Enhancing Make-to-Order Manufacturing Agility: When Flexible Capacity Meets Dynamic Pricing

Abstract: The rise of online marketplaces has raised customer expectations regarding customization and lead time, posing significant challenges to manufacturing firms and prompting a move from make-to-stock to a more flexible make-to-order system. A major challenge in make-to-order manufacturing is that fluctuations in demand cannot be smoothed by available stock. Many manufacturing firms can create capacity flexibility in addition to using dynamic pricing schemes to maintain a balance between demand and supply. In that scenario, system costs could be cut by managing capacity and demand simultaneously. In this paper, we consider a make-to-order production environment with base and surge capacity as well as the ability to adjust product pricing. Our main focus is on operational decision-making, assuming that base and surge capacity are fixed, but activating the surge capacity incurs a setup cost. Initially, we propose a stochastic control model to reflect this complex decision problem. However, our initial model leads to an intractable stochastic control problem. To overcome this, we convert the problem to a more tractable diffusion control problem. This approach helps to reveal the conditions under which utilizing flexible capacity is more advantageous than relying solely on fixed capacity in traditional contexts. When flexible capacity is advantageous, we provide a solution to the diffusion control problem that can guide optimal capacity and price adjustments. We discover a rich interplay between capacity adjustment and dynamic pricing. In particular, we find that the price, which aims at reducing congestion, may not monotonically increase with the congestion level when a setup cost is present. We also demonstrate how to expand the single-product model to accommodate multiple products and show that the model extends to a make-to-stock paradigm as well.

Key words: make-to-order production; flexible capacity; dynamic pricing; queueing; diffusion models

1 Introduction

The rise of online marketplaces such as Amazon, eBay, and Walmart has brought significant challenges to manufacturing firms. For one thing, customers are now expecting highly customized items with even shorter lead times. This is also true in business-to-business sectors, where individualized, engineering-to-order approaches are becoming more common. Another trend is the ongoing digitization of manufacturing, which results in mass customization and a shorter product life cycle. For those reasons, manufacturers are urged to move from a traditional make-to-stock (MTS) manufacturing environment to a more flexible make-to-order (MTO) manufacturing environment because demand for highly customized products cannot be economically produced to stock (similar to customer services that cannot be physically stored). However, a major challenge in implementing the MTO approach is matching supply with demand. This is because fluctuations in demand cannot be smoothed by available stock, potentially causing long delays in order processing in periods of high
demand and idle capacity in periods of low demand. While longer wait times lead to customer dissatisfaction and/or loss of customer lifetime value, idle capacity results in investment waste.

From a demand-control perspective, dynamic pricing has been recognized as a popular strategy for increasing supply chain efficiency in MTO production setups. For instance, Tesla adjusts the price of its electric vehicles based on supply and demand, in addition to customers’ selected features and options. Similarly, Nike offers custom-designed sneakers on its website, where prices vary based on colors, material choices, and market conditions. To further balance demand and supply, many manufacturing firms have capacity flexibility at their disposal. A common approach to achieving capacity flexibility in manufacturing is to combine base capacity with surge capacity. Base capacity refers to the amount of production that can be sustained over a long period of time, while surge capacity allows for temporary increases in production to meet unexpected demand or short-term spikes in orders. In the automotive industry, car manufacturers can achieve capacity flexibility by opening and closing production lines. For instance, a factory with \( n + 1 \) production lines can run between \( n \) and \( n + 1 \) lines. In this case, \( n \) of the lines constitute the base capacity, while the remaining line can be turned on and off as needed, thereby serving as surge capacity (Wu and Chao 2014). Alternatively, surge capacity can be procured by commissioning an external capacity provider (ECP) who offers manufacturing as a service.\(^1\) For example, in the shoe manufacturing industry, an ECP can be a 3D printing bureau that accepts outsourced orders from a shoe factory, enabling the shoe factory to expand its otherwise rigid internal production capacity as needed.

In this paper, we study a queueing model that serves as a parsimonious mathematical representation of an MTO manufacturing system offering a single or multiple products. The system features both base and surge production capacity and adapts pricing to balance supply with demand. We model the base capacity as the primary production server and the surge capacity as the secondary server. The primary server is continuously operational, while the secondary server can be switched on and off as needed to meet fluctuating demand. However, activating the secondary server incurs a fixed setup cost of \( C \), necessitating careful management of switching operations. As the system does not keep inventory, waiting costs also pose a concern. In addition, the product price can be dynamically adjusted, and the incoming order rate is a function of the posted price. The goal is to identify a joint capacity adjustment and dynamic pricing strategy that maximizes long-term average profit, calculated as revenue from product sales minus operating costs. In particular, capacity adjustments involve determining the timing of activating and deactivating the secondary server. These are path-dependent decisions that can be further compounded by dynamic pricing. To tackle these challenges, we adopt a rigorous stochastic control framework that accounts for the stochastic uncertainties in both demand and production.

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\(^1\) An ECP provides on-demand flexible production capabilities through its own production facilities, allowing for remote production if it is logistically feasible (Rauschecker et al. 2014).
The problem of finding an optimal joint capacity adjustment and dynamic pricing strategy is generally complex and not amenable to exact analysis. Therefore, we apply standard approximation techniques commonly used in the literature. We assume that the manufacturing firm operates with a large demand volume and is functioning at a critical capacity, which we will define later. In this scenario, we can use the traditional heavy-traffic regime as an approximation to the original stochastic control problem. By analyzing the solution to the diffusion control problem (DCP) that arises in the heavy-traffic regime, we can readily convert it into a control strategy for the original MTO system. We use a diffusion approximation to analyze the queueing system in our model for two main reasons. First, the manufacturing firm we model operates at a high demand volume with in-house capacity nearly matching the demand, which is a suitable parameter regime to apply heavy-traffic approximation. Second, a diffusion approximation allows decisions to be optimized “under very weak distributional restrictions on model input” (Bradley and Glynn 2002).

We outline the contributions of this paper in three aspects.

**Modeling:** We utilize a formal stochastic control framework to investigate joint capacity adjustment and dynamic pricing. We show that the two control levers, capacity adjustment and pricing, generate complex feedback loops in the system dynamics, resulting in a challenging decision problem. We overcome this challenge by deriving and solving an approximating DCP and then translating its solution into an easy-to-implement joint capacity adjustment and pricing strategy based on the current congestion level. We model a fixed cost of expediting and stochastic expedited processing times as well (in contrast to the previous literature). Further, we generalize the model to allow for negative $X(t)$, in which case $-X(t)$ represents the on-hand inventory, enabling us to model a make-to-stock setting.

**Methodology:** This paper establishes the well-posedness of the solution to the Bellman equation that characterizes the optimal policy for the DCP. Distinct from prior work on optimal switching where optimal policies can be characterized by solving a pair of linear differential equations (thus admitting explicit solutions), we overcome the technical challenge of establishing structural properties of the solutions to two nonlinear differential equations that do not satisfy the global Lipschitz continuity condition and cannot be explicitly solved. This technical hurdle methodologically distinguishes our paper from prior works on optimal switching.

**Managerial Insights:** Our model and analysis provide several useful insights into the control of MTO systems that have access to flexible capacity and pricing capability in real-time. We find that surge capacity is economically valuable only when the fixed cost is not prohibitively high. Specifically, using surge capacity benefits the manufacturing firm only when $C < \bar{C}$, with $\bar{C}$ being some threshold determined by model primitives. When $C < \bar{C}$, the manufacturing firm should use a capacity adjustment strategy that activates surge capacity when the number of outstanding orders exceeds a certain level $b$ and turns it off when it falls below another level $a$. Furthermore, we uncover a rich interplay between on-demand capacity adjustments and dynamic pricing decisions. Interestingly, we show that the price does not always increase with the
congestion level. Instead, the manager may choose to raise prices first to curb customer demand as the congestion level rises to a critical point, after which the manager’s best interests are served by lowering the price. To our knowledge, this is the first paper showing non-monotonic structures of pricing policies with unobservable queues. We attribute this “abnormality” to the capacity expansion and shrinkage mechanisms and elaborate on it later in the paper.

The remainder of this paper is structured as follows. Section 2 summarizes the relevant literature. Section 3 introduces our model in a single-product setting. Due to the intractability, we formulate and solve an approximating DCP in Section 4, accompanied with the main theoretical results. We also briefly demonstrate how these results carry over to an MTS system in the same section. Section 5 introduces and studies a DCP in a multi-product setting. Section 6 presents the proposed policies and an atypical pricing structure. Extensive numerical studies are presented in Section 7, followed by concluding remarks in Section 8.

2 Literature Review

Pricing in queues. Our approach to modeling dynamic pricing and its impact on demand is based on a stream of research studying dynamic pricing for queues. Low (1974) investigates the optimal control of a Markovian queue with a finite buffer with the objective of maximizing the long-run average reward by serving customers. In this system, the system manager can only control the arrival rates through prices, while having no control over the capacity level. Customers who arrive at the queue are independent, rational decision-makers who base their decision to join the queue on the prices set by the system manager and their net utilities. Yoon and Lewis (2004) explore the problem of dynamic pricing and admission control in a system where arrival and service rates are nonstationary, and customers are sensitive to prices. The authors establish several structural properties of the optimal policy, including the monotonicity of the optimal prices in the state of the system under various cost structures. Ata and Shneorson (2006) investigate the problem of dynamically controlling the arrival and service rates in a service facility to optimize long-run average system welfare. Their study proposes a solution to determine the optimal dynamic prices and service rates a system manager should set when serving delay-sensitive, rational customers. Contrary to the findings in Low (1974), the paper by Ata and Shneorson (2006) shows that the optimal price that induces the optimal arrival rate need not be monotonic with the number of service requests waiting in the system, even though the arrival rate monotonically decreases with the number of service requests waiting in the system. Çelik and Maglaras (2008) study the problem of dynamic pricing, outsourcing, and scheduling in an MTO manufacturing system. By approximating the original problem with a drift-rate control problem, they develop an effective dynamic control policy for the original manufacturing system. In a related study, Besbes and Maglaras (2009) investigate the optimal pricing policy in situations where the dynamically changing market sizes are either

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2 Ata and Shneorson (2006) also see a non-monotonic pricing policy, but that structure is closely related to the observable queue.
observable or unobservable. They show a simple policy of always choosing the maximum of the optimal price in the absent of capacity constraint and the price inducing full capacity utilization is asymptotically optimal. Ata and Olsen (2013) consider the problem of dynamically quoting price and lead-time menus for customers competing for a given resource and derive an asymptotically optimal policy in the heavy traffic limit. Kim and Randhawa (2018) examine the value of dynamic pricing to maximize revenues in queueing systems. They demonstrate that when demand volume is on the order of \( n \), static pricing causes a revenue loss of the order of \( n^{1/2} \) whereas dynamic pricing can reduce the loss to the order of \( n^{1/3} \), and a two-price policy can achieve most of the benefits of dynamic pricing. In an earlier work, Hall et al. (2009) observe a similar result: what they call “constant price up to cutoff state” (a special two-price policy in Kim and Randhawa (2018)) results in revenue very close to that gained by a general dynamic pricing policy. More recently, Ata and Barjesteh (2023) consider the dynamic control of a multiclass make-to-stock queue, where multiple types of products are produced and stored in inventory to satisfy customer demand. The paper incorporates pricing and outsourcing decisions, extending the classic model studied in Wein (1992). Gao and Huang (2023) consider a model slightly more general than that in Ata and Barjesteh (2023) and establish the existence of a unique smooth solution to the associated Bellman equation. They also prove the asymptotic optimality of the proposed policy. In an innovative queueing model where service time is chosen by customers, Lin et al. (2023) characterize optimal pricing policies through a set of delay differential equations. Our paper differs from the aforementioned papers in that we consider flexible capacity that can expand and shrink on demand, whereas the other papers assume the total processing capacity is fixed. Several papers look at pricing and capacity sizing in a queueing context from a design perspective and obtain valuable insights into design-phase pricing and capacity sizing decisions. For example, Maglaras and Zeevi (2005), Kumar and Randhawa (2010), Lee and Ward (2019) all explore pricing and capacity sizing in queueing systems. Our paper differs from these papers in both scope and methodology, as we focus on operational decision-making.

Our study is most closely related to Çelik and Maglaras (2008) in that both their paper and ours consider a control problem faced by an MTO manufacturer who offers multiple products to customers to maximize profit through dynamic pricing, scheduling, and using surge capacity. In addition, both their paper and ours conduct analysis based on the approximating DCPs. Hence, it is worth making careful comparison and highlighting the differences between the two works. First, Çelik and Maglaras (2008) assume that the orders can be instantaneously expedited by the surge capacity, which is an important assumption in their paper to satisfy the lead-time quotation constraint. In contrast, we assume the order processing time on the surge capacity follows some general distributions. This key difference leads to very different DCPs: the controlled process in Çelik and Maglaras (2008) is a single-dimensional workload process and is upper bounded by a reflecting barrier, whereas the controlled process in our paper consists of both a single-dimensional workload process and a stochastic process taking binary values indicating the status of the surge capacity. Second, we model a setup cost and a per-unit–of-time manufacturing cost related to the surge capacity, while in Çelik
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and Maglaras (2008) the system is assumed to incur cost per expedited order. The consideration of setup cost $C$ also leads to an atypical structure of pricing scheme shown in §6.2, which is absent in Çelik and Maglaras (2008) and other previous literature. Third, the approximation to the revenue function by a Taylor expansion in Çelik and Maglaras (2008) makes their Bellman equation a Riccati equation, whereas we do not make approximations to our profit function. This difference adds another level of complexity to our analysis but meanwhile improves the approximation errors, as shown in §7.1.1.

