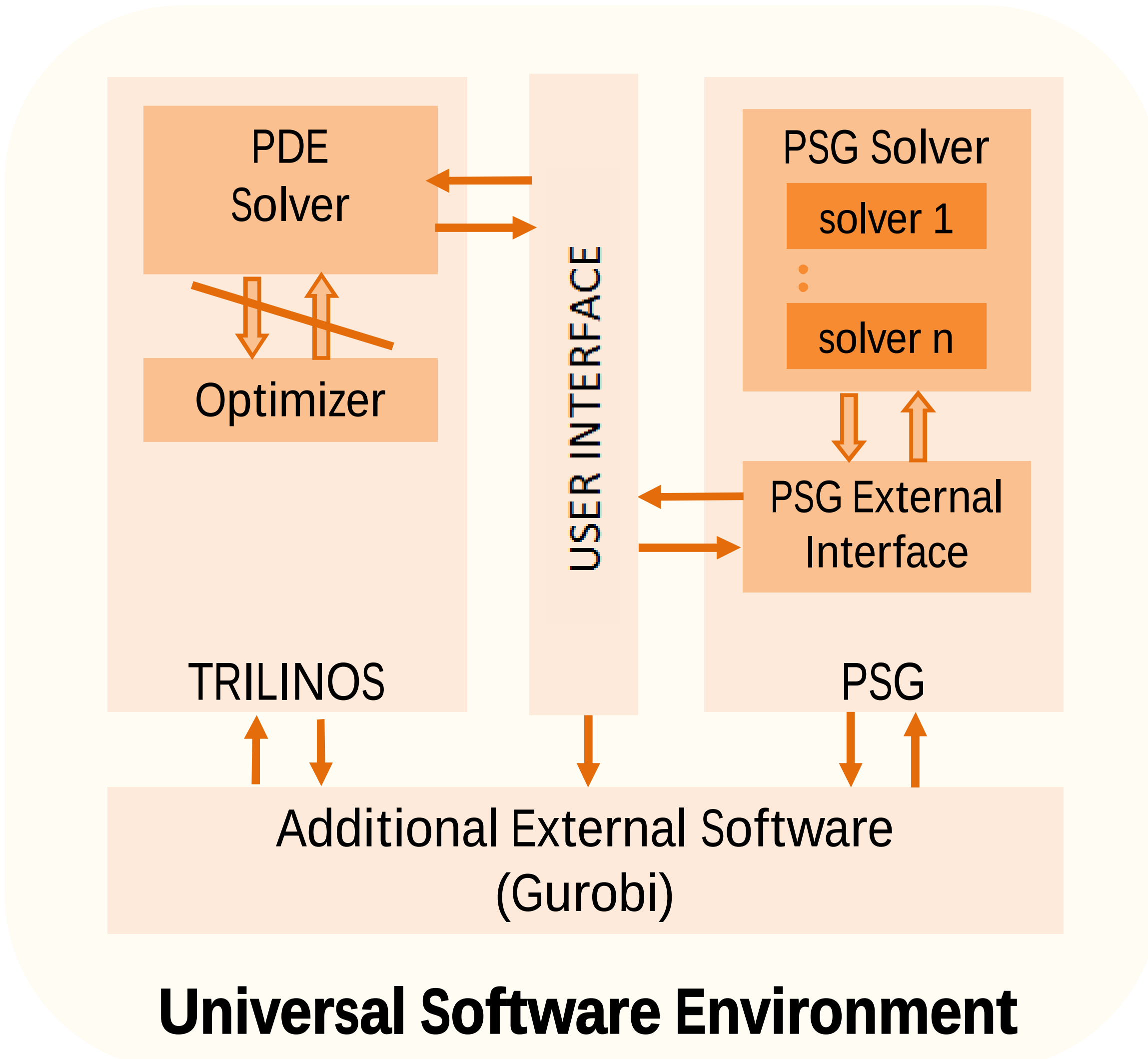
- Existence and uniqueness of the solution of boundary value problems with deterministic parameters
- Methods for solving boundary value problems with deterministic parameters
- Investigation of optimization control problems for boundary value problems with deterministic parameters
- Selection of the optimization method
- Building optimization algorithm
- Numerical implementation
- Analysis and verification of the solution

Software integration

- splitting the general problem into subproblems
- solving subproblems using specialized software
- development of interfaces for software integration
- application of a common universal programming environment for integration

Example: Trilinos + PSG

Example of integration Trilinos + PSG



Burgers Problem

$$\min_{z \in L^2(0,1)} J(z) = 1/2\, CVaR_{\beta}\left[\int_0^1 \left(u(\theta;x;z) - \overline{w}(x)\right)^2 dx\right] + \alpha/2 \int_0^1 z(x)^2\, dx$$

$$u = u(\theta; x; z):$$

$$-\nu(\theta) u_{xx}(\theta;x) + u(\theta;x) u_x(\theta;x) = f(\theta;x) + z(x)$$

$$x \in (0,1),\ \theta \in \Theta$$

$$u(\theta;0) = d_0(\theta),\ \ u(\theta;1) = d_1(\theta),\ \ \theta \in \Theta$$

$$\Theta = [-1,1]^4$$

$$\nu(\theta) = 10^{\theta_1 - 2}, \qquad f(\theta;x) = \theta_2/100$$

$$d_0(\theta) = 1 + \theta_3/1000, \qquad d_1(\theta) = \theta_4/1000$$

Risk functions: $CVaR \rightarrow VaR,\ PM, \ldots\ (PSG\ Risk\ Functions)$

[1] [D. P. KOURI, M. HEINKENSCHLOSS, D. RIDZAL , AND B. G. VAN BLOEMEN WAANDERS. *A trust-region algorithm with adaptive stochastic collocation for PDE optimization under uncertainty.* SIAM J. SCI. COMPUT. 2013 Society for Industrial and Applied Mathematics. Vol. 35, No. 4, pp. A1847–A1879]

PSG Codes

CVaR minimization code

```
Minimize
    cvar_risk(0.95, fnextdir_R(point_arg))
```

VaR minimization code

```
Minimize
  var_risk(0.9, fnextdir_R(point_arg))
Solver: VANGRB
```

