CASE STUDY: Optimal Position Liquidation with Risk Constraints (avg_g, max_risk, cvar_risk, avg_max_risk_g)

Background

This case study demonstrates a sample-path based stochastic programming model for solving optimal trading execution problem with and without risk constraints in a perfect frictionless market.

This case study reproduces and improves the optimization formulations in the paper by Krokhmal and Uryasev (2007) on sample-path approaches to optimal position liquidation. The paper utilized historical price data to generate the sample-path scenarios. Also, the paper presented optimal position liquidation strategies with no market impact, with temporary impact, and with a permanent impact. In order to eliminate the anticipativity of the model, the paper introduced a sample-path grouping method and a "lawn-mower" decision rule by adding nonanticipativity constraints. After analyzing the feasibility and optimality of the problem, the paper proposed a difference convex (DC) programming approach to evaluate the lower bound of the original problem. Finally, many numerical experiments are presented along with the analysis of the solutions.

We first conducted the numerical experiment by following the lower-bound evaluation optimization problem from the paper by Krokhmal and Uryasev (2007). Then we reformulated this problem and reduced the number of decision variables by replacing the sample-path variables with group variables. This modification eliminated the tremendous number of redundant decision variables and constraints. We implemented both formulations in PSG run-file environment. The first formulation includes PSG functions Avg_g and Max_risk. The second formulation includes PSG function Avg_max_risk_g and works much faster than the first one.

Similar to Krokhmal and Uryasev (2007) we added the CVaR constraints to the first formulation and solved the problem with PSG. The computational results were not encouraging for this problem formulation. The adding CVaR constraint significantly increased the solving time of the fist problem. Unfortunately, specialized function for CVaR risk is not available in the current version of PSG to include the CVaR constraint, similar to the second formulation containing only group variables (without CVaR risk constraint).

Three problems were presented in this case study: Problem 1 corresponds to the first formulation without risk constraint; Problem 2 corresponds to the second formulation without risk constraint; and Problem 3 corresponds to the first formulation with risk constraint.

References

- Krokhmal, P. and Uryasev, S. (2007). A sample-path approach to optimal position liquidation. *Annals of Operations Research,* 152, 193-225.
- Bertimas, D., Lo, A.W. and Hummel, P. (1999). Optimal control of execution costs for portfolios. *Computing in Science & Engineering*, 1, 40-53.

- Almgren, R. and Chriss, N. (2000). Optimal Execution of Portfolio Transactions. *Journal of Risk*, 3, 5–39.
- Birge, J.R. and Louveaux, F. (1997). Introduction to stochastic programming. *New York, Springer.*

Notations

J *= number of sample paths in the scenario model;* $j = \{1,\dots,J\}$ *is index of sample paths;*

T *= number of time periods;* $t = \{1,\dots,T\}$ *is index of time periods;*

K *= number of groups;* $k = \{1,\dots,K\}$ *is index of groups;*

$k(j,t)$ *is the group number function returning the group number for sample path j at time t.*

Sample paths are grouped according to the price S_t^j *such that: 1) every group has equal number of scenarios; 2)* ξ_t^j *with a higher value of* S_t^j *is assigned to a group with a higher group number;*

$\boldsymbol{S} = (S_1^1,\dots,S_T^1,\dots,S_t^j,\dots,S_1^J,\dots,S_T^J)$ *is vector of normalized prices,* $t = 1,\dots,T,\ \ j = 1,\dots,J;$

S_t^j *= normalized price on path j at time* t, $t = 1,\dots,T, j = 1,\dots,J;$

$\boldsymbol{\xi} = (\xi_1^1,\dots,\xi_T^1,\dots,\xi_t^j,\dots,\xi_1^J,\dots,\xi_T^J)$ *is vector of fractions of positions,* $t = 1,\dots,T,\ \ j = 1,\dots,J;$

ξ_t^j *= fraction of position on path j at time* t, $t = 1,\dots,T,\ \ j = 1,\dots,J;$

$\xi_0^j = 1, \xi_T^j = 0,\ \ j = 1,\dots,J;$

$\boldsymbol{x} = (x_1^1,\dots,x_T^1,\dots,x_t^k,\dots,x_1^K,\dots,x_T^K)$ *is vector of fractions of positions,* $t = 1,\dots,T,\ \ k = 1,\dots,K;$

x_t^j *= fraction position in group* k *at time* t, $t = 1,\dots,T,\ \ k = 1,\dots,K;$

$x_0^j = 1, x_T^j = 0,\ j = 1,\dots,J;$

$L(\boldsymbol{x},\boldsymbol{\xi})$ *is the random loss function with outcomes* $-\sum_{t=1}^{T} S_t^j \left(\xi_{t-1}^j - x_t^{k(j,t)}\right)$*, j =1,…,J having*

equal probabilities, 1/J ;

$cvar\_risk\big(L(\boldsymbol{x},\boldsymbol{\xi})\big) = CVaR_\alpha(L(\mathbf{x},\boldsymbol{\xi}))$*; it approximately equals to the average of (1-*α*)* $*\, 100\%$

worst-case scenarios. See exact definition in the PSG manual;