**Optimal switching.** This paper falls into a problem category known as “optimal switching”, which involves switching costs considered as fixed investments required to realize the operational advantages of an appropriate regime. These costs require the controller to look beyond the immediate advantages to ensure that a regime switch will accrue sufficient benefits over time to merit the fixed investment. Duckworth and Zervos (2001) derive the Bellman equation using a dynamic programming principle and apply a verification approach to an optimal two-regime switching problem. Bayraktar and Egami (2010) use a sequential approximation method to study a two-regime switching problem with discounted cost criterion. They establish a dynamic programming principle for the value function as two coupled optimal stopping problems, drawing upon results from optimal stopping of one-dimensional diffusion processes. Zervos et al. (2013) examine an investor who seeks to maximize the expected discounted cash-flow generated by sequentially buying and selling one share of an asset, taking into account fixed transaction costs. They model the underlying asset price using a general one-dimensional Itô diffusion and solve the resulting stochastic control problem in a closed analytic form. The aforementioned papers consider the discounted cost criterion and therefore are not applicable to the average cost case, which is the focus of this paper. Wu and Chao (2014) investigate a control problem for a stochastic production/inventory system with two production modes. They model the problem as an optimal switching problem for Brownian motion under the average cost criteria. They establish that the optimal production policy is of the $(s, S)$ type. To solve the Bellman equation, Wu and Chao (2014) use a similar approach to ours by considering a parametric family of function pairs and identifying a pair whose area of intersection is precisely the switching cost. However, unlike our approach, which involves solutions to nonlinear differential equations, the pair of functions in their analysis are solutions to two linear differential equations that are explicitly solvable. In a nutshell, our study advances the methodology by providing an explicit construction of a solution to the Bellman equation that does not rely on explicit formulas. Along the way, we establish useful properties of the solution to the Bellman equation. These properties, in turn, facilitate the identification of important structures of our control policy.

**Controlling queues in heavy traffic.** From a methodological standpoint, this paper is related to the stream of research that studies queueing controls in heavy traffic, which can be seen in various works such as Chevalier and Wein (1993), Plambeck (2004), Ata et al. (2005), Ghosh and Weerasinghe (2007, 2010), Ghamami and Ward (2013), Huang et al. (2015), Ata et al. (2019). These works often utilize heavy traffic approximations that result in either drift-rate or singular control problems for diffusion models, which are more amenable to analysis.
Demand and/or capacity management in make-to-stock systems. Last but not least, this paper is related to the literature on demand and/or capacity management in make-to-stock systems. We remark that we allow \( X(u) \) to take negative values, which extends our model to that of a make-to-stock system. Earlier research has examined adjustable capacity in a discrete-time setting. In those studies, the flexibility associated with capacity was indirectly captured through a piecewise linear ordering cost. For instance, the per-unit ordering cost as a function of quantity ordered may be \( A \) up to a cutoff quantity and \( B > A \) for order quantities greater than the cutoff. With this modeling approach, \( A \) may be interpreted as regular capacity, and orders for quantities greater than \( A \) tap into the reserves of flexible capacity, at a cost. Wang et al. (1996) and Henig et al. (1997) are two examples of flexible capacity models. Bradley and Glynn (2002) consider a manufacturing system that produces a single product using a make-to-stock approach. In their model, inventory is managed through a base-stock policy, and capacity and inventory are treated as joint decision variables. Specifically, they first treat the optimal operating cost as a function of capacity and then compute the optimal capacity and inventory policies to minimize the long-term average operating cost. Although capacity and inventory decisions are optimized jointly in Bradley and Glynn (2002), they occur on different time scales: capacity decisions are regarded as strategic decisions, whereas inventory decisions are operational. By contrast, in our model, capacity adjustments fall under operational decision-making. Some make-to-stock systems can prioritize orders based on profitability and other factors, which is yet another type of capacity management. For example, in Janakiraman et al. (2018), fixed capacity is allocated among a finite set of products, making the allocation of scarce capacity among a suite of products a key ingredient of their problem. Demand control, particularly in the form of pricing, has also been investigated in the context of a make-to-stock environment. For example, Federgruen and Heching (1999) examine the joint determination of pricing and inventory replenishment strategies in both finite and infinite horizon models. Their objective is to maximize total expected discounted profit or its time average value. Yao (2017) analyzes an infinite horizon, continuous-review, stochastic inventory system with price-dependent cumulative customer demand modeled as a Brownian motion and excess demand backlogged, and shows that inventory control follows the \((s^*, S^*)\) policy and characterizes the optimal state-dependent pricing strategy analytically. Cao and Yao (2018) investigate a stochastic inventory system where pricing and inventory controls are modeled as rate and impulse controls, respectively. Using stochastic control techniques, they demonstrate that inventory control is of the control-band type and that the optimal drift rate is first increasing and then decreasing in the relevant domain.

3 Single-Product Model

Notations. Let us first introduce the notations used in this paper. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a filtered probability space. \(1_{\cdot} \) denotes an indicator function. \( \Delta X(t) := X(t) - X(t^-) \) denotes the jump of some càdlàg process \( X \) at time \( t \). \([x]^+ := \max(0, x)\) and \([x]^− := -\min(0, x)\) represent the positive and negative
parts of \( x \), respectively. The positive jumps of \( X \) accumulated by time \( t \) are given by \( \sum_{u \leq t} [X(u)]^+ := \sup_{\mathcal{P}} \sum_{i=1}^{n_{\mathcal{P}}} [X(t_i) - X(t^-_i)]^+ \) where \( \sup \) is over all partitions \( \mathcal{P} := \{P := \{t_i\}_{i=0}^{np} : 0 = t_0 < t_1 < \ldots < t_{np} = t \} \). In addition, for two continuous functions \( f \) and \( g \), we say \( x \in \mathbb{R} \) is an “intersect point” if \( f(x) = g(x) \), and \( x \) is a “cross point” if \( x \) is an “intersect point” and \( f - g \) changes signs at \( x \).

We first introduce our model in a single-product setting. Consider an MTO production system producing a single customizable product. Customer orders arrive according to a non-homogeneous Poisson process with instantaneous rate \( \lambda(t) \); that is, the number of orders that have arrived up to time \( t \) is \( A(t) = N \int_0^t \lambda(u)du \), where \( N \) is a unit-rate Poisson process independent of everything else. The time taken to produce a product follows a general distribution having mean \( 1/\mu \) and squared coefficient of variation \( c_p^2 \). The production process is equivalently characterized by the renewal process \( S(t) \), which is the number of orders processed up to time \( t \) if the production machine (called the “primary server”) was continuously working in the time interval \([0, t]\). Let \( T(t) \) denote the cumulative amount of time the primary server is busy. Then the number of order completions by the primary server up to \( t \) can be expressed as \( S(T(t)) \). Because order delivery time in a typical MTO system is usually much shorter than waiting time plus manufacturing time, we assume delivery time is negligible in this paper. In other words, an order is considered “complete” when its production is completed.

The manager can determine \( \lambda(t) \) by deciding the price at time \( t \), which is denoted as \( p(t) \). The price-sensitivity of demand is captured by a non-negative demand function \( \Xi \) that maps each price to an instantaneous demand rate; i.e., \( \lambda(t) = \Xi(p(t)) \) for \( t \geq 0 \). As a matter of convention, we assume the function \( \Xi \) has an inverse function, denoted as \( \Xi^{-1} \), that maps each achievable demand rate to a price lying within the domain of the function \( \Xi \). Thus, we can define the profit rate \( \pi \) as a function of the demand rate \( \pi(\lambda) := \lambda \cdot (\Xi^{-1}(\lambda) - q) \), where \( q \) denotes the manufacturer’s cost to produce one item. We assume \( \pi \) admits a unique maximum point, denoted as \( \bar{\lambda} \), and we refer to \( \bar{\lambda} \) as the nominal demand. In addition, since \( \Xi^{-1} \) can be used to infer the price from the demand rate, we will interchangeably refer to \( \lambda \) and \( p \) as the pricing decisions.

Aside from the pricing decisions, the system manager can adjust short-term capacity by activating and deactivating the surge capacity, modelled as a secondary server. In the rest of this paper, we use secondary server and surge capacity interchangeably. Order processing times at the secondary server follow some distribution with mean \( 1/\gamma \), and these times are equivalently characterized by the renewal process \( \Gamma(t) \), which denotes the number of orders processed by the surge capacity up to \( t \) if the secondary server produces continuously from 0 through \( t \).

The short-term capacity adjustment results in two operating modes: “on” and “off”: The system is said to be in its “on” mode if the secondary server is operational and in its “off” mode if the secondary server is
deactivated. The transition from one mode to the other is instantaneous and these transitions form a sequence of decisions made by the system manager, which are modeled by an adapted, finite variation, càdlàg process $Y$ taking values in $\{0, 1\}$. Specifically, $Y(t) = 0$ means that the system is in the off mode, whereas $Y(t) = 1$ means that the system is in the on mode. Switching from the off to the on mode triggers a setup cost of $C$, whereas switching from the on to the off mode is free. For greater generality, we assume using surge capacity incurs additional manufacturing costs of $c$ (possibly 0) per unit of time. Let $X(t)$ denote the number of orders in the system at time $t$. It follows from the conservation of flow that
\[
X(t) = X(0) + A(t) - S(T(t)) - \Gamma \left( \int_0^t Y(u)du - L_2(t) \right),
\]
where $L_2(t)$ represents the cumulative amount of time the secondary server is on but idle. In addition, we define $L_1(t) := t - T(t)$ to be the cumulative amount of time the primary server is idle up to time $t$. It is worth mentioning that the cumulative busy times of the primary and the secondary server (i.e., $T(t)$ and $\int_0^t Y(u)du - L_2(t)$ respectively) are non-decreasing stochastic processes which do not increase during the time when $X(t) = 0$; that is, for any $t \geq 0$:
\[
\int_0^t 1_{\{X(u) = 0\}}dT(u) = 0, \quad \text{and} \quad \int_0^t 1_{\{X(u) = 0\}}d \left( \int_0^u Y(s)ds - L_2(u) \right) = 0.
\]
The equation (2) implies that $X(t) \geq 0$ for all $t \geq 0$.

Furthermore, the system incurs waiting costs at the rate of $h(x)$ whenever the number of pending orders equals $x$, where $h(x)$ grows to infinity as $x \to \infty$ with $h(0) = 0$. We also impose the following assumption purely for technical reasons.

**Assumption 1.** For any $a > 0$, there exists some $\tilde{a} > 0$ such that $h(x) > ah'(x) - \tilde{a}$ holds for any $x > 0$.

We make the above assumption because we need the cost rate function $h$ to grow faster than its derivative at some point in our mathematical proof. To understand the implications of this assumption, we note that it is satisfied by all polynomial functions of the form $h(x) = a_1 |x|^{a_2}$ for $a_1 > 0$ and $a_2 \geq 1$. Models in the classical inventory control literature often assume linear or increasing convex holding cost functions to derive optimal policies; for instance, the optimality of base stock policies in a multiple period inventory model with backlogging requires this assumption (see Theorem 3, Chapter 10 by Karlin and Scarf (1958)).

To proceed, the objective is to seek $(\lambda, T, Y)$ to maximize:
\[
\lim_{t \to +\infty} \frac{1}{t} \mathbb{E} \left[ \int_0^t \pi(\lambda(u))du - \int_0^t h(X(u))du - c \int_0^t Y(u)du - C \sum_{u \leq t} [\Delta Y(u)]^+ \right],
\]
where based on our notations $\sum_{u \leq t} [\Delta Y(u)]^+$ represent the total number of times the secondary server is activated through time $t$. In (3), the first term represent the profit, the second term describes the cost of holding orders in queues, the third term is the surge capacity cost, and the last term captures expenses associated with switching operations, all in the sense of long-run average.
Problem (3) is complex and does not lend itself to tractable solutions via exact analysis. As a result, it makes sense to consider approximation techniques. As a starting point, notice that if we temporarily ignore all randomness in the system and any potential capacity constraints, the system manager will simply use the “fluid-optimal” price \( \bar{p} = \Xi^{-1}(\bar{\lambda}) \), as this price maximizes revenue while not incurring any cost. This motivates the consideration of an operating regime under which the base capacity “matches” nominal demand, resulting in moderate congestion caused by stochastic variability. Consequently, congestion effects are considered to be of second order. To refine the fluid-optimal pricing strategy, a second-order correction is needed to approximate the stochastic control problem described by (3). The formulation and solution of the DCP that approximates the problem will be the main focus of the next section.

**Remark 1.** The fixed setup cost is a key feature in our model; this feature differentiates our model from many other models of make-to-stock production with a secondary capacity source. We recall that in the classical multiple-period inventory control literature, there is a clear dividing line between models that incorporate a fixed order cost and those that do not; the optimal policies in the former class of models are generally of the \((s, S)\) type, whereas the optimal policies in the latter class are generally of the order-up-to type. Analogously, we find that our fixed cost model yields some insights that are absent from similar models that omit a fixed setup cost for the surge capacity, as we will discuss in the sequel.