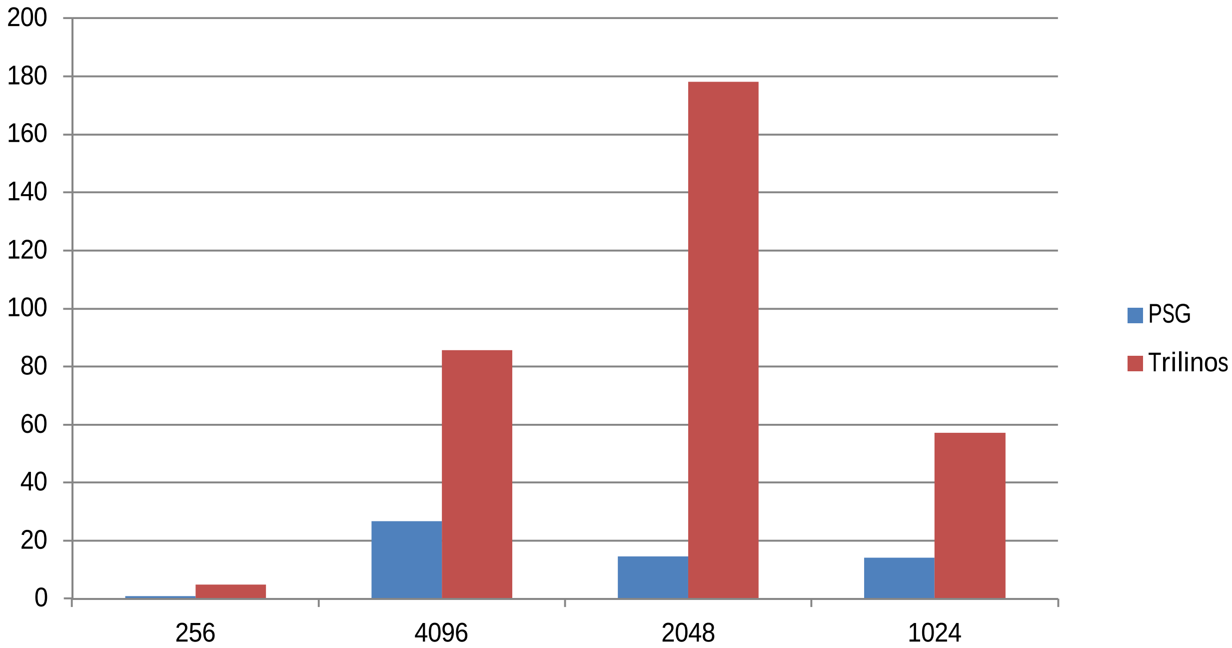
Partial Moment in constraint

```
minimize
  linear(matrix_1)
constraint < 100
  pm_pen(0.9, fnextdir_R(point_arg))
```

Numerical Experiments

Solution times: PSG vs Trilinos

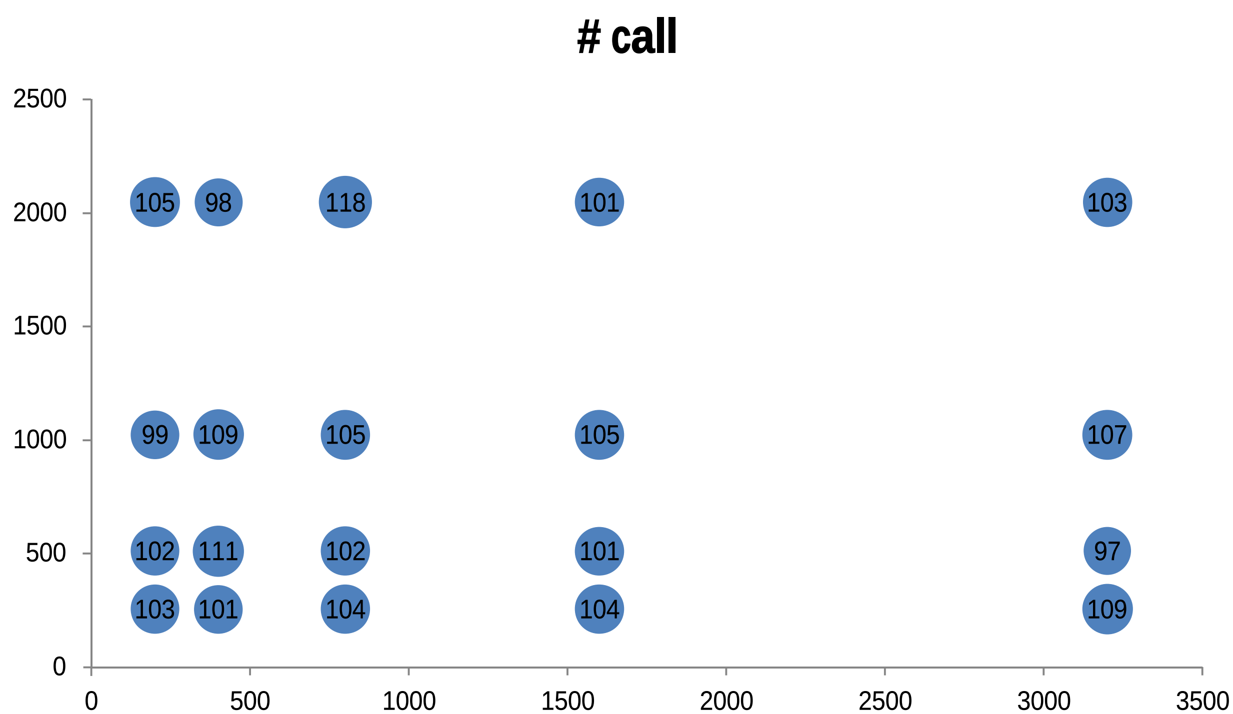
Problem	Q	N	PSG	Trilinos
min CVaR(0.99)	200	256	1	4.8
min CVaR(0.9)	200	4096	26.71	85.7
min CVaR(0.9)	400	2048	14.52	178.23
min CVaR(0.99)	800	1024	14.08	57.2



Solution time depends on the number of grid nodes N

Number of PSG calls: min CVaR(0.9)

Q	N	Fun calls
200	256	103
200	512	102
200	1024	99
200	2048	105
400	256	101
400	512	111
400	1024	109
400	2048	98
800	256	104
800	512	102
800	1024	105
800	2048	118
1600	256	104
1600	512	101
1600	1024	105
1600	2048	101
3200	256	109
3200	512	97
3200	1024	107
3200	2048	103



