CASE STUDY: Omega Portfolio Rebalancing (omega, pm_pen, avg)

Background

This case study demonstrates an Omega optimization setup for a portfolio optimization problem. A fund of funds blends the risk-return profiles of various hedge fund managers/strategies to meet investor requirements. The data for this case study are prepared with the Converter_Omega_Portfolio. To install this converter you should download installation file “Converter_Omega_Portfolio_setup.zip” in the Client Area (www.aorda.com) from the Download page by selecting the “Case Studies” downloading option.

References


Notations

I = number of instruments in the portfolio; i={1,...,I} index of instruments in the portfolio;

\( y_i \) = exposure to instrument i-th (in currency);

\( w_i = \frac{y_i}{\sum_{i=1}^{I} y_i} \) = weight of i-th instrument in the portfolio;

\( w = (w_1, ..., w_I) \) = vector of weights of instruments in the portfolio;

\( r_i \) = rate of return of i-th instrument in time period t;

\( r_h \) = hurdle rate of return;

\( L(y, t) = \sum_{i=1}^{I} (r_i - r_i^h) y_i \) = loss function (underperformance of the portfolio) at time t as a function of y (in currency);

\( L(w, t) = \frac{\sum_{i=1}^{I} (r_i - r_i^h) y_i}{\sum_{i=1}^{I} y_i} = \sum_{i=1}^{I} (r_i - r_i^h) \frac{y_i}{\sum_{i=1}^{I} y_i} = \sum_{i=1}^{I} (r_i - r_i^h) w_i \) = loss function (underperformance of the portfolio) at time t as a function of w (in fractions);

\( \eta(y) = \frac{1}{T} \sum_{t=1}^{T} L(y, t) \) = low partial moment for loss function \( L(y, t) \) with respect to benchmark 0 (in currency);
\[ \eta(w) = \frac{1}{T} \sum_{t=1}^{T} L(w, t) \] is low partial moment for loss function \( L(w, t) \) with respect to benchmark 0 (in fractions);

\[ \xi(y) = \frac{1}{T} \sum_{t=1}^{T} -L(y, t) \] is upper partial moment for loss function \( L(y, t) \) with respect to benchmark 0 (in currency);

\[ \xi(w) = \frac{1}{T} \sum_{t=1}^{T} -L(w, t) \] is upper partial moment for loss function \( L(w, t) \) with respect to benchmark 0 (in fractions);

\[ q(y) = \frac{1}{T} \sum_{t=1}^{T} -L(y, t) \] is expected gain with respect to hurdle rate as a function of \( y \) (in currency);

\[ q(w) = \frac{1}{T} \sum_{t=1}^{T} -L(w, t) \] is expected gain with respect to hurdle rate as a function of \( w \) (in currency);

\[ \Omega(w) = \frac{\xi(y)}{\xi(y)} = \frac{\xi(w) - \eta(w) + \eta(w)}{\xi(w)} = 1 + \frac{q(w)}{\xi(w)} = \text{Omega function}; \]

\( M \) is number of strategies;

\( J_m \) is set of managers using \( m \)-th strategy, \( m = 1, \ldots, M \);

\( b^m_l \) is lower bound in the constraints on allocations to strategies, \( m = 1, \ldots, M \);

\( b^u_m \) is upper bound in the constraints on allocations to strategies, \( m = 1, \ldots, M \);

\( l_i \) is lower bound in the box constraints for individual positions, \( i = 1, 2, \ldots, J \);

\( u_i \) is upper bound in the box constraints for individual positions, \( i = 1, 2, \ldots, J \);

**Omega Maximization Problem**

\[
\max \Omega(w) = 1 + \max_{m} \frac{q(w)}{\xi(w)} \tag{CS.1}
\]

subject to

**budget constraint**

\[
\sum_{j=1}^{J} w_j = 1 \tag{CS.2}
\]

**constraints on allocations to strategies**
box constraints for individual positions
\[ l_i \leq w_i \leq u_i, \quad i = 1, 2, \ldots, I \] (CS.4)

Reduction Omega Optimization to Partial Moment Optimization

Problem (CS.1)-(CS.4) can be reduced to a convex optimization programming problem by introducing an additional variable and restriction. Let \( x_0 > 0 \) such that
\[ x_0 \eta(w) = \eta(x_0w) = 1. \] (CS.5)
The following equality is valid
\[ \frac{q(w)}{\eta(w)} = \frac{x_0q(w)}{x_0\eta(w)} = \frac{q(x_0w)}{\eta(x_0w)}. \] (CS.6)
Denote \( x_i = x_0w_i, \quad i = 1, 2, \ldots, I, \quad x = (x_0, x_1, \ldots, x_I). \) The problem (CS.1)-(CS.4) can be reduced to the following problem.