## 4 DCP of the Single-Product Model

In §4.1, we formulate a DCP by identifying the diffusion process that approximates the number of orders in the system. In §4.2, we apply the Bellman equation approach to solve the DCP and interpret the solution as an admissible policy. §4.3 shows how our results can be carried over to an MTS system.

### 4.1 The DCP formulation

To better understand our approximation framework, consider our original decision problem as a member of a series of problems indexed by \( \bar{\lambda} \), with revenue and cost rates scaled so that

\[
\Xi^{-1}(\cdot) := \sqrt{\bar{\lambda}} \hat{\Xi}^{-1}(\cdot/\bar{\lambda}), \quad q := \sqrt{\bar{\lambda}} \hat{q}, \quad h(\cdot) := \sqrt{\bar{\lambda}} \hat{h}(\cdot/\sqrt{\bar{\lambda}}), \quad c := \sqrt{\bar{\lambda}} \hat{c} \quad \text{and} \quad C := \sqrt{\bar{\lambda}} \hat{C},
\]

where \( \hat{\Xi}^{-1}, \hat{q}, \hat{h}, \hat{c}, \) and \( \hat{C} \) are corresponding baseline quantities/functions that do not scale with \( \bar{\lambda} \). The main purpose of applying the scaling conditions mentioned above is to establish an appropriate objective function, where each term in (3) is of equal order, as \( \bar{\lambda} \to \infty \). If some terms are of smaller order, meaning that they are not given equal importance, they will be dominated and thus become insignificant as \( \bar{\lambda} \) grows large. For further understanding, a discussion on the order of each term in the objective as well as the baseline quantities/functions is provided in EC.3. Because we are mostly concerned with the derivation of the approximating DCP rather than formally establishing a proper notion of asymptotic optimality, in what follows, we continue to use \( h, c, C, \Xi^{-1}, \) and \( q \) (hence \( \pi \)) with the understanding that they need to satisfy
appropriate scaling conditions so that they are compatible with the scaling imposed on $\mu$ and $\gamma$ that will be introduced in a moment.

The MTO system has high customer demand, large base production capacity, and is critically loaded in the following sense:

$$\mu = \bar{\lambda} + \beta \sqrt{\bar{\lambda}}$$  \hspace{1cm} (5)

for some constant $\beta \in \mathbb{R}$, so that the capacity approximately balances supply and demand (under the fluid-optimal price). To see that the scaling condition (5) can arise naturally in practice, consider a manufacturer who has a primary goal of maximizing revenue and a secondary goal of making efficient use of the production capacity. Then it would make sense for the manufacturer to set a base capacity level that approximately matches $\bar{\lambda}$. Indeed, as noted by Bradley and Glynn (2002), the high cost of capacity can force a firm to set capacity at a level that leads to high utilization at the production facility, forcing the facility into a “heavy-traffic” regime, where the system can be approximated by a Brownian motion based model.

The key feature of (5) is that $\mu = \bar{\lambda} + o(\bar{\lambda})$; the particular choice $o(\bar{\lambda}) = \beta \bar{\lambda}$ results in traffic intensity equal to

$$\frac{\bar{\lambda}}{\mu} = 1 - \frac{\beta}{\beta + \sqrt{\bar{\lambda}}}$$

which is a standard choice in modeling the heavy-traffic operating regime (see p. 1136 in Çelik and Maglaras (2008)). Hence, the value of $\bar{\lambda}$, in the spirit of (5), indicates the first-order size of the system. A key insight is that when using the fluid-optimal price and capacity level (5), the system will naturally operate in heavy traffic with a mild level of congestion overall, and the resulting congestion-related costs will be moderate. This means that we will need to fine-tune the corresponding pricing decisions and develop an appropriate pricing scheme. Specifically, we consider pricing schemes of the following form:

$$\lambda(t) = \bar{\lambda} - \vartheta(t) \quad t \geq 0,$$  \hspace{1cm} (6)

where $\vartheta$ is a correction term seen to be an order of magnitude smaller than $\bar{\lambda}$. More formally, we let $\vartheta := \sqrt{\bar{\lambda}} \Theta$, where $\Theta$ is a control process independent of the scaling parameter $\bar{\lambda}$. We also assume that the order processing rate of the secondary server is $\gamma := \kappa \sqrt{\bar{\lambda}}$, where $\kappa$ is a constant that does not scale with $\bar{\lambda}$. In plain words, our solution framework requires the processing speed of the secondary server to be in the second order. This is reasonable because the base capacity already matches the nominal demand in the first order.

To find the diffusion process that approximates $X$, we apply the strong approximation to $A$ and $S$ in a manner similar to that in Çelik and Maglaras (2008) to get

$$A(t) = \bar{\lambda} t - \int_0^t \vartheta(u) du + \sqrt{\bar{\lambda}} \hat{A}(t) + \varepsilon_a(t) \quad \text{and} \quad S(T(t)) = \mu t - \mu \hat{L}_1(t) + c_p \sqrt{\bar{\lambda}} \hat{S}(t) + \varepsilon_p(t),$$  \hspace{1cm} (7)
where \( \hat{A} \) and \( \hat{S} \) are two independent, standard Brownian motions capturing stochastic fluctuations in order arrivals and completions; in Equation (7), \( \varepsilon_a(t) \) and \( \varepsilon_p(t) \) are error terms from the strong approximation, and they are an order of magnitude smaller than \( \sqrt{\lambda} \) (for every fixed \( t \)). Next, by applying the functional strong law of large numbers type approximation for renewal processes, we get
\[
\Gamma \left( \int_0^t Y(u)du - L_2(t) \right) = \kappa \sqrt{\lambda} \int_0^t Y(u)du - \kappa \sqrt{\lambda} L_2(t) + \varepsilon_r(t),
\]
where \( \varepsilon_r(t) \) is an approximation error term that has an order of magnitude smaller than \( \sqrt{\lambda} \). Plugging (5)–(8) into (1) and ignoring all the error terms, we arrive at the desired diffusion approximation for \( X \) (which we recall is the process that tracks the number of orders over time and is defined in (1)). In particular, the approximating diffusion process, which we denote by \( Z \), is given as the solution to the following stochastic integral equation:
\[
Z(t) = Z(0) - \beta \sqrt{\lambda} t - \int_0^t \vartheta(u)du - \kappa \sqrt{\lambda} \int_0^t Y(u)du + \sigma B(t) + L(t),
\]
where \( \sigma := \sqrt{\lambda(1 + c_p^2)} \), \( Y \) tracks the operating mode of the system, \( B \) is a standard Brownian motion, and the last term in (9), which serves as the diffusion approximation for \( \mu L_1(t) + \kappa \sqrt{\lambda} L_2(t) \), is a “regulator” ensuring that \( Z \) is always non-negative, i.e., \( Z(t) \geq 0 \) for all \( t \geq 0 \). It is known from the Skorokhod map that for a fixed control strategy \( Y \), there exists a regulator \( L \) which is a non-decreasing process satisfying
\[
\int_0^t 1_{\{Z(u) > 0\}} dL(u) = 0, \quad \text{for any } t \geq 0,
\]
yields the minimal idleness in order to keep \( Z \) non-negative (see, e.g., Chapter 2.2 on pp. 20-21 (Harrison 2013) for the reference).

Define \( \delta(\vartheta) := \pi(\lambda) - \pi(\lambda - \vartheta) \), which can be regarded as the profit loss due to the deviation from the nominal demand rate. Then, using \( Z \) in (9) to approximate \( X \), we arrive at the diffusion approximation for the objective function in (3) which is given by
\[
\limsup_{t \to \infty} \frac{1}{t} \mathbb{E} \left[ \int_0^t \delta(\vartheta(u))du + \int_0^t h(Z(u))du + c \int_0^t Y(u)du + C \sum_{u \leq t} [\Delta Y(u)]^+ \right].
\]
(11)
So the DCP further simplifies to the one that seeks a pair \((\vartheta, Y)\) to minimize (11) subject to (9) and (10).

### 4.2 The DCP Solution

In what follows, we apply Bellman’s principle of optimality to deduce the optimal policy for the DCP. To that end, let \( v \) denote the (relative) value function associated with the DCP and define the convex conjugate function \( g(x) := \sup_{\vartheta} \{ x\vartheta - \delta(\vartheta) \} \). Note that \( g(x) \geq 0 \) for any \( x \in \mathbb{R} \), because \( \vartheta = 0 \) is a feasible solution to \( \sup_{\vartheta} \{ x\vartheta - \delta(\vartheta) \} \) and \( \delta(0) = 0 \). We will assume that \( g \) is locally Lipschitz continuous and exhibits super-linear growth (i.e., for any \( l \in \mathbb{R} \), we have \( g(x) \geq lx \) for all \( x \) large enough). With reference to the general
control theory, we expect that the value function \( v \), in conjunction with some constant \( \eta^* \), solve the following Bellman equation which takes the form of the quasi-variational inequality

\[
\min \left\{ \frac{\sigma^2}{2} v_z(y, z) - \left( \beta \sqrt{\lambda} + \kappa \sqrt{\lambda y} \right) v_z(y, z) - g(v_z(y, z)) + h(z) + cy - \eta^* \right\} = 0,
\]

subject to the boundary condition \( v_z(y, 0) = 0 \) and the requirement that \( v_z(y, z) \) exhibits polynomial growth as \( z \to \infty \). In average cost dynamic programming, \( \eta^* \) is interpreted as a guess for the optimal average cost. Intuitively speaking, the first term in the quasi-variational inequality (12) states the optimality condition if the status of the secondary server does not change at the state \((y, z)\), whereas the second and the third terms establish the optimality conditions for the system’s transition from off to on and from on to off, respectively.

Moreover, if a solution \( v \) to Equation (12) exists, then we can extract an optimal \( \theta^* \), given as

\[
\theta^*(y, z) := \arg \max_\theta \left\{ v_z(y, z) \theta - \delta(\theta) \right\}.
\]

We next draw from our intuition to postulate the structure of the Bellman equation solution. Note that if costs associated with on-demand production capacity are not exorbitant, it would clearly be advantageous for the system manager to use surge capacity on a temporary basis. The intuition leads us to conjecture that the optimal control strategy is a sequential switching policy comprising the following actions. If the system is currently operating in its off mode, then it is optimal to remain in that mode if \( Z \) is below a threshold, say \( z_1^* \), and switch to its on mode once \( Z \) rises above \( z_1^* \). On the other hand, if the system is currently operating in its on mode, then it is optimal to remain in that mode if \( Z \) is above a certain level, say \( z_0^* \), and switch to its off mode as soon as \( Z \) drops below \( z_0^* \). Clearly, this strategy is well-defined if \( z_0^* < z_1^* \). Moreover, if this strategy, henceforth denoted as \( Y^* \), is indeed optimal, we should be able to find \( v_0(z) := v(0, z) \), \( v_1(z) := v(1, z) \) and \( \eta^* > 0 \), such that

\[
\frac{\sigma^2}{2} v'_0(z) - \beta \sqrt{\lambda} v'_0(z) - g(v'_0(z)) + h(z) = \eta^* \quad \text{for} \quad z \in [0, z_1^*),
\]

\[
\frac{\sigma^2}{2} v'_1(z) - (\beta + \kappa) \sqrt{\lambda} v'_1(z) - g(v'_1(z)) + h(z) + c = \eta^* \quad \text{for} \quad z > z_0^*,
\]

\[
v_0(z) = v_1(z) \quad \text{for} \quad z \in [0, z_0^*], \quad \text{and} \quad v_0(z) = v_1(z) + C \quad \text{for} \quad z \geq z_1^*,
\]

with the boundary condition

\[
v'_0(0) = 0 \quad \text{and the requirement that} \ v'_1(z) \text{ grows polynomially as} \ z \to \infty,
\]

plus a set of optimality conditions derived from the “principle of smooth fit”:

\[
v'_0(z_0^*) = v'_1(z_0^*) \quad \text{and} \quad v'_0(z_1^*) = v'_1(z_1^*).
\]
Because (14) and (15) do not involve the zero-order term (the unknown function itself), they can be reduced to a pair of first-order differential equations for \( f_0(z) := v_0'(z) \) and \( f_1(z) := v_1'(z) \). The observation leads to the consideration of the class of functions \( \{ f_0(\cdot, \eta); \eta \in \mathbb{R} \} \) where \( f_0(\cdot, \eta) \) solves

\[
\frac{\sigma^2}{2} f_0'(z) - \beta \sqrt{\lambda} f_0(z) - g(f_0(z)) + h(z) - \eta = 0,
\]

subject to the boundary condition \( f_0(0) = 0 \), and the function class \( \{ f_1(\cdot, \eta); \eta \in \mathbb{R} \} \) where \( f_1(\cdot, \eta) \) is the solution to the following differential equation

\[
\frac{\sigma^2}{2} f_1'(z) - (\beta + \kappa) \sqrt{\lambda} f_1(z) - g(f_1(z)) + h(z) + c - \eta = 0,
\]

subject to the requirement that \( f_1(z) \) exhibits polynomial growth as \( z \to \infty \). For those equations specified by (19), we also let \( z_{\eta, \infty} := \inf \{ z : \lim_{x \to z} f_0(x) = \pm \infty \} \). Also, it is straightforward to verify that the requirement (16) leads to

\[
\int_{z_0^*}^{z_1^*} [f_0(z, \eta^*) - f_1(z, \eta^*)] \, dz = C;
\]

also, by appealing to (18), we obtain

\[
f_0(z_0^*, \eta^*) = f_1(z_0^*, \eta^*) \quad \text{and} \quad f_0(z_1^*, \eta^*) = f_1(z_1^*, \eta^*).\]

To summarize, the mission of constructing a solution to the Bellman equation (12) boils down to seeking variables \( \eta^*, z_0^*, \text{ and } z_1^* \) such that (21) and (22) hold. Our next result is concerned with the properties of \( f_0 \) and \( f_1 \).