Number of function calls
X: number of samples Q
Y: number of grid nodes N

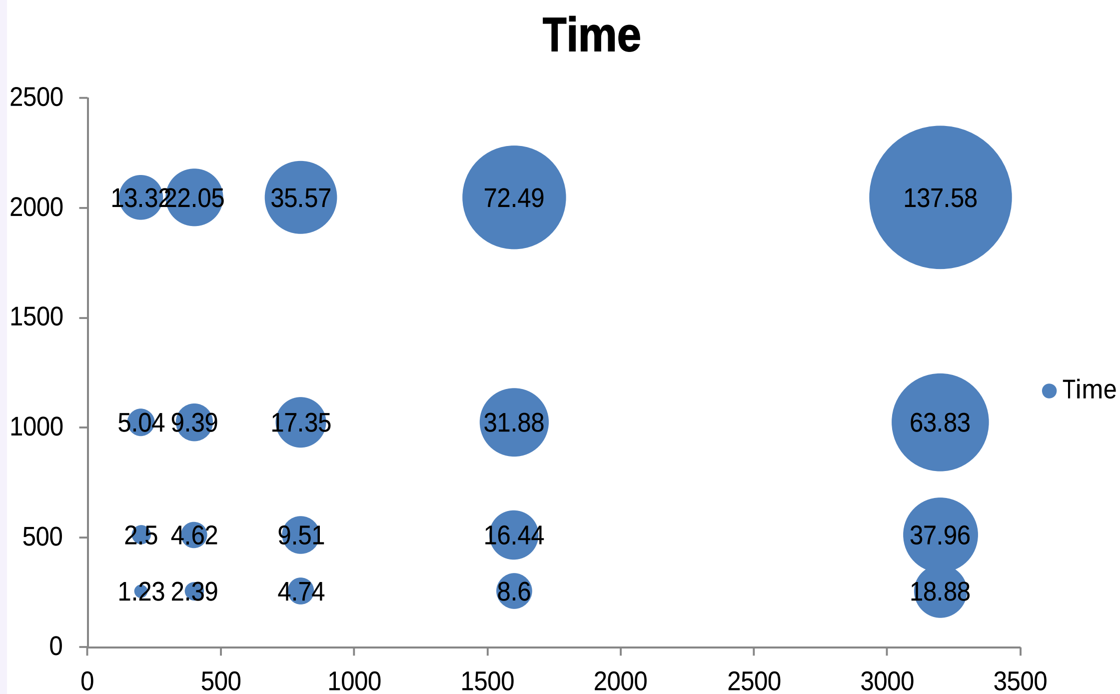
Number of calls of PSG solver for CVaR minimization

PSG vs Trilinos: min CVaR(0.99), VaR is quite close

Q	N	PSG (10^2)	Trilinos (10^2)	diff (10^-7)
400	1024	1.16880	1.16891	11.330
1600	256	1.08303	1.08308	5.364
1600	512	1.13167	1.13170	2.697
400	256	1.09247	1.09249	2.477
400	2048	1.17994	1.17996	1.967
1600	2048	1.17181	1.17182	1.475
200	1024	1.22173	1.22174	1.286
200	4096	1.24552	1.24553	0.797
800	256	1.06685	1.06686	0.741
800	1024	1.14483	1.14484	0.649
3200	1024	1.18049	1.18049	0.163
200	2048	1.23753	1.23753	-0.107
400	512	1.14021	1.14021	-0.327
200	256	1.15937	1.15936	-0.506
200	512	1.19149	1.19148	-0.743
800	512	1.11398	1.11397	-0.818
800	2048	1.16099	1.16098	-0.897
3200	256	1.10513	1.10511	-1.892
1600	1024	1.15617	1.15615	-2.456
3200	2048	1.19644	1.19639	-4.744

min PM(0) = average,   VaR is reported

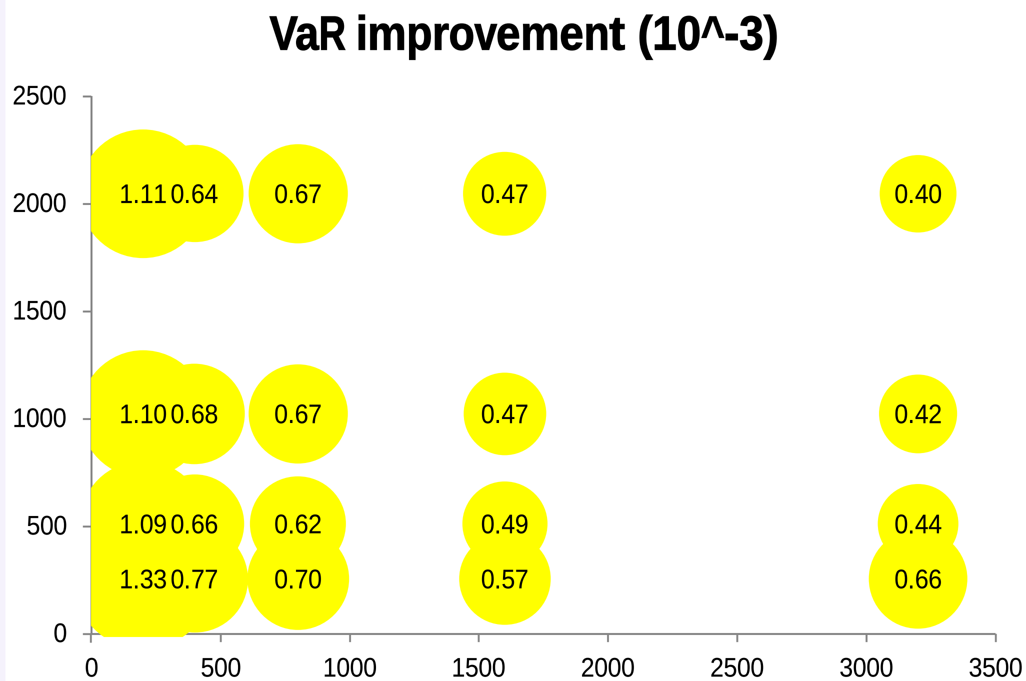
Q	N	Time	# call	Difference (10^-3)	PSG VaR	Trilinos VaR
200	256	1.23	112	-2.8848	0.01448	0.01159
200	512	2.5	115	-3.0705	0.01499	0.01191
200	1024	5.04	108	-3.0517	0.01527	0.01222
200	2048	13.32	120	-3.0436	0.01542	0.01238
400	256	2.39	112	-3.3305	0.01426	0.01092
400	512	4.62	113	-3.3656	0.01477	0.01140
400	1024	9.39	112	-3.3611	0.01505	0.01169
400	2048	22.05	120	-3.4035	0.01520	0.01180
800	256	4.74	109	-3.2661	0.01393	0.01067
800	512	9.51	117	-3.3044	0.01444	0.01114
800	1024	17.35	110	-3.2792	0.01473	0.01145
800	2048	35.57	108	-3.2671	0.01488	0.01161
1600	256	8.6	110	-3.6901	0.01452	0.01083
1600	512	16.44	111	-3.7127	0.01503	0.01132
1600	1024	31.88	106	-3.7476	0.01531	0.01156
1600	2048	72.49	120	-3.7421	0.01546	0.01172
3200	256	18.88	114	-4.0277	0.01508	0.01105
3200	512	37.96	123	-4.0606	0.01558	0.01152
3200	1024	63.83	107	-4.0598	0.01586	0.01180
3200	2048	137.58	117	-4.0534	0.01602	0.01196