$avg\_g\big(L(\boldsymbol{x},\boldsymbol{\xi})\big) = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} -S_t^j \left(\xi_{t-1}^j - x_t^{k(j,t)}\right)$ = average gain function;

$D(\boldsymbol{x},\boldsymbol{\xi}) = \{\xi_t^j - x_t^{k(j,t)} | t = 1,\dots,T-,\ j = 1,\dots,J\}$ *is the set of differences* $\xi_t^j - x_t^{k(j,t)}$*;*

$\max\_risk\big(D(\boldsymbol{x},\boldsymbol{\xi})\big) = \max_{t=1,\dots,T-1,\ \ j=1,\dots,J}\{\xi_t^j - x_t^{k(j,t)}\}$ *;*

$C(\boldsymbol{\xi}) = \{\xi_{t+1}^j - \xi_t^j | t = 1, \ldots, T-1, \;\; j = 1, \ldots, J\}$ *is the set of differences* $\xi_{t+1}^j - \xi_t^j$;

$\max\_risk(C(\boldsymbol{\xi})) = \max_{t=1,\ldots,T-1, \;\; j=1,\ldots,J}\{-(\xi_{t+1}^j - \xi_t^j)\}$;

$B(\boldsymbol{x}) = \left\{x_\tau^{k(j,\tau)} - x_t^{k(j,t)} | \; \tau = 0,1, \ldots, t, \;\; t = 1, \ldots, T, \;\; j = 1, \ldots, J \right\}$ *is the set of differences*

$x_\tau^{k(j,\tau)} - x_t^{k(j,t)}$;

$avg\_max\_risk\_g(B(\boldsymbol{x})) = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\max_\tau\{-(x_\tau^{k(j,\tau)} - x_t^{k(j,t)})\}$ *is the average max risk for gain function;*

$w =$ *threshold in risk constraint.*

Optimization Problem 1

maximizing expected sum of payoffs:

$$\max \; avg\_g(L(\boldsymbol{x}, \boldsymbol{\xi})) \quad \text{CS.1}$$

subject to

nonanticipativity constraint:

$$max\_risk(D(\boldsymbol{x}, \boldsymbol{\xi})) \leq 0 \quad \text{CS.2}$$

monotonicity constraint:

$$max\_risk(C(\boldsymbol{\xi})) \leq 0 \quad \text{CS.3}$$

box constraints:

$$0 \leq x_t^{k(j,t)} \leq 1, \; t = 1, \ldots, T, \; j = 1, \ldots, J \quad \text{CS.4}$$

To eliminate the redundant variables ξ_t^j in *Optimization Problem 1* we use the following notation:

$$\xi_t^j = \min\{\xi_{t-1}^j, x_t^{k(j,t)}\} \, . \quad \text{CS.5}$$

We equivalently present ξ_{t-1}^j using CS. 5:

$$\xi_t^j = \min\left\{\min\left\{\xi_{t-2}^j, x_{t-1}^{k(j,t-1)}\right\}, x_t^{k(j,t)}\right\}$$
$$= \min\left\{\xi_{t-2}^j, x_{t-1}^{k(j,t-1)}, x_t^{k(j,t)}\right\} .$$

By repeating this substitution, we get:

$$\xi_t^j = \min\left\{\xi_0^j, x_1^{k(j,1)}, \dots, x_t^{k(j,t)}\right\}.$$

Since $\xi_0^j = 1$, we obtain:

$$\xi_t^j = \min\left\{1, x_1^{k(j,1)}, \dots, x_t^{k(j,t)}\right\}. \qquad \text{CS.6}$$

Optimization Problem 1 is equivalently reformulated as follows:

Optimization Problem 2

maximizing expected sum of payoffs:

$$\max\ -avg\_max\_risk\_g\big(B(\boldsymbol{x})\big) \qquad \text{CS. 7}$$

subject to

box constraints:

$$0 \le x_t^{k(j,t)} \le 1, \quad t = 1, \dots, T, \quad j = 1, \dots, J \qquad \text{CS.8}$$

Optimization Problem 3

maximizing expected sum of payoffs:

$$\max avg\_g\big(L(\boldsymbol{x}, \boldsymbol{\xi})\big) \qquad \text{CS.9}$$

subject to

CVaR constraint:

$$CVaR\big(L(\boldsymbol{x}, \boldsymbol{\xi})\big) \le w \qquad \text{CS.10}$$

nonanticipativity constraint:

$$max\_risk\big(D(\boldsymbol{x}, \boldsymbol{\xi})\big) \le 0 \qquad \text{CS.11}$$

monotonicity constraint:

$$max\_risk\big(C(\boldsymbol{\xi})\big) \le 0 \qquad \text{CS.12}$$

box constraints:

$$0 \le x_t^{k(j,t)} \le 1, \quad t = 1, \dots, T, \quad j = 1, \dots, J \qquad \text{CS.13}$$