**Proposition 1.** (i) For all \( z < z_{\eta, \infty} \wedge z_{\eta', \infty} \), we have \( f_0(z, \eta) < f_0(z, \eta') \) if \( \eta < \eta' \); in particular, there exists a unique \( \eta_0 > 0 \) such that \( f_0(z, \eta_0) \) grows to infinity at a polynomial rate with \( z_{\eta_0, \infty} = \infty \), and \( \lim_{z \to z_{\eta, \infty}} f_0(z, \eta) = -\infty \) for any \( \eta < \eta_0 \). (ii) For all \( z \), we have \( f_1(z, \eta) > f_1(z, \eta') \) if \( \eta < \eta' \); particularly, there exists a unique \( \eta_1 > 0 \) such that \( f_1(0, \eta_1) = 0 \), and \( f_1(0, \eta) > 0 \) for any \( \eta < \eta_1 \).

Our starting point to spell out the condition under which the triple \( (\eta^*, z_0^*, z_1^*) \) exists is establishing the existence of two benchmark policies corresponding to \( f_0(\cdot, \eta_0) \) and \( f_1(\cdot, \eta_1) \) in Proposition 1. The first policy, denoted as \( S_0 \), requires \( Y \equiv 0 \) regardless of the system state; this corresponds to the case where the production system relies entirely on in-house capacity, thereby forgoing the option of using on-demand capacity. The second policy, denoted as \( S_1 \), sets \( Y \equiv 1 \); this describes the scenario where the system always retains the on-demand production capacity, thereby incurring costs associated with flexible capacity at all times. With respect to ergodic control of diffusion processes, we can interpret \( \eta_0 \) as the long-run average cost incurred by the production system under the static policy \( S_0 \) and \( \eta_1 \) as the long-run average cost under the static policy \( S_1 \). A key intuition is that the constant \( \eta^* \), if it exists, ought to satisfy \( \eta^* \leq \bar{\eta} := \min(\eta_0, \eta_1) \), because both \( S_0 \) and \( S_1 \) are admissible policies. The following result is concerned with the number of crossings that \( f_0(\cdot, \eta) \) and \( f_1(\cdot, \eta) \) can have for each \( \eta \) that is less than \( \bar{\eta} \).
LEMMA 1. There exists $\bar{\eta} \leq \bar{\eta}$ such that $f_0(\cdot, \eta)$ and $f_1(\cdot, \eta)$ do not intersect on $\mathbb{R}^+$ for $\eta < \bar{\eta}$, intersect on $\mathbb{R}^+$ but do not cross for $\eta = \bar{\eta}$, and cross at exactly two points on $\mathbb{R}^+$ for $\eta \in (\bar{\eta}, \bar{\eta})$.

Based on Lemma 1, if $\eta = \bar{\eta}$, then there is no such $\eta < \bar{\eta}$ that $f_0(\cdot, \eta)$ and $f_1(\cdot, \eta)$ will ever cross. We interpret this case to mean that no matter how small the fixed cost $C$ is, sequential switching is never optimal. If, however, $\eta < \bar{\eta}$, then by Lemma 1, we know that $f_0(\cdot, \eta)$ and $f_1(\cdot, \eta)$ will cross at exactly two points for all $\eta \in (\bar{\eta}, \bar{\eta})$. We interpret this case to mean that, for a sufficiently small fixed cost, it is optimal to switch between the different modes.

To avoid triviality, in the rest of the paper we will focus on the latter scenario by further assuming that $\eta < \bar{\eta}$. It is noteworthy that in this case the two functions, $f_0(\cdot, \bar{\eta})$ and $f_1(\cdot, \bar{\eta})$ will not only coincide at $z = 0$ or $z = \infty$ but also cross at some finite point; we denote these two points of intersection by $\bar{z}_0$ and $\bar{z}_1$ with $\bar{z}_0 < \bar{z}_1$. The next result provides the condition under which an optimal sequential switching policy exists.

THEOREM 1. Suppose $C < \bar{C} := \int_{\bar{z}_0}^{\bar{z}_1} [f_0(z, \bar{\eta}) - f_1(z, \bar{\eta})]dz$. Then the following statements are true. (i) There exists a triplet $(\eta^*, z_0, z_1)$ satisfying (19)-(22). (ii) The “sequential switching policy” characterized by $(z_0, z_1)$ and the pricing scheme $\hat{\nu}^*$ given by (13) are jointly optimal for the DCP (11). On the other hand, when $C \geq \bar{C}$, then (iii) $\eta^* = \bar{\eta} = \min(\eta_0, \eta_1)$, and if $\eta_0 < \eta_1$ then the static policy $S_0$ is optimal and otherwise the static policy $S_1$ is optimal.

Theorem 1 states the conditions under which the optimal policy for managing the surge capacity is of sequential switching or static type. To be more precise, the sequential switching policies arise to be optimal for the DCP if the switching cost is not too high, i.e., $C < \bar{C}$. When $C \geq \bar{C}$, an optimal policy for managing the surge capacity is to set the static policy $S_0$ by always sticking to the off mode if $\eta_0 < \eta_1$, and to set $S_1$ by always sticking to the on mode if $\eta_0 \geq \eta_1$.

4.3 Allowing For On-Hand Inventory

When $X(t) < 0$ (where $X(t)$ is defined in (1)), the model will allow for a positive inventory of completed orders (represented by $[X(t)]^-$), which characterizes a make-to-stock model. In particular, we assume that $X(t) \geq -b$, where $-b$ is a negative threshold and $b > 0$ represents the maximal inventory the manufacturer would like to hold. Accordingly, we modify the meaning of $h$ to let it be a non-negative continuous function with $h(0) = 0$ that is strictly increasing in positive $x$ and strictly decreasing in negative $x$. When $x < 0$, $h(x)$ represents the cost rate of holding inventory $-x$.

We can approximate $X(t)$ by an approximating diffusion process $Z(t)$ which is given as the solution to the following stochastic integral equation:

$$Z(t) = Z(0) - \beta \sqrt{\lambda} t - \int_0^t \hat{\nu}(u)du - \kappa \sqrt{\lambda} \int_0^t Y(u)du + \sigma B(t) + L_0(t),$$

(23)
where the last term in (23) is a “regulator” ensuring that $Z$ is always greater than or equal to the negative threshold, i.e.,

$$Z(t) \geq -b \quad \text{for all} \quad t \geq 0. \quad (24)$$

In particular, $L_b$ is a one-sided regulator that is non-decreasing and continuous with $L_b(0) = 0$. It is known that for a fixed control strategy $Y$, a regulator $L_b$ satisfying

$$\int_0^t 1_{\{Z(u) > -b\}} dL(u) = 0, \quad \text{for any} \quad t \geq 0,$$

yields the minimal idleness in order to keep $Z \geq -b$. So under the make-to-stock extension, the DCP further simplifies to the one that seeks a pair $(\vartheta, Y)$ to minimize (11) subject to (23), (24) and (25).

As in § 4.2, the DCP solution satisfies the quasi-variational inequality (12) subject to the boundary condition $v_z(y, -b) = 0$ and the requirement that $v_z(y, z)$ exhibits polynomial growth as $z \to \infty$. We can show that the main result (Theorem 1) still holds under the make-to-stock extension.

**Theorem 2.** The conclusion in Theorem 1 holds under the extension to the make-to-stock model.

We will provide the proof of Theorem 2 in the e-companion. The main technical challenge is that under the make-to-stock extension, $Z(t)$ can take negative values, although the waiting cost function $h(x)$ is strictly increasing in positive $x$, we no longer have the monotonicity in $x \geq -b$. Indeed, $h(x)$ is strictly decreasing in negative $x$. In addition, the control over the moments of the $Z$ process gets more complicated when $Z(t)$ can also take negative values.

## 5 Multi-Product Model and the DCP

We have assumed that the manufacturing firm only offers a single product. In reality, there may be more than one product to be offered. Here, we extend our analysis to treat a decision problem with multiple products.

For that purpose, consider an MTO firm that offers multiple products, indexed by $k = 1, \ldots, K$, to a market of price-sensitive customers. A request for product $k$ will now be referred to as a class $k$ job. The time it takes for the primary server to process a class $k$ job follows a general distribution with a mean of $1/\mu_k$ and a squared coefficient of variation of $\nu_k^2$. Class $k$ jobs enter the system via a non-homogeneous Poisson process with an instantaneous rate of $\lambda_k(t)$. We call $\lambda(t) := (\lambda_k(t))$ the instantaneous demand rate vector at time $t$, and we use $\lambda := \{\lambda(t); t \geq 0\}$ to represent the $K$-dimensional demand rate process. The demand rate at time $t$ is determined by the vector $p(t) := (p_k(t))$, where $p_k(t)$ represents the price for product $k$ at time $t$. A demand function $\Xi$ captures the price-sensitivity of demand by mapping the price vector to an instantaneous demand rate vector. Assuming $\Xi$ has an inverse function (that maps each achievable demand rate vector to a price vector lying within the domain of the function $\Xi$) and slightly abusing the notation, we can define the profit rate $\pi$ as a function of the demand rate vector; that is $\pi(\lambda) := \langle \lambda, \Xi^{-1}(\lambda) - q \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product operator and $q := (q_k)$ is a vector collecting the product-specific production costs.
Apart from pricing, the system manager can adjust production capacity by turning on and off the secondary server. The time it takes the secondary server to process a class \( k \) job follows a general distribution with a mean of \( 1/\bar{Y}_k \). We assume the two vectors, \( \mu := (\mu_k) \) and \( \gamma := (\gamma_k) \), are proportional. So there exists some positive constant \( \bar{Y} \), such that

\[
\gamma_k = \mu_k \bar{Y}
\]

for all \( k \). This assumption simply means that if a product requires more production time from the primary server, it will also consume more machine time on the secondary server. This makes sense in practice because the length of time it takes to manufacture a product is typically determined by its inherited characteristics, such as manufacturing complexity. Let \( Y := (Y_k) \) where \( Y_k \) is an adapted, finite variation, càdlàg process taking values in \( \{0, 1\} \) with 1 indicating the secondary server is currently serving class \( k \) and 0 otherwise. Hence, at each time \( t \), there are at most one \( k \) such that \( Y_k(t) = 1 \).

Next, let \( Y(t) \) denote the process of turning on and off the secondary server with \( Y(t) = 1 \) denoting that the secondary server is in the on mode and \( Y(t) = 0 \) the off mode. Then from the definition we have \( Y(t) - \sum_{k=1}^{K} Y_k(t) \geq 0 \) for all \( t \geq 0 \). Also, similar to the single-product setting, it is assumed that switching from the off to the on mode triggers a setup cost of \( C \), whereas switching from the on to the off mode is free. Moreover, using the secondary server incurs additional manufacturing costs of \( c \) per unit of time.

In the presence of multiple products, the system manager needs to make scheduling decisions in addition to pricing and capacity adjustment decisions. Such decisions can be described by two \( K \)-dimensional allocation processes, \( T := (T_k) \) and \( Y \), with \( T_k(t) \) and \( \int_{0}^{t} Y_k(u)\,du \) denoting the total amount of time devoted to product \( k \) by the primary and secondary servers, respectively. It is worth noting that the primary server’s cumulative idle time up to \( t \) can be calculated as \( t - \sum_{k=1}^{K} T_k(t) \) and the secondary server’s cumulative idle time up to \( t \) can be expressed as

\[
\int_{0}^{t} \left( Y(u) - \sum_{k=1}^{K} Y_k(u) \right) \, du.
\]

To formally describe the manager’s objective, we use \( X_k(t) \) to denote the number of class \( k \) jobs in the system at time \( t \), and denote by \( h_k(\cdot) \) the waiting cost rate function associated with product \( k \). Then the managerial objective is to seek \( (\lambda, T, Y) \) to maximize

\[
\liminf_{t \to \infty} \frac{1}{t} \mathbb{E} \left[ \int_{0}^{t} \pi(\lambda(u))\,du - \sum_{k=1}^{K} h_k(X_k(u))\,du - c \int_{0}^{t} Y(u)\,du - C \sum_{u \leq t} [\Delta Y(u)]^+ \right].
\]

As before, we consider a deterministic relaxation of the problem that seeks \( \lambda \) to maximize \( \pi(\lambda) \), and we assume the deterministic optimization problem admits a unique solution \( \bar{\lambda} := (\bar{\lambda}_k) \). With a slight abuse of notation, let \( \bar{\lambda} := \sum_k \bar{\lambda}_k \).