Time
X: number of samples Q
Y: number of grid nodes N

VaR Risk minimization with PSG

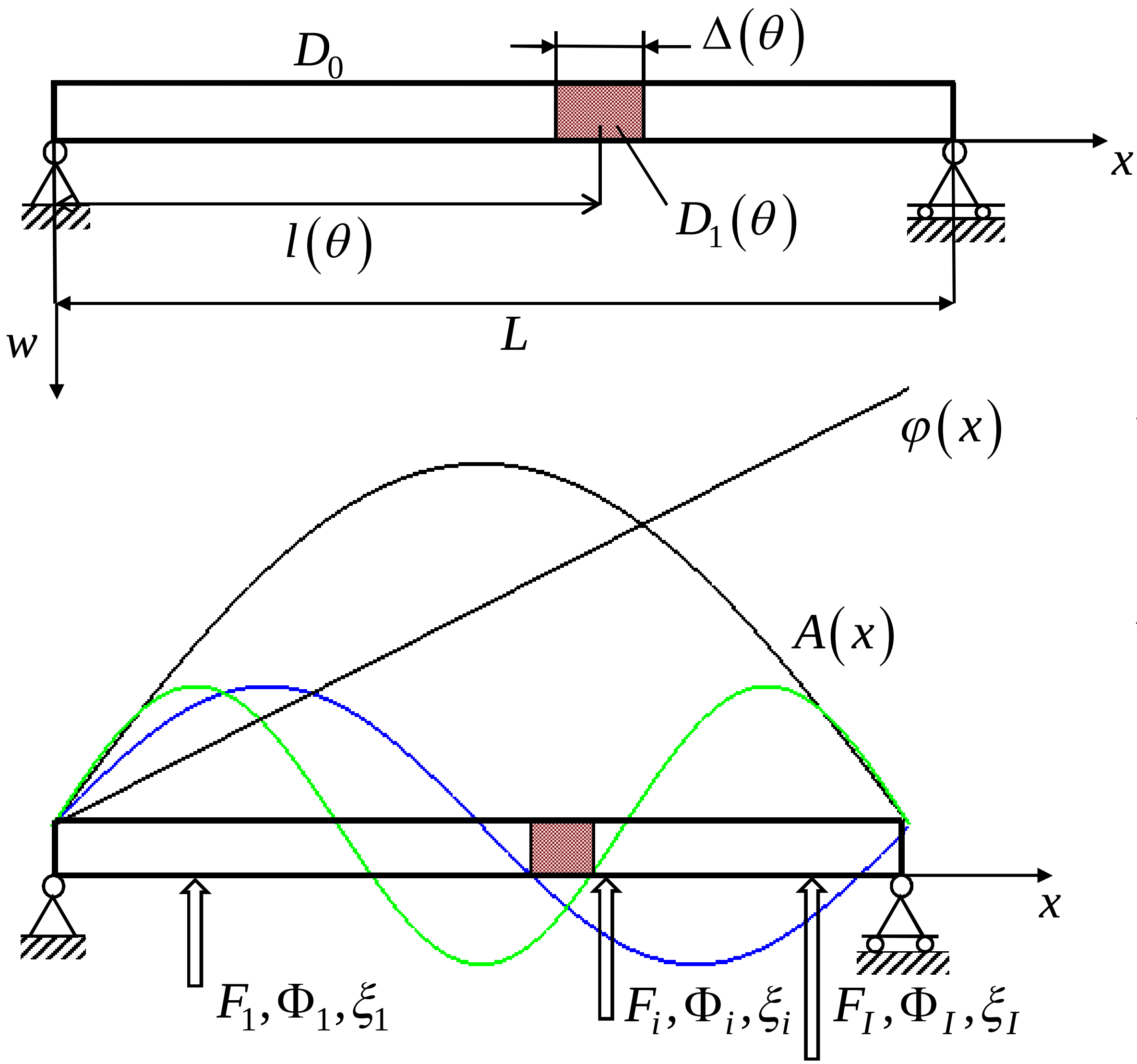
Q	N	Time	# call	Diff. (10^-3)	VaR	true VaR
200	256	2.35	269	1.3322	0.01026	0.01159
200	512	3.37	198	1.0888	0.01083	0.01191
200	1024	6.61	184	1.1023	0.01112	0.01222
200	2048	15.52	187	1.1134	0.01126	0.01238
400	256	4.14	249	0.7651	0.01016	0.01092
400	512	5.97	185	0.6615	0.01074	0.01140
400	1024	11.92	182	0.6838	0.01101	0.01169
400	2048	28.13	198	0.6417	0.01116	0.01180
800	256	8.15	249	0.7009	0.00997	0.01067
800	512	12.12	191	0.6242	0.01052	0.01114
800	1024	28	223	0.6726	0.01078	0.01145
800	2048	48.12	190	0.6672	0.01094	0.01161
1600	256	16.18	256	0.569	0.01026	0.01083
1600	512	23.99	198	0.489	0.01083	0.01132
1600	1024	44.61	187	0.465	0.01110	0.01156
1600	2048	95.89	195	0.4713	0.01125	0.01172
3200	256	43.19	334	0.6609	0.01039	0.01105
3200	512	57	232	0.4367	0.01109	0.01152
3200	1024	101.36	210	0.4155	0.01139	0.01180
3200	2048	187.34	194	0.4025	0.01156	0.01196



Difference of VaRs
X: number of samples Q
Y: number of grid nodes N

PSG solver minimizes VaR, therefore, VaR values are lower compared to Trilinos.
PSG solver works especially well for a small number of scenarios.

Excitation of Beam Oscillation (1D)



$$D_1 = D_0\left(1 + 10^{\theta_1 - 2}\right)$$

$$l = L\left(3/4 + \theta_2/1000\right)$$

$$\Delta = L \cdot \theta_3 / 1000$$

Boundary Value Problem

$$\frac{\partial^2}{\partial x^2}\left[ED\frac{\partial^2 w(x,t)}{\partial x^2}\right] + \rho\frac{\partial^2 w(x,t)}{\partial t^2} = \sum_{i=1}^{I} F_i \cos(\omega t + \delta_i)\delta(x-\xi_i)$$

$$x \in (0,L),\ t \in (-\infty,\infty)$$

Boundary conditions:

$$\begin{cases} w(0,t) = w(L,t) = 0 \\ \dfrac{\partial^2 w(0,t)}{\partial x^2} = \dfrac{\partial^2 w(L,t)}{\partial x^2} = 0 \end{cases},\ t \in (-\infty,\infty)$$

Choose parameters of forces to make vibration of the beam (according to some criterion) and achieve specified waveform.