Maximizing Expected Gain Subject to Constraint on Partial Moment

\[
\max [1 + q(x)] = 1 + \max AVG\_GAIN(L(x,t))
\]
subject to
downside loss constraint
\[ \eta(x) = PM\_PEN(L(x,t)) \leq 1, \] (CS.8)
budget constraint
\[ \sum_{i=1}^{I} x_i = x_0, \] (CS.9)
constraints on allocations to strategies
\[ b_m^l x_0 \leq \sum_{i \in J \cap M^l} x_i \leq b_m^u x_0, \quad m = 1, 2, \ldots, M, \] (CS.10)
constraints on allocations to individual managers
\[ x_0 l_i \leq x_i \leq x_0 u_i, \quad i = 1, 2, \ldots, I. \] (CS.11)
constraints on variables
\[ x_0 > 0, \quad x_i \geq 0, \quad i = 1, 2, \ldots, I. \] (CS.12)

Let us denote by \( x^* = (x_0^*, x_1^*, \ldots, x_I^*) \) an optimal solution of problem (CS.7)-(CS.12). An optimal solution of the original problem (CS.1)-(CS.4) equals \( w_i^* = \frac{x_i^*}{x_0^*}, \quad i = 1, 2, \ldots, I. \)

With expression (CS.9) we excluded variable \( x_0^* \) from problem (CS.7)-(CS.12) and obtained the following simplified formulation:
Maximizing Expected Gain Subject to Constraint on Partial Moment (Simplified Formulation)

\[
\max [1 + q(x)] = 1 + \max \text{AVG\_GAIN}(L(x, t)) \quad \text{(CS.13)}
\]

subject to

downside loss constraint

\[
\eta(x) = \text{PM\_PEN}(L(x, t)) \leq 1, \quad \text{(CS.14)}
\]

constraints on allocations to strategies

\[
b_m \sum_{i=1}^{I} x_i \leq \sum_{j=1}^{I} x_j \leq b_m' \sum_{i=1}^{I} x_i, \quad m = 1, 2, \ldots M, \quad \text{(CS.15)}
\]

constraints on allocations to individual managers

\[
l_i \sum_{i=1}^{I} x_i \leq x_i \leq u_i \sum_{i=1}^{I} x_i, \quad i = 1, 2, \ldots I. \quad \text{(CS.16)}
\]

constraints on variables

\[
x_i \geq 0, \quad i = 1, 2, \ldots, I. \quad \text{(CS.17)}
\]

An optimal decision vector of the original problem (CS.1)-(CS.4) equals

\[
w_i^* = \frac{x_i^*}{\sum_{i=1}^{I} x_i^*}, \quad i = 1, 2, \ldots, I. \quad \text{(CS.18)}
\]

Constraints (CS.15) can be transformed to the following linear constraints

\[
\sum_{i=1}^{I} (\delta_m - b'_m) x_i \geq 0, \quad m = 1, 2, \ldots M, \quad \text{(CS.15-1)}
\]

\[
\sum_{i=1}^{I} (\delta_m - b''_m) x_i \leq 0, \quad m = 1, 2, \ldots M, \quad \text{(CS.15-2)}
\]

where

\[
\delta_m = \begin{cases} 
1, & \text{if } i \in J_m \\
0, & \text{otherwise}
\end{cases}
\]

Constraints (CS.16) can be transformed to the following linear constraints

\[
\sum_{k=1}^{I} (\delta_k - l_k) x_i \geq 0, \quad i = 1, 2, \ldots I \quad \text{(CS.16-1)}
\]

\[
\sum_{k=1}^{I} (\delta_k - u_k) x_i \leq 0, \quad i = 1, 2, \ldots I, \quad \text{(CS.16-2)}
\]

where

\[
\delta_k = \begin{cases} 
1, & \text{if } i = k \\
0, & \text{otherwise}
\end{cases}
\]