Similar to the single-product setting, we can deduce a DCP that approximates the foregoing decision problem. To this end, we again consider the problem as a member of a sequence of problems indexed by \( \bar{\lambda} \), with revenue and cost rates scaled as in (4) so that

\[
\Xi^{-1}(\cdot) := \sqrt{\bar{\lambda}} \hat{\Xi}^{-1}(\cdot/\bar{\lambda}), \quad q_k := \sqrt{\bar{\lambda}} \hat{q}_k, \quad h_k(\cdot) := \sqrt{\bar{\lambda}} \hat{h}_k(\cdot/\bar{\lambda}), \quad c := \sqrt{\bar{\lambda}} \hat{c} \quad \text{and} \quad C := \sqrt{\bar{\lambda}} \hat{C},
\]
where $\hat{\mathcal{E}}^{-1}, \hat{q}_k, \hat{h}_k, \hat{c}$ and $\hat{C}$ are corresponding baseline quantities/functions that do not scale with $\bar{\lambda}$.

Base capacity is chosen so that

$$
\sum_{k=1}^{K} \rho_k = 1 - \psi \quad \text{for} \quad \rho_k := \bar{\lambda}_k / \mu_k, \quad (28)
$$

where $\psi$ is a quantity perceived to be of order $1/\sqrt{\bar{\lambda}}$. More formally, we can let $\psi := \bar{\psi} / \sqrt{\bar{\lambda}}$, where $\bar{\psi}$ is a constant that does not scale with $\bar{\lambda}$. Additionally, let $\gamma$ be scaled with $\bar{\lambda}$ so that

$$
\gamma := \hat{\gamma} / \sqrt{\bar{\lambda}}, \quad (29)
$$

for some constant $\hat{\gamma}$ that does not scale with $\bar{\lambda}$. For reasons similar to those in the single-product setting, this critical-loading condition motivates the consideration of instantaneous demand rate vectors of the following form:

$$
\lambda(t) = \hat{\lambda} - \bar{\theta}(t) \quad \text{for} \quad t \geq 0,
$$

where $\bar{\theta}$ is perceived to be in the second order, which can be formalized by letting $\bar{\theta} := \sqrt{\bar{\lambda}} \theta$.

With a slight abuse of notation, we let $\delta(\bar{\theta}) := \pi(\bar{\lambda}) - \pi(\hat{\lambda} - \bar{\theta})$. Using the techniques pioneered by Harrison (1988) and similar to Çelik and Maglaras (2008), we can derive a DCP that approximates the problem posed by (27). Formally, the DCP seeks $(\bar{\theta}, \mathbf{L}, \mathbf{Y})$ to

$$
\begin{align*}
\min \quad & \limsup_{t \to \infty} \frac{1}{t} \mathbb{E} \left[ \int_0^t \delta(\bar{\theta}(u)) du + \sum_{k=1}^{K} \int_0^t h_k(Z_k(u)) du + c \int_0^t Y(u) du + C \sum_{u \leq t} |\Delta Y(u)| \right] \\
\text{subject to} \quad & Z_k(t) = Z_k(0) + (\bar{\lambda}_k - \mu_k) t - \int_0^t \bar{\theta}_k(u) du + \mu_k L_k(t) - \gamma_k \int_0^t Y_k(u) du + \sigma_k B_k(t), \quad (31) \\
& Z_k(t) \geq 0 \quad \text{for} \quad k = 1, \ldots, K, \quad \text{and} \\
& I(t) := \sum_{k=1}^{K} L_k(t) + \gamma \int_0^t \left( Y(u) - \sum_{k=1}^{K} Y_k(u) \right) du \quad \text{is non-decreasing with} \quad I(0) = 0, \quad (32)
\end{align*}
$$

where $Z_k$ is the diffusion approximation for $X_k$, $\sigma_k$ is defined as $\sigma_k := \sqrt{\bar{\lambda}_k (1 + \psi_k^2)}$, $(B_k)$ are $K$ independent standard Brownian motions, and the process $I$ is the diffusion approximation for the primary server’s cumulative idleness process.

The DCP is not easy to solve because it is a multidimensional stochastic control problem. To achieve dimensional reduction, we next develop a one-dimensional formulation that is equivalent to the DCP and we refer to it as the workload problem. To that end, let $m_k := 1/\mu_k$ and define the one-dimensional workload process as

$$
W(t) := \sum_{k=1}^{K} m_k Z_k(t) \quad \text{for} \quad t \geq 0.
$$

Next, multiplying (31) by $m_k$ and adding over $k = 1, \ldots, K$, plus using (32)–(33), we obtain

$$
W(t) = W(0) - \psi t - \sum_{k=1}^{K} \int_0^t m_k \bar{\theta}_k(u) du - \bar{\gamma} \int_0^t Y(u) du + \sigma B(t) + I(t) \quad \text{for} \quad t \geq 0, \quad (34)
$$
where \( \sigma = \sqrt{\sum_{k=1}^{K} \bar{h}_k (1 + v_i^2) m_i^2} \).

To formally state the workload problem, let \( \bar{h}(\cdot) \) be defined as

\[
\bar{h}(w) := \min \left\{ \sum_{k=1}^{K} h_k(z_k) : \sum_{k=1}^{K} m_k z_k = w \right\}.
\]

so that \( \bar{h} \) is the work-based waiting cost function. With these preparations, we can state the workload problem as follows: we seek the triplet \((\hat{\Phi}, I, Y)\) that solves the constrained optimization problem

\[
\begin{align*}
\min_{\hat{\Phi}} & \quad \limsup_{t \to \infty} \frac{1}{t} \mathbb{E} \left[ \int_0^t \delta(\hat{\Phi}(u)) du + \int_0^t h(W(u)) du + c \int_0^t Y(u) du + C \sum_{u \leq t} [\Delta Y(u)]^+ \right] \\
\text{subject to} & \quad (34), \quad W(t) \geq 0, \quad \text{and} \quad I(t) \text{ is non-decreasing with } I(0) = 0.
\end{align*}
\]

Overloading the notation a bit, let \( g(x) := \sup \Theta (m x - \delta(\hat{\Phi})) \), where \( m := (m_k) \). Similar to our analysis for the single-product setting, the variational inequality characterizing the solution to the workload problem can be deduced as

\[
\min \left\{ \frac{\sigma^2}{2} v_{ww}(y, w) - (\psi + \bar{y}) v_w(y, w) - g(v_w(y, w)) + \bar{h}(w) + cy - \eta, \right\}
\]

\[
v(1, w) + C - v(0, w), \quad v(0, w) - v(1, w) \right\} = 0.
\]

Provided a solution \( v(y, w) \) to (38) exists, a candidate for the optimal pricing policy is given as

\[
\hat{\Phi}^*(y, w) = \arg \sup_{\hat{\Phi}} \left\{ \Theta^T m v_w(y, w) - \delta(\hat{\Phi}) \right\}.
\]

To construct a solution to (38), as before, we can define \( f_0(w, \eta) \) as the solution to

\[
\frac{\sigma^2}{2} f_0(w) - \psi f_0(w) - g(f_0(w)) + \bar{h}(w) - \eta = 0,
\]

with the left boundary condition \( f_0(0, \eta) = 0 \); similarly, define \( f_1(w, \eta) \) as the solution to

\[
\frac{\sigma^2}{2} f_1(w) - (\psi + \bar{y}) f_1(w) - g(f_1(w)) + \bar{h}(w) + c - \eta = 0,
\]

with the right boundary condition that \( f_1(w, \eta) \) exhibits polynomial growth as \( w \to \infty \). With a slight abuse of notation, let \( \eta_0 \) and \( \eta_1 \) be two constants defined similarly as in Proposition 1 such that \( f_0(\cdot, \eta_0) \) exhibits polynomial growth and \( f_1(0, \eta_1) = 0 \), and let \( \bar{\eta} = \min(\eta_0, \eta_1) \). To avoid uninteresting cases, we stipulate through the remainder of this section that \( f_0(\cdot, \bar{\eta}) \) and \( f_1(\cdot, \bar{\eta}) \) intersect at two points \( \bar{w}_0 \) and \( \bar{w}_1 \). Paralleling Theorem 1, we have the following theorem:

**Theorem 3.** Suppose \( C < \bar{C} := \int_{\eta_0}^{\eta_1} [f_0(w, \bar{\eta}) - f_1(w, \bar{\eta})] dz \). Then the following statements are true. (i) There exists a triplet \((\eta^*, \bar{w}_0^*, \bar{w}_1^*)\) satisfying (38); and (ii) the “sequential switching policy” characterized by \((\bar{w}_0^*, \bar{w}_1^*)\) and the pricing scheme \( \hat{\Phi}^* \) given by (39) are jointly optimal for the DCP (36)–(37). On the other hand, when \( C \geq \bar{C} \), then (iii) \( \eta^* = \bar{\eta} = \min(\eta_0, \eta_1) \), and if \( \eta_0 < \eta_1 \) then the static policy \( S_0 \) is optimal and otherwise the static policy \( S_1 \) is optimal.

The multi-product setting is more prevalent in practice. In the next section, we will use this result to construct a joint pricing, capacity adjustment, and scheduling policy.
6 Policy Recommendation

In this section, we interpret the solution to a DCP in the context of the corresponding original system. Since the multi-product problem (36)–(37) includes the single-product problem (11) as a special case, we mainly focus on this more generalized multi-product system.

Specifically, we propose a dynamic control policy based on the results in Theorem 3. Note that we can find the solution to the DCP by numerically solving the equations (40) and (41) through the finite difference method. The solution to the DCP can then be interpreted in a way as some “near-optimal” admissible policy to the original control problem for the MTO system, i.e., problem (27). Intuitively, because the MTO system is facing relatively high demands, the optimal control policy for the original control problem should not be “far” from those of the DCP. This framework for interpreting the DCP solution in the context of the original queueing system was pioneered by Harrison (1988) and is widely used in the studies of control problems in queueing systems (Çelik and Maglaras 2008, Kim and Ward 2013, Ata et al. 2019).

There are three control levers related to (27): dynamic pricing, capacity management rules, and scheduling. In the sequel, we first interpret the three control levers separately in the original MTO system in §6.1. Next, we show an interesting structure of the pricing scheme under this system in §6.2.

6.1 Interpretation of the DCP Solutions

**Dynamic Pricing.** Given the solution \( v_w(y,w) \) to the variational inequality and the current system state \((y,w)\), we derive the optimal demand rate adjustment term \( \vartheta^*(y,w) \), which is the solution to the equations

\[
mv_w(y,w) - \nabla \delta(\vartheta) = 0.
\]

We can then derive a pricing policy:

\[
p^*(y,w) = \Xi^{-1} \left( \tilde{\lambda} - \vartheta^*(y,w) \right).
\]

**Capacity Management Rule.** As we have discussed in Theorem 1, depending on whether \( C < \bar{C} \) or \( C \geq \bar{C} \), a dynamic or static capacity management policy is optimal for the DCP. Hence, we propose the following capacity management rule \( Y^*(\cdot) \):

- If \( C \geq \bar{C} \), we apply a static capacity management rule. Let \( Y^*(y,w) \equiv \arg \min \{ \eta_i : i = 0, 1 \} \) (let \( Y^*(y,w) \equiv 1 \) if \( \eta_0 = \eta_1 \)), where \( \eta_0 \) and \( \eta_1 \) are defined in Proposition 1, such that the secondary server is always off/on.
  - If \( C < \bar{C} \), we apply a sequential switching rule defined by \( w_0^* \) and \( w_1^* \) as in Theorem 3: \( Y^*(y,w) = 1 \) if \( y = 0 \) and \( w > w_1^* \); \( Y^*(y,w) = 0 \) if \( y = 1 \) and \( w < w_0^* \).
Scheduling Rule. The workload problem defined in (35) indicates the optimal allocation of the total workload into different classes for the DCP. Therefore, given the workload $w$, the solution

$$z^*(w) = \arg \min \left\{ \sum_{k=1}^{K} h_k(z_k) : \sum_{k=1}^{K} m_k z_k = w \right\}$$ (42)

(with ties broken consistently and arbitrarily) suggests we try to keep the queue length of each class as close to $z^*(w)$ as possible. We propose the following scheduling rule:

- At each time whenever the primary server is available or the secondary server is both on and available, and there are jobs waiting in some queues, serve the head-of-the-line job from class

$$i \in \arg \max_k \{ X_k(t) - z_k^*(w) \}$$

with ties broken consistently and arbitrarily.

As in Ata and Tongarlak (2013) and a few related studies, different waiting cost functions result in different scheduling rules. For instance, when all waiting cost functions are linear, the proposed scheduling rule reduces to the celebrated $c\mu$ rule. As another example, when all waiting cost functions are quadratic, the schedule becomes a queue-ratio rule that strives to maintain various queue lengths at some target ratio.