$$\overline{W}(x,t) = A(x)\cos(\omega t + D(x))$$

Solution is represented as follows:

$$w(x,t)=\sum_{i=1}^{I}F_i G(x;\xi_i;k)\cos(\omega t+\delta_i)$$

where $G(x,t;\xi)=G(x;\xi;k)\begin{bmatrix}\cos(\omega t)\\ \sin(\omega t)\end{bmatrix}$ = the Green's function,

$G(x;\xi;k)$ satisfies equation:

$$\frac{d^4G(x;\xi;k)}{dx^4}-k^2=\frac{1}{ED}\delta(x-\xi),\qquad k^2=\frac{\omega^2\rho}{ED}$$

$$x\in(0,L),\ \xi\in(0,L)$$

With homogeneous boundary conditions:

$$\begin{cases}G(0,t;\xi)=G(L,t;\xi)=0\\ \dfrac{\partial^2G(0,t;\xi)}{\partial x^2}=\dfrac{\partial^2G(L,t;\xi)}{\partial x^2},\end{cases}\qquad t\in(-\infty,\infty),\ \xi\in(0,L)$$

$$R_\Omega(x,t) = w(x,t) - \overline{W}(x,t) =$$

$$= \left\{ \sum_{i=1}^{I} G(x;\xi_i;k) F_i \cos\delta_i - A(x)\cos D(x) \right\} \cos\omega t -$$

$$- \left\{ \sum_{i=1}^{I} G(x;\xi_i;k) F_i \sin\delta_i - A(x)\sin D(x) \right\} \sin\omega t,$$

Since we consider steady oscillations:

$$\begin{cases} \left\{ \sum_{i=1}^{I} G(x;\xi_i;k) F_i \cos\delta_i - A(x)\cos D(x) \right\} \to \min \\ \left\{ \sum_{i=1}^{I} G(x;\xi_i;k) F_i \sin\delta_i - A(x)\sin D(x) \right\} \to \min \end{cases}$$

Approximation criteria = mean square error functional

$$\begin{cases} I^c(\vec{a}^c, \vec{\xi}, k, I) = \int\limits_0^1 \left(\sum\limits_{i=1}^{I} G(x;\xi_i;k) a_i^c - A(x) \cos D(x) \right)^2 dx \\ I^s(\vec{a}^s, \vec{\xi}, k, I) = \int\limits_0^1 \left(\sum\limits_{i=1}^{I} G(x;\xi_i;k) a_i^s - A(x) \sin D(x) \right)^2 dx, \end{cases}$$

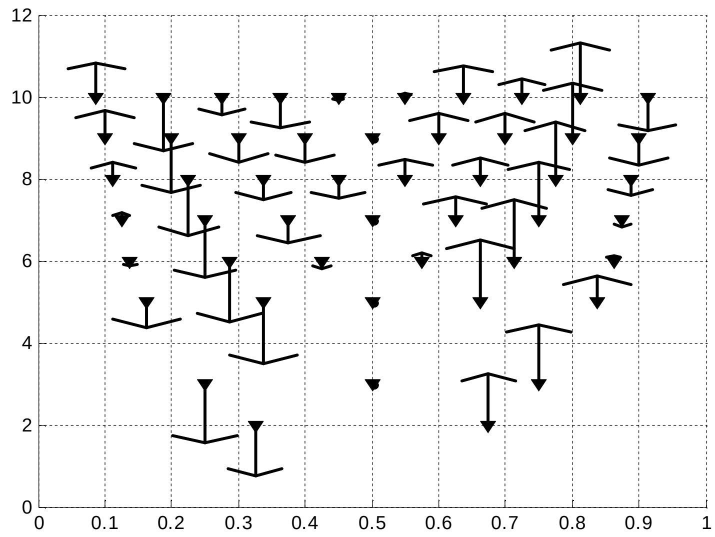
where

$$\begin{cases} a_i^c = F_i \cos \delta_i \\ a_i^s = F_i \sin \delta_i \end{cases}$$

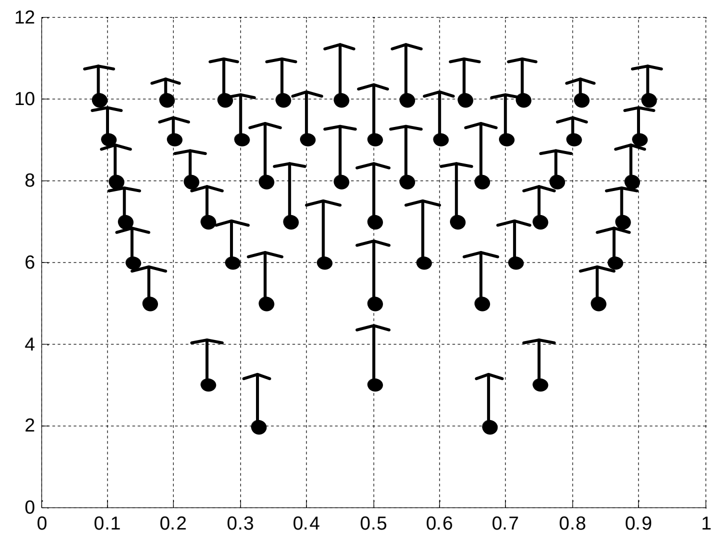
Example

Find:

number of forces, points of application, characteristics in the specified frequency range.

