CASE STUDY: Basic CVaR Optimization Problem, Beyond Black-Litterman (cvar_risk)

Background

This case study demonstrates a simple (basic) setup of single-period portfolio optimization problem when risk is measured by CVaR. Portfolio optimization has come a long way from when Markowitz's (1952) seminal work suggested the mean/variance framework. The Markowitz problem addresses the trade-off between mean and variance of a portfolio return. In the original formulation, the problem is to find a minimum-variance portfolio under the constraint of mean return. Variance is not a good measure of risk for asymmetric distributions. Conditional Value-at-Risk (CVaR) (see Rockafellar and Uryasev (2000, 2002)) takes into account only the downside part of the distribution and it has nice mathematical properties (a coherent risk measure). One of the advantages of the CVaR approach is that it solves the portfolio optimization problem without normal assumption on distribution of market risk factors. This assumption violates in most markets, where distribution of market risk factors is characterized by fat tails, skewness and high dependence among extreme events. Moreover, the CVaR approach can be easily implemented within more complex approach to the portfolio optimization problem, where the portfolio manager blends his subjective views on the market with a prior market distribution (see, Meucci (2005a, 2005b)). This case study implements the portfolio optimization problem described by Meucci (2005b). This project also demonstrates how to handle possible infeasibility of the constraints by adding the compensation variables.

References

- Rockafellar, R. T. and S. Uryasev (2000): Optimization of Conditional Value-At-Risk, *The Journal of Risk*, Vol. 2, No. 4, pp. 21-51.
- Rockafellar, R.T. and S. Uryasev (2002): Conditional Value-at-Risk for General Loss Distributions, Journal *of Banking and Finance*, 27/7.
- Meucci, A. (2005a): Beyond Black-Litterman: Views on Non-Normal Markets, (November 2005), download: http://ssrn.com/abstract=848407.
- Meucci, A. (2005b): Beyond Black-Litterman in Practice: A Five-Step Recipe to Input Views on Non-Normal Markets, (December 28, 2005), download: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=872577 .

Notations

I = number of instruments in the portfolio; $i=\{1,\ldots,I\}$ index of instrument in the portfolio;

J = number of scenarios; $j=\{1,\ldots,J\}$ index of scenarios;

$\mathbf{x} = (x_1, \ldots, x_I)$ = vector of weights of instruments in the portfolio, $i=1\ldots I$;

θ_{ij} = rate of return of instrument i under scenario j;

$\boldsymbol{\theta} = (\theta_0, \theta_1, \ldots, \theta_I)$ = random return vector;

$L(\boldsymbol{x}, \boldsymbol{\theta}) = -\sum_{i=1}^{I} \theta_i x_i$ = loss function;

$r_i = \frac{1}{J}\sum_{j=1}^{J} \theta_{ij}$ = average rate of return of the i-th instrument over set of scenarios $j=\{1,\ldots,J\}$;

r = target level of rate of return of the portfolio;

w = bound on the portfolio CVaR;

λ = penalty coefficient;

α = confidence level in CVaR.

Further, we provide three equivalent formulations for the generation of the efficient frontier: mean return versus risk (measured by $\boldsymbol{CVaR(L(x,\theta))}$) of a portfolio.

Optimization Problem 1

minimizing portfolio CVaR

$$\min_{\boldsymbol{x}} \boldsymbol{CVaR(L(x,\theta))} \tag{CS.1}$$

subject to

budget constraint

$$\sum_{i=1}^{I} x_i = 1, \tag{CS.2}$$

constraint on the portfolio mean return

$$\sum_{i=1}^{I} r_i x_i \geq r \tag{CS.3}$$

no short constraints

$$x_i \geq 0, \quad i = 1, \ldots, I. \tag{CS.4}$$

The constraint (CS.3) may be infeasible. To prevent possible infeasibility we have added to the performance function a compensation variable assuring feasibility.

Optimization Problem 2

minimizing weighted average of risk and penalty

$$\min_{x, x^a} \left[\lambda x^a + CVaR(L(x,\theta)) \right] \tag{CS.5}$$

subject to

budget constraint

$$\sum_{i=1}^{I} x_i = 1, \tag{CS.6}$$

constraint on portfolio mean return combined with penalty variable x^a

$$\sum_{i=1}^{I} r_i x_i + x^a \geq r \tag{CS.7}$$

no short constraints and non-negativity constraint on penalty variable x^a

$$x_i \geq 0, \quad i = 1, \ldots, I, \quad x^a \geq 0. \tag{CS.8}$$

Optimization Problem 3

maximizing portfolio mean return

$$\max_{x} \sum_{i=1}^{I} r_i x_i, \tag{CS.9}$$

subject to

CVaR constraint

$$CVaR(L(x,\theta)) \leq w, \tag{CS.10}$$

budget constraint

$$\sum_{i=1}^{I} x_i = 1 , \tag{CS.11}$$

no short constraints

$$x_i \geq 0, \quad i = 1, \ldots, I . \tag{CS.12}$$