6.2 An Atypical Pricing Scheme

In this subsection, we provide concrete examples to illustrate our proposed policy. To make things simple, we consider a single-product MTO system. Potential customers come to the system following a Poisson process with a rate of $\Lambda$. Each customer is assumed to have a willingness-to-pay $v$ that follows a logistic distribution $G(x) = 1/(1 + e^{-(x-m)/s})$ as described in Çelik and Maglaras (2008), where $m$ and $s$ are two model parameters representing location and scale, respectively. The demand rate function is then $\Xi(p) = \Lambda G^*(p)$, where $G^*(p) := 1 - G(p)$ denotes the tail probability. The inverse demand rate function is derived as $\Xi^{-1}(\lambda) = m + s \log((\Lambda - \lambda)/\lambda)$. We choose the following model primitives: $\Lambda = 78.327$, $m = 500$, $s = 30$, $\beta = -1$, $\kappa = 2$, $c = 200$, $C = 600$, and $q = 400$. The nominal demand rate is $\bar{\lambda} = \arg \max \lambda \cdot (\Xi^{-1}(\lambda) - q) = 50$. The production times of both the primary and secondary servers are assumed to be exponentially distributed, such that the production rates can be derived accordingly as $\mu = 42.929$ and $\gamma = 14.142$, respectively.

Figures 1(a) and 1(b) illustrate the pricing schemes when the waiting cost functions are $h(x) = x$ and $h(x) = 0.1x^2$, respectively. To compare with the optimal control policy, we plot both the pricing policies derived from our DCP and a Markov Decision Process (MDP) formulation (please refer to EC.4 for details in the MDP formulation). The solid/dash lines represent the pricing policies under MDP/DCP, with different colors differentiating whether the surge capacity is on or off. In particular, the legend “price (base capacity)” indicates the price curve when the surge capacity is off whereas “price (surge capacity)” indicates the price curve when the surge capacity is on. Furthermore, the up-pointing and down-pointing triangles denote the
thresholds of switching (i.e., $z_0^*$ and $z_1^*$) derived from the solutions of the MDP and DCP, respectively. One can see that the pricing policy derived from the DCP is very close to their optimal counterparts and shares the same structure. The performance gap between the two policies will be studied in Section 7. It is worth mentioning that the optimal policies for managing surge capacity are of threshold or static types (Theorems 1-3) is proved only for DCPs. However, our extensive numerical experiments on solving the optimal solutions to the MDPs show the optimal policies share the same structures, as illustrated in the two figures.

The most intriguing finding in panels 1(a) and 1(b) is that when the surge capacity is off, the pricing strategy does not result in a monotonic increase in congestion level. Rather, it suggests that as the congestion level increases but is not too high, the manager should raise prices to limit customer demand. When the system is too congested, however, it is in the manager’s best interest to lower the price. Our findings reveal a strong interaction between dynamic pricing and the setup cost of the surge capacity: In the absence of the setup cost, the pricing policies are monotonically increasing functions in system congestion levels (see, e.g., Çelik and Maglaras (2008), Ata and Barjesteh (2023)). To our knowledge, this is the first paper revealing non-monotonic structures of pricing policies with unobservable queues. Rich insights from this atypical pricing scheme are explained in more detail in the rest of this section.

Structural Insights. To explain the atypical structure of the pricing policies, consider the manager’s ideal scenario: rely on the primary server, set the price at $\bar{p}$, and maintain a low level of congestion. However, this perfect situation is not achievable because, with a base capacity that matches nominal demand, stochasticity can often overload the system, leading to significant waiting costs. To combat congestion, managers have two levers of control: they can raise the product price to lower the effective arrival rate or activate the secondary server to increase capacity. However, the existence of the setup cost discourages switching the surge capacity

![Figure 1](image-url)
Enhancing Make-to-Order Manufacturing Agility

on and off too frequently, and the higher $C$ is, the less frequently the surge capacity should be activated. This intuition implies that if the system manager is facing a nontrivial setup cost, the secondary server should only be activated when the system is heavily congested, which further implies that pricing should be the main device to adjust demand when the congestion level is low or moderate. In addition, with the base capacity, as the system becomes more congested, a higher price should be set to prevent the congestion level from further increasing. However, the higher price also implies a higher profit loss. As a result, reducing system congestion by raising prices becomes more cost-ineffective as the system congestion level increases. On the other hand, because capacity expansion is costly, the secondary server should be activated only when the congestion level is sufficiently high. When the congestion level falls short but is near the activating threshold, it is a reasonable strategy for the system manager to lower the price to both reduce profit loss and accumulate enough backlog to make the best use of the activation of surge capacity. Although lowering the price makes the system even more congested in the short term, the long-term holding cost would not be too high, provided that the system could accumulate enough backlog and activate the surge capacity in a brief period of time.

When we compare Figures 1(a) and 1(b), we can see that the quadratic waiting costs tend to result in more aggressive pricing strategies: The pricing curves under both base and surge capacities in Figure 1(b) are steeper than those in Figure 1(a). Additionally, the thresholds for the sequential switching policy in Figure 1(b) are lower than those shown in Figure 1(a). These findings reinforce our intuition about reducing waiting costs through dynamic pricing and expanding capacity. When the waiting cost function is quadratic, the system waiting cost rate grows rapidly with the congestion level, requiring a higher price and lower thresholds for activating the surge capacity to effectively prevent system congestion levels from growing too high. In addition, the downward slope of pricing policy with base capacity in Figure 1(b) is much steeper than that shown in Figure 1(a), implying the system should quickly accumulate backlog and thus spend less time in the highly congested but not-so-congested-to-activate-surge-capacity states.

7 Simulation Studies

In this section, we run simulation experiments to examine system performance under the proposed policies. §7.1 presents the simulation results based on a single-product system we have studied in §6.2. A two-product system is studied in §7.2.

7.1 Single Product

The basic parameter settings of the single-product systems we study in this subsection are the same as those described in §6.2. In §7.1.1, we numerically study the performance gap of policies derived from the DCP in this paper and a Taylor series-based DCP (T-DCP) in the literature, compared to optimal policies derived from an exact MDP formulation. §7.1.2 elaborates on the effect of the setup cost $C$ on the system. We next study the effect of the load parameter $\beta$ in §7.1.3. Finally, in §7.1.4 we show how different values of per-unit surge capacity cost $c$ affect the system performance.
7.1.1 Performance Gap of DCP and T-DCP Compared to MDP

One noteworthy fact in this paper is that we do not use any function approximation techniques to the profit loss function $\delta$, in contrast to a stream of papers using second-order Taylor expansions as the approximation to the revenue/profit functions (Çelik and Maglaras 2008, Kim and Randhawa 2018, Ata and Barjesteh 2023).

More specifically, if $|\lambda - \bar{\lambda}|$ is not too large, then

$$\pi(\lambda) \approx \tilde{\pi}(\lambda) := \pi(\bar{\lambda}) + \pi'(\bar{\lambda})(\lambda - \bar{\lambda}) + \frac{\pi''(\bar{\lambda})}{2}(\lambda - \bar{\lambda})^2,$$

where (with a little abuse of notation) $\bar{\lambda}$ is the optimizer of a static revenue/profit maximization problem, and it is reasonable to substitute $\pi$ by $\tilde{\pi}$ in the objective function. The upside of this approximation is that the resulting Bellman equation is a Riccati equation, which may be solved explicitly. However, it may cause a non-negligible error if the actual arrival rates deviate from $\bar{\lambda}$ too much. We henceforth call the DCP derived from this second-order Taylor expansion as the Taylor series-based DCP (T-DCP).

To study the performance gap of the policies generated by our DCP and T-DCP, we make comparison of the system performance under the two policies with the optimal policy computed through an MDP formulation. The details of how to formulate our problem as an MDP are provided in EC.4. Tables 1 and 2 present the numerical results, corresponding to the holding cost function $h(x) = x$ and $h(x) = 0.1x^2$, respectively. The optimal long-run average cost can be derived by solving the MDP, which is denoted as $\bar{\eta}^*$ in the tables. $\hat{\eta}_{DCP}$ and $\hat{\eta}_{T-DCP}$ represent the estimated long-run average cost through conducting Monte Carlo simulation programs using the corresponding policies, and the values in the parentheses indicate 95% confidence levels. “Gap” is defined as $(\hat{\eta}_k - \bar{\eta}^*)/\hat{\eta}_k$ for $k = \text{DCP}, \text{T-DCP}$ to measure the differences between the two policies and the optimal ones.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Gaps of DCP and T-DCP Compared to MDP With $h(x) = x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>MDP $\bar{\eta}^*$</td>
</tr>
<tr>
<td>0</td>
<td>100.294</td>
</tr>
<tr>
<td>200</td>
<td>120.616</td>
</tr>
<tr>
<td>400</td>
<td>130.028</td>
</tr>
<tr>
<td>600</td>
<td>136.503</td>
</tr>
<tr>
<td>800</td>
<td>141.083</td>
</tr>
<tr>
<td>1,000</td>
<td>144.217</td>
</tr>
</tbody>
</table>

“Static Off” 148.319 148.238(0.675) -0.05 149.34(0.624) 0.68

“Static On” 207.348 207.827(0.779) 0.23 207.409(0.808) 0.03

Tables 1 and 2 show that the performance of the policies derived from our DCP is nearly indistinguishable from the optimal ones because the gaps are all below about 0.5%, under various values of $C$ and static policies. In contrast, the performance gap between the T-DCP and MDP policies enlarges as $C$ increases, and the worst gap is approximately 3% when $C = 1,000$ with the quadratic waiting cost function $h(x) = 0.1x^2$. 
Table 2  Gaps of DCP and T-DCP Compared to MDP With $h(x) = 0.1x^2$

<table>
<thead>
<tr>
<th></th>
<th>MDP</th>
<th>DCP $\bar{\eta}$</th>
<th>T-DCP $\bar{\eta}$</th>
<th>Gap (%)</th>
<th>T-DCP Gap (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>108.001</td>
<td>108.034(0.749)</td>
<td>108.091(0.805)</td>
<td>0.03</td>
<td>0.08</td>
</tr>
<tr>
<td>200</td>
<td>143.942</td>
<td>144.072(0.718)</td>
<td>144.774(0.632)</td>
<td>0.09</td>
<td>0.57</td>
</tr>
<tr>
<td>400</td>
<td>160.031</td>
<td>160.161(0.718)</td>
<td>161.617(0.668)</td>
<td>0.08</td>
<td>0.98</td>
</tr>
<tr>
<td>600</td>
<td>169.403</td>
<td>169.839(0.629)</td>
<td>172.341(0.632)</td>
<td>0.26</td>
<td>1.70</td>
</tr>
<tr>
<td>800</td>
<td>174.1</td>
<td>174.388(0.632)</td>
<td>178.714(0.705)</td>
<td>0.17</td>
<td>2.58</td>
</tr>
<tr>
<td>1,000</td>
<td>175.896</td>
<td>176.832(0.693)</td>
<td>181.394(0.636)</td>
<td>0.53</td>
<td>3.03</td>
</tr>
<tr>
<td>“Static Off”</td>
<td>176.495</td>
<td>176.509(0.679)</td>
<td>176.967(0.683)</td>
<td>0.01</td>
<td>0.26</td>
</tr>
<tr>
<td>“Static On”</td>
<td>209.386</td>
<td>208.669(0.811)</td>
<td>209.289(0.831)</td>
<td>-0.34</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Our numerical results show that solving the DCP without approximating profit functions can keep the performance gaps extremely small under various scenarios. It is also worth mentioning that gaps should always be non-negative, but due to variances in simulation results, we may sometimes obtain negative estimated gaps, and the negative gaps typically imply the performance of the derived policies is very close to that of optimal ones.

7.1.2 The Effect of Setup Cost $C$

We next study the effect of the setup cost parameter $C$, which is an important feature in our model to study the surge capacity. The simulation results of the DCP are reported in Tables 3 and 4 for waiting cost functions $h(x) = x$ and $h(x) = 0.1x^2$, respectively. We present how the thresholds $\bar{z}_0^*, \bar{z}_1^*$ and average costs $\bar{\eta}_{\text{DCP}}$ are affected by various values of $C$. In addition, the columns “Profit Loss”, “Waiting Cost”, “Surge Capacity Cost”, and “Setup Cost” report the estimated values of individual cost rates of the first to the last terms in (30), with $Z_k$ replaced by $X_k$, the state variables in pre-limit systems.

In practice, the setup cost could be related to some physical costs. For example, $C$ can be used to model the fixed fee component of contractual arrangements with a contract manufacturer that take the form of a fixed fee plus a variable fee. The setup cost can also be regarded as a system design parameter that prevents the surge capacity from switching on and off too frequently. Indeed, our formulation is closely related to the following constrained problem:

$$\begin{align*}
\text{minimize} & \quad \limsup_{t \to \infty} \frac{1}{t} \mathbb{E} \left[ \int_0^\tau \delta(\Theta(u)) du + \sum_{k=1}^K \int_0^\tau h_k(Z_k(u)) du + c \int_0^\tau Y(u) du + Z_t \right] \\
\text{subject to the constraint} & \quad \limsup_{t \to \infty} \frac{\mathbb{E} \left[ \sum_{u \leq t} [\Delta Y(u)]^+ \right]}{t} \leq \nu
\end{align*}$$

(43)

for some constant $\nu$, and (31), (32), and (33). The constraint (44) requires the long-run average switching frequency of the surge capacity to be bounded from above by some “budget” $\nu$. This is the so-called constrained Markov decision processes (Altman 1999), and roughly speaking, our formulation (30)-(33)
can be regarded as the corresponding Lagrangian problem where \( C \) is the Lagrange multiplier. We do not rigorously justify the equivalence between the constrained and the Lagrangian problem, but encourage interested readers to see Ata et al. (2005), Chen et al. (2021), Chai et al. (2023) for the relevant studies developed in the same vein.