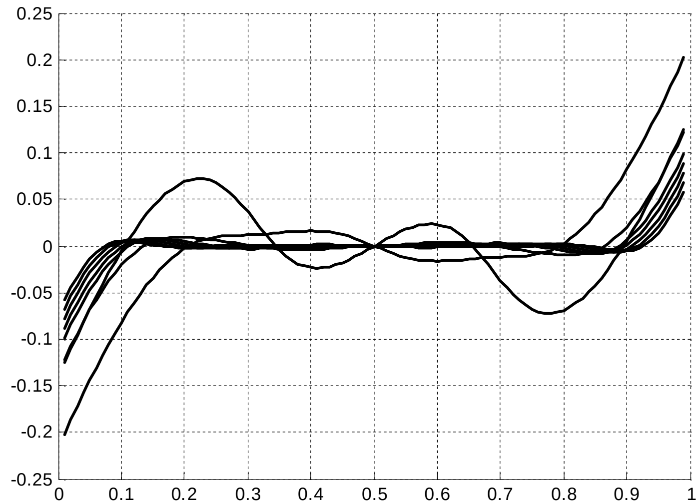


$$F_i \cos(\delta_i)$$

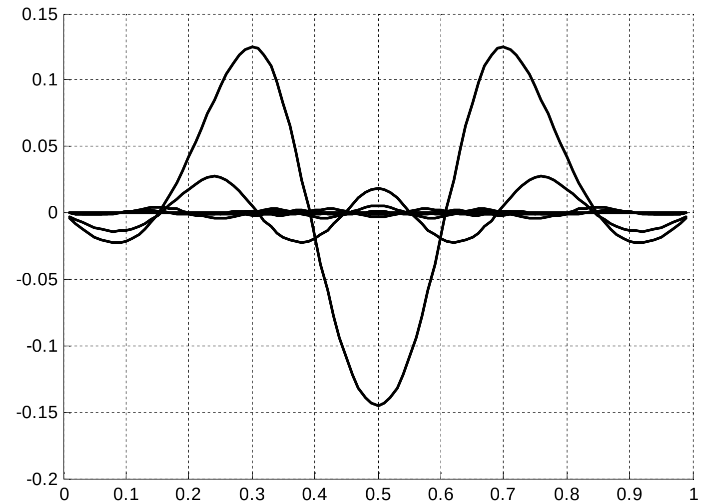


$$F_i \sin(\delta_i)$$

Increase in the number of forces significantly improves the quality of approximation:



Phase error

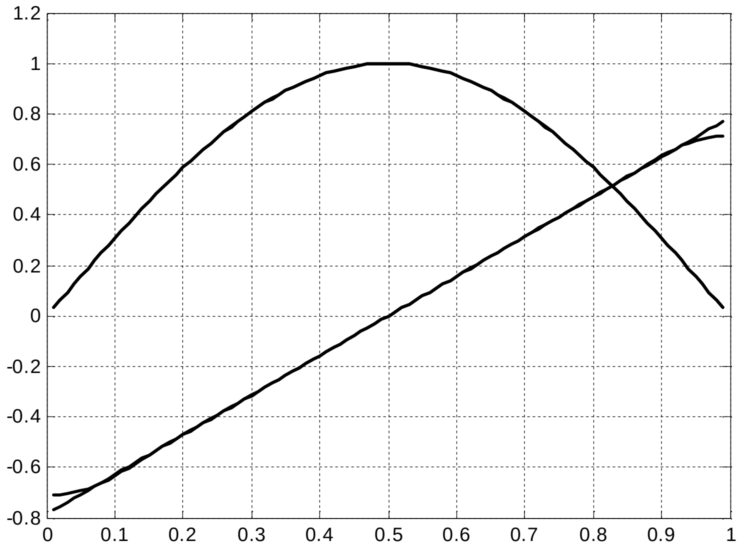


Amplitude error

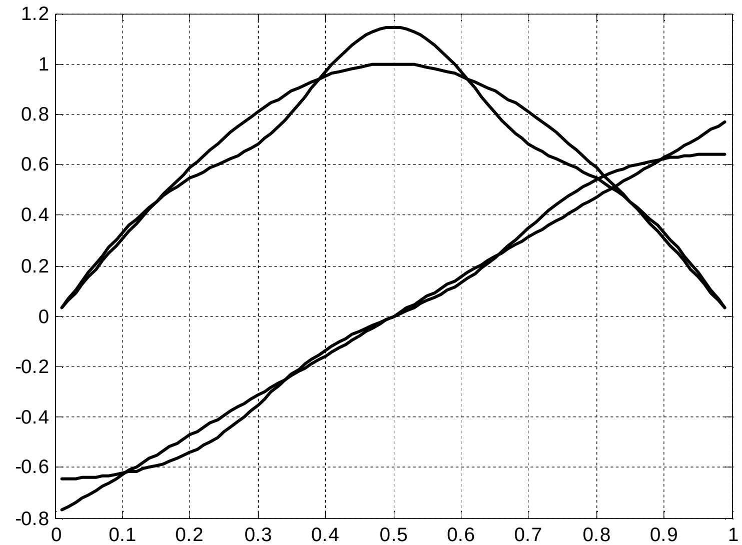
Curves with different number of applied forces

(vertical): deviation of phase and amplitude from specified values
(horizontal axis): coordinate of point along the beam

The amplitudes and phases of optimal forces depend strongly on the number of forces:

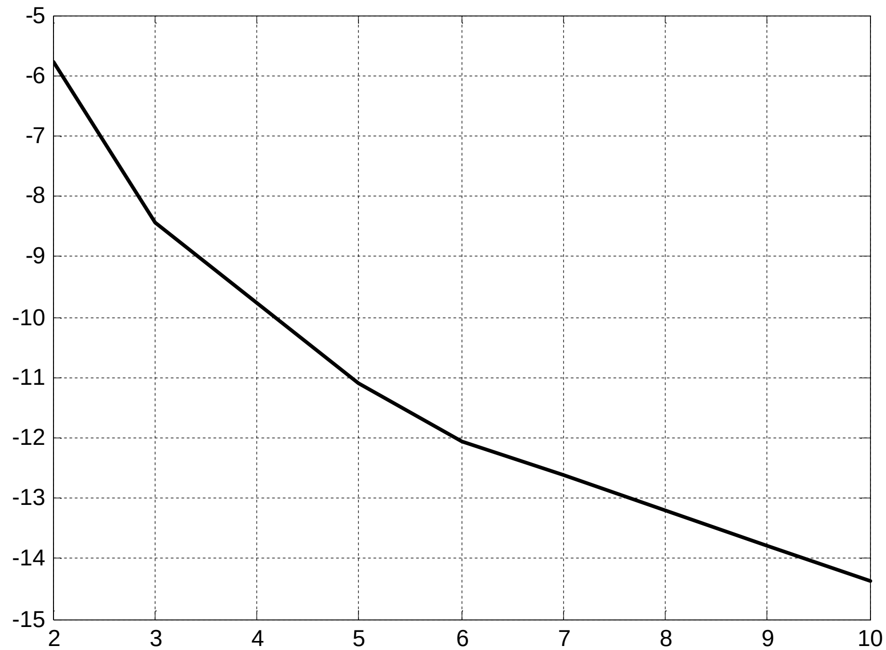


10 forces



2 forces

Specified and achieved values of amplitude and phase of the vibrations of the beam for 10 and 2 forces.