### Table 3 Numerical Results With \( h(x) = x \)

<table>
<thead>
<tr>
<th>( C )</th>
<th>((z_0^<em>, z_1^</em>))</th>
<th>( \hat{\eta}_{\text{DCP}} )</th>
<th>Profit</th>
<th>Loss</th>
<th>Waiting</th>
<th>Surge Capacity</th>
<th>Cost</th>
<th>Setup</th>
<th>Cost</th>
<th>Switching</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(19.42, 19.42)</td>
<td>100.399(0.705)</td>
<td>20.302</td>
<td>16.784</td>
<td>63.313</td>
<td>0</td>
<td>1.665</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>(6.607, 53.153)</td>
<td>120.787(0.743)</td>
<td>31.803</td>
<td>25.978</td>
<td>51.906</td>
<td>11.1</td>
<td>0.055</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>(5.105, 64.565)</td>
<td>130.227(0.643)</td>
<td>41.835</td>
<td>29.631</td>
<td>43.908</td>
<td>14.852</td>
<td>0.037</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>600</td>
<td>(4.204, 72.973)</td>
<td>136.855(0.681)</td>
<td>52.604</td>
<td>31.643</td>
<td>36.597</td>
<td>16.012</td>
<td>0.027</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>(3.604, 79.279)</td>
<td>141.654(0.695)</td>
<td>64.682</td>
<td>32.672</td>
<td>29.113</td>
<td>15.188</td>
<td>0.019</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>(3.303, 84.985)</td>
<td>144.559(0.685)</td>
<td>77.761</td>
<td>32.629</td>
<td>21.212</td>
<td>12.956</td>
<td>0.013</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

“Static off” - 148.238(0.675) 119.506 28.733 0 0 0

“Static on” - 207.827(0.779) 0.687 7.14 200 0 0

### Table 4 Numerical Results With \( h(x) = 0.1x^2 \)

<table>
<thead>
<tr>
<th>( C )</th>
<th>((z_0^<em>, z_1^</em>))</th>
<th>( \hat{\eta}_{\text{DCP}} )</th>
<th>Profit</th>
<th>Loss</th>
<th>Waiting</th>
<th>Surge Capacity</th>
<th>Cost</th>
<th>Setup</th>
<th>Cost</th>
<th>Switching</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(12.514, 12.514)</td>
<td>108.034(0.749)</td>
<td>20.29</td>
<td>16.48</td>
<td>71.264</td>
<td>0</td>
<td>1.935</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>(3.303, 31.231)</td>
<td>144.072(0.718)</td>
<td>38.72</td>
<td>30.812</td>
<td>54.696</td>
<td>19.845</td>
<td>0.099</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>(2.102, 36.937)</td>
<td>160.161(0.718)</td>
<td>59.314</td>
<td>35.102</td>
<td>41.208</td>
<td>24.537</td>
<td>0.061</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>600</td>
<td>(1.502, 41.141)</td>
<td>169.839(0.629)</td>
<td>85.412</td>
<td>37.672</td>
<td>26.581</td>
<td>20.174</td>
<td>0.034</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>(1.201, 45.345)</td>
<td>174.388(0.632)</td>
<td>111.33</td>
<td>36.952</td>
<td>13.514</td>
<td>12.592</td>
<td>0.016</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>(0.901, 49.249)</td>
<td>176.832(0.693)</td>
<td>130.749</td>
<td>35.114</td>
<td>5.447</td>
<td>5.521</td>
<td>0.006</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

“Static off” - 176.509(0.679) 143.945 32.564 0 0 0

“Static on” - 208.669(0.811) 0.429 8.24 200 0 0

We emphasize that if the system manager has some budget on the switching frequency of the surge capacity, it can be achieved by carefully fine-tuning the parameter \( C \): a higher value of \( C \) implies a higher penalty cost per switch and hence discourages switching on and off the surge capacity too frequently, while a lower \( C \) would stimulate switches. We corroborate the above intuition through simulation programs summarized in the “Switching Frequency” columns in both Tables 3 and 4. One noteworthy result is that the switching frequency becomes prohibitively high when the setup cost is zero. For example, our results in Table 4 show that the optimal policy requires the system to switch on and off the surge capacity nearly twice a day when \( C = 0 \) and \( h(x) = 0.1x^2 \). This may not be a viable policy for some manufacturing systems, especially those that require extra efforts during each capacity expansion, such as staffing, training, etc. To make a comparison, under the linear waiting cost, if \( C = 600 \), then the switching frequency is only 0.027 and 1/0.027 ≈ 37, so that on average the system only activates the secondary server once per 37 days.
Tables 3 and 4 also provide insights on how the individual costs are affected by $C$. We notice that both the profit loss and waiting cost increase in $C$, but the surge capacity cost decreases in $C$. This result is intuitive because a higher penalty leads to a lower frequency of switching, which makes the system less flexible to reduce waiting costs through dynamic capacity adjustment and hence more reliable on pricing policies. Less flexibility also encourages the manager to be less reliant on the surge capacity, so its usage also decreases. Finally, our numerical results show the setup cost may not be a monotone function with respect to $C$, as it is a product of $C$ and a term (switching frequency) decreasing in $C$.

### 7.1.3 The Effect of Load Parameter $\beta$

We next investigate how the load parameter $\beta$ affects the system performance by fixing $C = 600$ and keeping other parameters unchanged. Recall that $\mu = \bar{\lambda} + \beta \sqrt{\bar{\lambda}}$, so that by varying $\beta$ we have different levels of nominal traffic intensity $\rho := \bar{\lambda}/\mu$ in the absence of pricing adjustment and surge capacity. Table 5 summarizes the numerical results as we vary $\beta$ from 0 to $-2$, resulting in $\rho$ changes from 1 to 1.39. We focus on the critically loaded and overloaded regimes because only in these regimes the surge capacity plays an important role in reducing waiting costs.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\rho$</th>
<th>$(\xi_0, \xi_1)$</th>
<th>$\bar{\eta}$</th>
<th>$\bar{\eta}_{DCP}$</th>
<th>Gap</th>
<th>Profit Loss</th>
<th>Waiting Cost</th>
<th>Surge Capacity Cost</th>
<th>Setup Cost</th>
<th>Switching Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.00</td>
<td>“Static Off”</td>
<td>27.719</td>
<td>27.876(0.174)</td>
<td>0.56</td>
<td>9.213</td>
<td>18.663</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-0.5</td>
<td>1.08</td>
<td>(12.913, 88.589)</td>
<td>69.05</td>
<td>68.98(0.367)</td>
<td>-0.10</td>
<td>40.9</td>
<td>25.276</td>
<td>1.808</td>
<td>0.996</td>
<td>0.002</td>
</tr>
<tr>
<td>-1</td>
<td>1.16</td>
<td>(4.204, 72.973)</td>
<td>136.503</td>
<td>136.855(0.681)</td>
<td>0.26</td>
<td>52.604</td>
<td>31.643</td>
<td>36.597</td>
<td>16.012</td>
<td>0.027</td>
</tr>
<tr>
<td>-1.5</td>
<td>1.27</td>
<td>(1.201, 70.871)</td>
<td>196.867</td>
<td>196.601(0.744)</td>
<td>-0.14</td>
<td>29.169</td>
<td>36.903</td>
<td>103.739</td>
<td>26.79</td>
<td>0.045</td>
</tr>
<tr>
<td>-2</td>
<td>1.39</td>
<td>“Static On”</td>
<td>228.24</td>
<td>227.871(0.759)</td>
<td>-0.16</td>
<td>9.153</td>
<td>18.717</td>
<td>200</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We can see from Table 5 that the nominal traffic intensity has a great impact on the average cost, where as $\rho$ increases from 1 to 1.39, $\bar{\eta}_{DCP}$ has more than eightfold increase from about 28 to about 228. In addition, as the nominal traffic intensity increases, the capacity adjustment strategy changes from “Static Off” to dynamic adjustment and finally to “Static On”. We also see that the long-run average profit loss, waiting cost, and setup cost are all non-monotonic functions in $\rho$, whereas the surge capacity cost increases rapidly as $\rho$ becomes large.

### 7.1.4 The Effect of Per-Unit Surge Capacity Cost $c$

Parameter $c$ measures how expensive it is to continuously use the surge capacity. Intuitively speaking, for any fixed setup cost $C$, we expect that the system should have a “Static On” policy if $c$ is too small and a “Static Off” policy if $c$ is too large. A dynamic capacity adjustment strategy should arise when $c$ is in some middle range. This intuition is verified through the numerical examination in Table 6, where we vary $c$ from 50 to 400 while fixing $C = 600$ and $\beta = -1$. 

Table 6 The Effect of Per-Unit Surge Capacity Cost $c$ With $h(x) = x$

<table>
<thead>
<tr>
<th>$c$</th>
<th>$(z_0, z_1)$</th>
<th>$\hat{\eta}$</th>
<th>$\hat{\eta}_{DCP}$</th>
<th>Gap (%)</th>
<th>Profit</th>
<th>Loss</th>
<th>Waiting Cost</th>
<th>Surge Capacity Cost</th>
<th>Setup Cost</th>
<th>Switching Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>“Static On”</td>
<td>57.348</td>
<td>57.494(0.388)</td>
<td>0.25</td>
<td>0.35</td>
<td>7.144</td>
<td>50</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>“Static On”</td>
<td>107.348</td>
<td>107.314(0.823)</td>
<td>-0.03</td>
<td>0.166</td>
<td>7.147</td>
<td>100</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>200</td>
<td>(4.204, 72.973)</td>
<td>136.503</td>
<td>137.091(0.756)</td>
<td>0.43</td>
<td>52.824</td>
<td>31.639</td>
<td>36.59</td>
<td>16.039</td>
<td>0.027</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>(10.511, 80.781)</td>
<td>146.784</td>
<td>147.022(0.665)</td>
<td>0.16</td>
<td>99.606</td>
<td>29.824</td>
<td>13.073</td>
<td>4.518</td>
<td>0.008</td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>“Static Off”</td>
<td>148.28</td>
<td>148.077(0.654)</td>
<td>-0.14</td>
<td>119.36</td>
<td>28.717</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

By comparing the numerical results in the first two rows when $c = 50, 100$ and the rest of the data in Table 6, we see an interesting implication indicated by negligible profit loss and unchanged waiting cost: Pricing policy keeps at a nearly static optimal level when $c$ is relatively small. In other words, the system behaves like an $M/M/2$ queue for a range of $c$ being small. If, however, $c$ increases beyond this range but is not too large, the manager should combine dynamic pricing and dynamic capacity adjustment to restrict waiting costs. Furthermore, when $c$ becomes too high, the manager relies more on the pricing policies as we see the profit loss increase, and eventually solely uses dynamic pricing while keeping the secondary server off. This result again shows the intricate interaction between dynamic pricing and the setup cost of surge capacity.

### 7.2 Two Products

In this subsection, we study a two-product MTO system to illustrate the performance of our model in multi-product settings. §7.2.1 presents the structures of pricing policies under a multinomial logit model in a multi-product setting. In §7.2.2, we study the effect of price sensitivity on system performance.

#### 7.2.1 Pricing Structure

We now consider an MTO manufacturer offering two products, indexed by $k = 1, 2$, with the price vector $p$. Following literature, we assume the probability of purchasing product $k$ follows a multinomial logit model, such that

$$P(\text{purchase } k|p) = \frac{e^{U_k}}{1 + \sum_{k=1}^{2} e^{U_k}},$$

where $U_k := b_k^0 - b_k^1 p_k$ for some model primitives $b_k^0, b_k^1$. Then we have the demand rate function $\Xi_k(p) = \Lambda P(\text{purchase } k|p)$ where we recall that $\Lambda$ denotes the arrival rate of potential customers. Through some algebraic manipulation, we can derive the inverse demand rate function

$$\Xi_k^{-1}(p) = -\frac{1}{b_k} \log\left(\frac{\lambda_k}{\Lambda - \sum_{k=1}^{2} \lambda_k} + \frac{b_k^0}{b_k^1}\right), \quad \text{for } k = 1, 2.$$

We set the basic parameters as follows: $\Lambda = 75$, $q_1 = q_2 = 400$, $c = 200$, $C = 600$, $\psi = -1$, $\gamma = 2$, $b_1^0 = b_2^0 = 15$, and $b_1^1 = b_2^1 = 0.03$. The nominal demand rate can be calculated as $\bar{\lambda} = (25, 25)$ through a gradient ascent method. Assuming the production rates of the two products are identical, then we can derive the service rates of the primary server $\mu = (43.8, 43.8)$ by (28), and $\gamma = (12.39, 12.39)$ by (26) and (29). The
waiting cost functions are assumed to be $h_1(x) = x$ and $h_2(x) = 1.2x$. It is worth mentioning that based on our assumption on the waiting cost functions, by (42) we have $z^*_1(W(t)) = \mu_1 W(t)$ and $z^*_2(W(t)) = 0$ such that the scheduling rule is to always give priority to class-2 products.