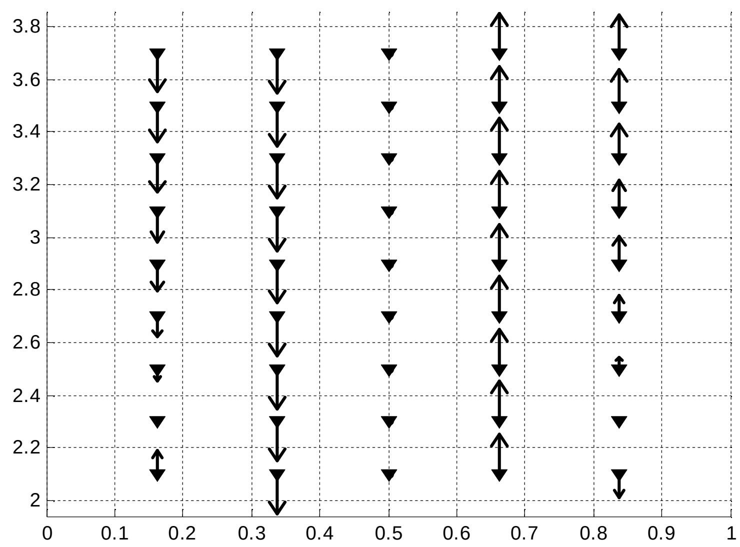
Logarithmic approximation error:

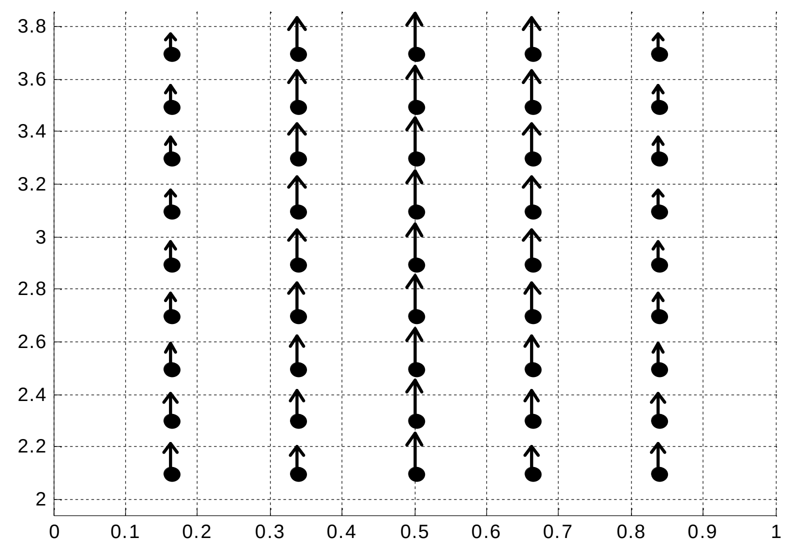
- the objective value (vertical axis)
- number of forces (horizontal axis)

Error is reduced exponentially for $I \geq 5$.

The optimum number of forces = 5.

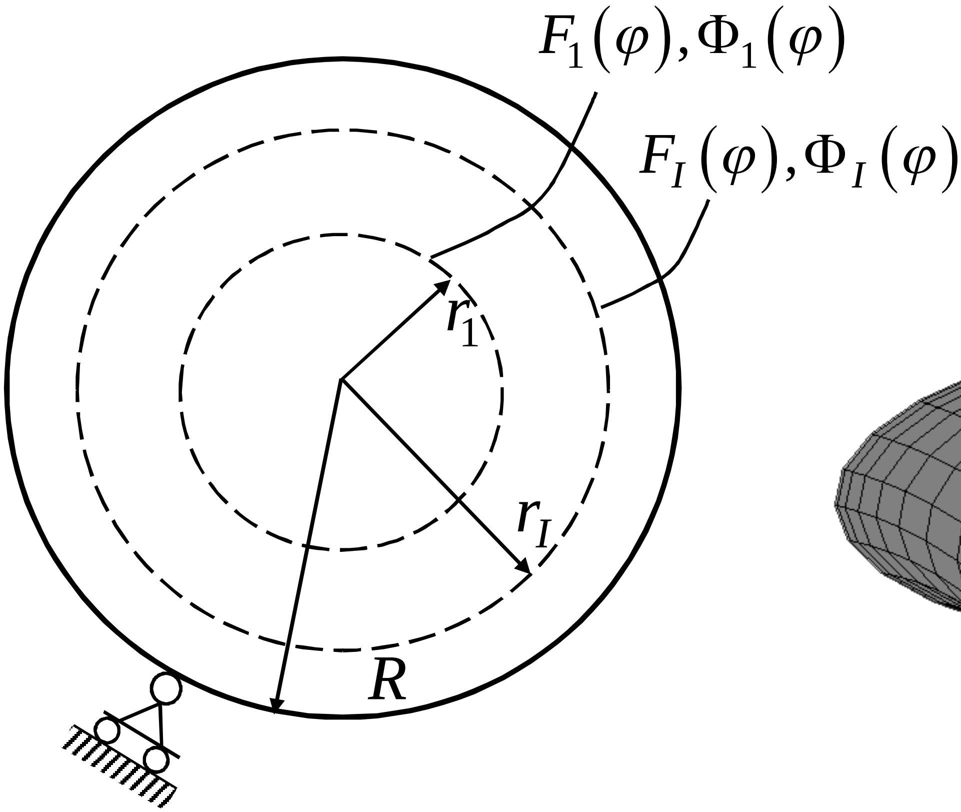
Optimal characteristics for 5 forces near resonance frequency.
Range: $k = 2.1 - 3.7, \quad (k = 3),$

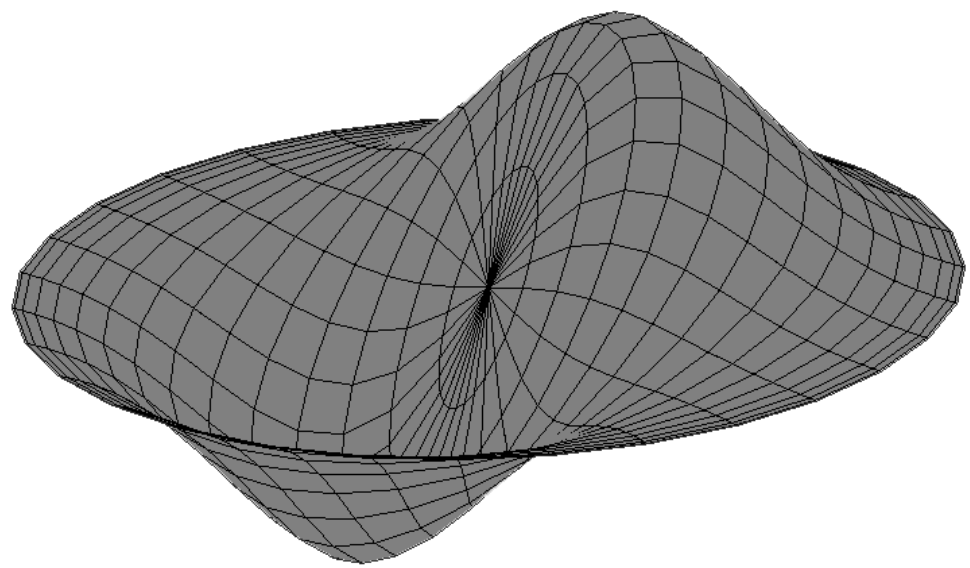




Vertical axis: frequency (with increments of 0.2)

Excitation of Circular Plate Oscillation (2D)





$$D = D_0\left(1 + 10^{\theta - 2}\right)$$

Control Problem

$$\min_{\vec{a},\vec{\xi},I} R\left\{\int_0^{2\pi}\int_0^R \left(w(r,\varphi,t)-\bar{W}(r,\varphi,t)\right)^2 r\,dr\,d\varphi+\gamma I\right\},\quad i=\overline{1,I}$$

$$\bar{W}(r,\varphi,t)=\bar{W}_s(r,\varphi)+\bar{W}_c(r,\varphi)\equiv$$

$$\equiv\sum_{n=0}^{\infty}\left(\bar{A}_{cn}(r)\cos n\varphi+\bar{A}_{sn}(r)\sin n\varphi\right)\sin\omega t+$$

$$+\sum_{n=0}^{\infty}\left(\bar{B}_{cn}(r)\cos n\varphi+\bar{B}_{sn}(r)\sin n\varphi\right)\cos\omega t$$

Boundary value problem (Kirchhoff model)

$$\Delta^2 w_\varsigma - k^2 w_\varsigma = \sum_{i=1}^{I} \Phi_i(\varsigma,\varphi)\Delta(r,\xi_i),$$

$$i = \overline{1,I},\ \xi_i, r \in (0,R),\ \varphi \in (0,2\pi]$$

$$\Delta(r,\xi_i) = \begin{cases} \delta(r-\xi_i),\ \xi_i \neq 0 \\ \delta(r)/r,\ \xi_i = 0 \end{cases},$$

$$\Phi_i(\varsigma,\varphi) = \sum_{n=0}^{\infty}\left(\varsigma_{cin}\cos n\varphi + \varsigma_{\sin}\sin n\varphi\right),$$

$$\varsigma = a \vee b$$

$k^2 = \rho h\omega^2 / D$ – wave number

Boundary Conditions (hinge support):

$$\begin{cases} w(R,\varphi)=0 \\ M_r(R,\varphi)=0, \ \ \varphi\in[0,2\pi) \end{cases}$$

Fourier components on the angular coordinate:

$$w_\varsigma(r,\varphi,\vec{\xi}) = w_{\varsigma 0}(r) +$$

$$+\sum_{n=0}^{\infty}(w_{\varsigma cn}(r,\vec{\xi})\cos n\varphi + w_{\varsigma sn}(r,\vec{\xi})\sin n\varphi) \equiv$$

$$\equiv \vartheta_0\tilde{G}(r) + \sum_{i=1}^{I}\sum_{n=0}^{\infty}(\vartheta_{cin}G_n(r,\xi_i)\cos n\varphi + \vartheta_{sin}G_n(r,\xi_i)\sin n\varphi),$$

$$\vartheta = \begin{cases} c, \varsigma = a \\ d, \varsigma = b \end{cases}.$$

$G_n(r,\xi_i)$ – Fourier components of Green's function:

$$r^2 \frac{d^2 G_n(r,\xi_i)}{dr^2} + r\frac{dG_n(r,\xi_i)}{dr} -$$

$$-(n^2 \pm (rk)^2)G_n(r,\xi_i)) = 0,\ \ r \in (0,R)$$

$$\begin{cases} G_n(R,\xi_i) = 0 \\ M_r G_n(R,\xi_i) = 0 \end{cases},\ \ n = \overline{0,\infty}$$

$$LG_n = 0, r \in (0,R),\ G_n\big|_{r=R} = 0 \wedge M_r G_n\big|_{r=R} = 0.$$

with a differential jump: $\begin{cases} \left[G_n(r,\xi_i)\right]_{\xi_i} = 0 \\ \left[\dfrac{\partial G_n(r,\xi_i)}{\partial r}\right] = 0 \\ \left[MG_r\right]_{\xi_i} = 0 \\ \left[QG_r\right]_{\xi_i} = 1/D \end{cases}$

Transformed Optimization Problem:

$$\min_{Z} R\left\{ I^{s}{}_{N}(\mathrm{Z}) + \lambda I^{c}{}_{N}(\mathrm{Z}) + \gamma I \right\},\ i = \overline{1, I}$$

$$\begin{cases} I^{c}{}_{N}(Z) = \int_{0}^{2\pi} \int_{0}^{R} \left(w_b(r, \varphi; \vec{\xi}) - \overline{W}_c(r, \varphi) \right)^2 r dr \\ I^{s}{}_{N}(\mathrm{Z}) = \int_{0}^{2\pi} \int_{0}^{R} \left(w_a(r, \varphi; \vec{\xi}) - \overline{W}_s(r, \varphi) \right)^2 r dr \end{cases}$$

$Z = \left\{ I,\ \xi_i, c_{cin}, d_{cin}, c_{sin}, d_{sin} \right\}, i = \overline{1, I},\ n = \overline{1, N}$ – set of control variables

Proposition 1

Transformed Optimization Problem is not convex, but for the fixed number of forces I the objective is continuous in control variables.

For fixed I and applications points $\{\xi_i\}_{i=1}^{N}$ the objective is convex and the problem has a unique solution that can be found from necessary condition of extremum.

Proposition 2

For fixed $I, \{\xi_i\}_{i=1}^{N}$ the problem is equivalent to:

$$\min_{\hat{Z}} R\left\{\hat{I}_N^s(\mathrm{Z}) + \hat{\lambda}\hat{I}_N^c(\mathrm{Z}) + \hat{\gamma} I\right\},\ i = \overline{1, I}$$

$$\begin{cases} \hat{I}_N^c(\mathrm{Z}) = \sum\limits_{n=0}^{N}\left\{\int\limits_0^R\left(\left(w_{d\,cn}(r,\vec{\xi}) - \overline{A}_{cn}(r)\right)^2 + \left(w_{d\,sn}(r,\vec{\xi}) - \overline{A}_{sn}(r)\right)^2\right) rdr\right\} + \\ + \int\limits_0^R\left(w_{d\,0}(r) - \overline{A}_{c0}(r)\right)^2 rdr \\ \hat{I}_N^s(\mathrm{Z}) = \sum\limits_{n=0}^{N}\left\{\int\limits_0^R\left(\left(w_{c\,cn}(r,\vec{\xi}) - \overline{B}_{cn}(r)\right)^2 + \left(w_{csn}(r,\vec{\xi}) - \overline{B}_{sn}(r)\right)^2\right) rdr\right\} + \\ + \int\limits_0^R\left(w_{d\,0}(r) - \overline{B}_{c0}(r)\right)^2 rdr \end{cases}$$

$\hat{Z}$ – reduced set of control variables