Based on our model parameter settings, we can see that the profit rate $\pi$ is a symmetric function with respect to demand rates $\lambda_1$ and $\lambda_2$, which implies the optimal demand rates and hence the optimal pricing policies of the DCP should be the same for products 1 and 2. Being aware of this, we only plot the pricing schemes of class-1 products in Figures 2 and 3, corresponding to the pricing policies with surge capacity off and on, respectively. In addition, the panels 2(a) and 3(a) present the curves of pricing policies, where the $x$- and $y$-axis represent the number of products of class 1 and 2 in the system, respectively. To better illustrate the pricing policies, panels 2(b) and 3(b) present two-dimensional color plots with brighter colors representing higher values.

![Pricing curve](image1.png)  ![Pricing colormap](image2.png)

**Figure 2** Pricing Schemes of Class 1 with Base Capacity

We can see from Figures 2 and 3 that the pricing policies share a similar structure as the ones illustrated in Figure 1: When the surge capacity is off, the proposed pricing policy first increases and then decreases with respect to the congestion level, but when the surge capacity is on, the proposed pricing policy is monotonically increasing with respect to the congestion level. Furthermore, it is easier to see from the panels 2(b) and 3(b) that the pricing schemes are the same along all lines parallel to a diagonal line, such that the total number of customers is the same along the lines. This is because the pricing policies under multi-product systems depend on system workload, which is proportional to the total number of customers in the system.
7.2.2 Effect of Price Sensitivity Between the Two Products

We now study how the price sensitivity between the two products affects the system performance. To this end, we keep all basic parameters fixed, but vary $b_2$ from 0.03 to 0.034. Notice that by changing the demand function, we need to recalculate $\bar{\lambda}$, $\mu$ and $\gamma$ accordingly. The numerical results are shown in Table 7.

<table>
<thead>
<tr>
<th>$b_2$</th>
<th>($\hat{w}_0^<em>, \hat{w}_1^</em>$)</th>
<th>$\hat{\eta}_{DCP}$</th>
<th>$\hat{\eta}_{TDCP}$</th>
<th>Profit Loss</th>
<th>Waiting Cost</th>
<th>Surge Capacity Cost</th>
<th>Setup Cost</th>
<th>Switching Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>(0.115, 1.672)</td>
<td>137.752(0.852)</td>
<td>138.123(0.827)</td>
<td>47.471</td>
<td>33.183</td>
<td>47.471</td>
<td>33.183</td>
<td>0.026</td>
</tr>
<tr>
<td>0.031</td>
<td>(0.13, 1.712)</td>
<td>131.712(0.756)</td>
<td>132.565(0.769)</td>
<td>52.576</td>
<td>31.927</td>
<td>52.576</td>
<td>31.927</td>
<td>0.022</td>
</tr>
<tr>
<td>0.032</td>
<td>(0.14, 1.747)</td>
<td>128.306(0.707)</td>
<td>128.904(0.759)</td>
<td>59.263</td>
<td>30.73</td>
<td>59.263</td>
<td>30.73</td>
<td>0.018</td>
</tr>
<tr>
<td>0.033</td>
<td>(0.155, 1.782)</td>
<td>123.905(0.709)</td>
<td>125.295(0.725)</td>
<td>61.735</td>
<td>29.918</td>
<td>61.735</td>
<td>29.918</td>
<td>0.016</td>
</tr>
<tr>
<td>0.034</td>
<td>(0.16, 1.812)</td>
<td>122.153(0.677)</td>
<td>122.128(0.724)</td>
<td>65.438</td>
<td>29.494</td>
<td>65.438</td>
<td>29.494</td>
<td>0.013</td>
</tr>
</tbody>
</table>

Table 7 The Effect of Different Price Sensitivity Across Two Products

By comparing the columns “$\hat{\eta}_{DCP}$” and “$\hat{\eta}_{TDCP}$” of Table 7, we first find that the policies derived from our DCP outperform the policies derived from the T-DCP in multi-product scenarios. Table 7 also shows that the long-run average cost $\hat{\eta}_{DCP}$ decreases as the customers become more price-sensitive to the product 2. This is because when customers are more price-sensitive, to achieve the same demand rates, the manager only needs to raise a lower price compared to the case of less price-sensitive customers, hence the higher profit rate loss. Moreover, as customers become more price-sensitive, the system manager should rely more on dynamic pricing and less on surge capacity, as suggested by the results in the columns “Profit Loss”, “Surge Capacity Cost”, and “Setup Cost”. The less dependency on the surge capacity also implies the switching frequency should decrease in price sensitivity, as expected in the simulation results shown in the last column of Table 7.
8 Concluding Remarks

Maintaining capacity flexibility can be a cost-effective option for manufacturers who produce make-to-order (MTO) products to meet the fluctuating demand of their customers. However, adjusting capacity levels while making pricing decisions is a challenging task. This paper introduces a formal stochastic control framework that jointly considers capacity adjustment with a setup cost and dynamic pricing, which, to the best of our knowledge, has not been done before. We present a nearly explicit joint capacity adjustment and pricing strategy that depends on the congestion level in the system. Capacity adjustments follow a switching strategy that determines when to use surge capacity and when to shut it down, based on a set of switching boundaries. The pricing rule, on the other hand, shows an unusual non-monotonic feature that highlights the complex interplay between the two control levers.

This model may be applied to a manufacturing environment where workforce size is the primary determinant of productivity, and capacity flexibility can be achieved by hiring temporary workers in addition to permanent employees. Permanent workers are paid on a payroll, while contingent workers are only paid for the time they work. The decision-making framework can also be extended to situations where permanent capacity is used for overtime production, determining when to initiate overtime and for how long.

References


Ata, Barış, Shiri Shneorson. 2006. Dynamic control of an M/M/1 service system with adjustable arrival and service rates. Management Science, 52 (11), 1778-1791.


E-Companion

EC.1  Proofs of Main Results

Proof of Proposition 1. To prove Proposition 1, we need following auxiliary lemmas:

Lemma EC.1. If $\eta_1 < \eta_2$, then $f_0(z, \eta_1) < f_0(z, \eta_2)$ for all $z \in (0, z_{\eta_1,\infty} \land z_{\eta_2,\infty})$.

Lemma EC.2. Let $L_0 := \{ \eta > 0 : \exists z \in (0, z_{\eta,\infty}), f_0(z, \eta) < 0 \}$. If $\eta \in L_0$, then $f_0(z, \eta)$ is quasi-concave and $\lim_{z \to z_{\eta,\infty}} f_0(z, \eta) = -\infty$. In addition, $L_0$ is nonempty.

Lemma EC.3. The set $U_0 := \{ \eta > 0 : \eta \notin L_0 \}$ is nonempty.

Lemma EC.4. Let $\eta_0 := \sup L_0$. Then $\eta_0 \in U_0$, and $f_0(z, \eta_0) \geq 0$ for all $z \geq 0$.

Lemma EC.5. For each $\eta \in L_0$, $f_0(z, \eta) < g^{-1}(2h(z) + a)$ for some positive constant $a$.

The proofs of the auxiliary lemmas are postponed to EC.2. Part (i) of Proposition 1 follows directly from Lemmas EC.1–EC.5.

Next, to prove part (ii) of Proposition 1, we first define a class of functions $\{ \xi_a(x, \eta) : \eta \in \mathbb{R} \}$ which are solutions to (20), but with the left boundary conditions $\xi_a(0, \eta) = \alpha$ and no restriction to the right boundary condition. Let $z_{\eta,\infty}^\alpha := \inf \{ z : \lim_{\eta, \xi} \xi_a(x, \eta) = \pm \infty \}$. We first need Proposition EC.1:

Proposition EC.1. For all $z < z_{\eta,\infty}^\alpha \land z_{\eta,\infty}^\alpha$, we have $\xi_a(z, \eta) < \xi_a(z, \eta')$ if $\eta < \eta'$; in particular, there exists a unique $\eta_a > 0$ such that $\xi_a(z, \eta_a)$ grows to infinity at a polynomial rate with $z_{\eta,\infty} = \infty$, and $\lim_{z \to z_{\eta,\infty}} \xi_a(z, \eta) = -\infty$ for any $\eta < \eta_a$.

Let $c_0 := c - (\beta + \kappa) \sqrt{\lambda} \alpha - g(\alpha)$. Analogously, the proof of Proposition EC.1 relies on the following lemmas:

Lemma EC.6. If $\eta_1 < \eta_2$, then $\xi_a(z, \eta_1) < \xi_a(z, \eta_2)$ for all $z \in (0, z_{\eta_1,\infty}^\alpha \land z_{\eta_2,\infty}^\alpha)$.

Lemma EC.7. Let $L_a := \{ \eta > c_0 : \exists z \in (0, z_{\eta,\infty}^\alpha), \xi_a(z, \eta) < 0 \}$. If $\eta \in L_a$, $\xi_a(z, \eta)$ is quasi-concave and $\lim_{z \to z_{\eta,\infty}} \xi_a(z, \eta) = -\infty$. In addition, $L_a$ is nonempty.

Lemma EC.8. The set $U_a := \{ \eta > c_0 : \eta \notin L_a \}$ is nonempty.

Lemma EC.9. Let $\eta_a := \sup L_a$. Then $\eta_a \in U_a$, and $\xi_a(z, \eta_a) \geq 0$ for all $z \geq 0$.

Lemma EC.10. For each $\eta \in L_a$, $\xi_a(z, \eta) < g^{-1}(2h(z) + a)$ for some positive constant $a$.

In the sequel, we denote $\eta_a$ as $\eta(\alpha)$ (with a little abuse of notation) to stress that $\eta(\alpha)$ is a function of $\alpha$. We next show that $\eta(\alpha)$ is a strictly decreasing and continuous function of $\alpha$.

Lemma EC.11. If $\alpha_1 > \alpha_2$, then $\eta(\alpha_1) < \eta(\alpha_2)$ and $\xi_a(z, \eta(\alpha_1)) < \xi_a(z, \eta(\alpha_2))$. In addition, $\eta(\cdot)$ is a continuous mapping and $\eta(\alpha) \to -\infty$ as $\alpha \to \infty$. 
From Lemma EC.11, the inverse function $\eta^{-1}(\cdot)$ is well defined. Let $f_1(z, \eta) = \zeta_{\eta^{-1}(\eta)}(z, \eta)$. We know from Proposition EC.1 that $f_1(z, \eta)$ corresponds to the desired class of functions in part (ii). Thus, the proof of the Proposition 1 is now complete.

Proof of Lemma 1. To begin with, we observe that the two functions $f_0(\cdot, \eta)$ and $f_1(\cdot, \eta)$ will touch at either $z = 0$ or $z = \infty$. On the other hand, $f_1(z, 0) > 0$ for any $z \in [0, \infty]$ and $f_0(z, 0) \leq 0$ for any $z \in [0, \infty]$, which implies that $f_1(z, 0) - f_0(z, 0) > 0$ for any $z \in [0, \infty]$. Moreover, by Proposition 1, for any $z$, $f_0(z, \eta)$ is increasing in $\eta$ and $f_1(z, \eta)$ is decreasing in $\eta$. Therefore, there exists some $\eta \in (0, \eta]$ such that for any $\eta < \eta$, $f_0(\cdot, \eta)$ and $f_1(\cdot, \eta)$ do not intersect, and for $\eta = \eta$, $f_0(\cdot, \eta)$ and $f_1(\cdot, \eta)$ intersect but do not cross each other, and for any $\eta \in (\eta, \eta)$, $f_0(\cdot, \eta)$ and $f_1(\cdot, \eta)$ cross at least twice. In the remainder of the proof, we will write $f_0$ and $f_1$ in place of $f_0(\cdot, \eta)$ and $f_1(\cdot, \eta)$, respectively, and let $\tilde{f} := f_0 - f_1$. We intend to argue that $f_0$ and $f_1$ can cross at most twice for any $\eta \in (\eta, \eta)$.

Suppose, by way of contradiction, that the two functions cross more than twice. Then $\tilde{f}$ must have crossed the horizontal line $y = 0$ at least four times, two times from below and two times from above. On the other hand, from (19) and (20) it follows that

$$\tilde{f}'(z) = \frac{2}{\sigma^2} [c - \kappa \lambda^{1/2} f_1(z)] \quad \text{for all } z \text{ such that } \tilde{f}(z) = 0. \quad \text{(EC.1)}$$

However, by Proposition 1, we know that $f_1$ is strictly increasing on $[0, \infty)$ for all $\eta \leq \eta_1$. This implies that $\tilde{f}'(z)$ can cross the horizontal line $y = 0$ at most twice thanks to (EC.1), leading to a contradiction. Therefore, $f_0$ and $f_1$ can cross at most twice for any $\eta \in (\eta, \eta)$. The proof is thus complete.

Proof of Theorem 1. We first prove (i). For any $\eta \in (\eta, \eta')$, by Lemma 1, the functions $f_0$ and $f_1$ cross at two points. Let $z_0(\eta) < z_1(\eta)$ denote two points where the functions $f_0$ and $f_1$ cross. Then $\eta \to \eta$, $z_1(\eta) - z_0(\eta) \to 0$, and $f_{z_0(\eta)}[f_0(z, \eta) - f_1(z, \eta)]dz \to 0$. When $\eta = \eta$, $z_0(\eta) = \tilde{z}_0$ and $z_1(\eta) = \tilde{z}_1$. By Proposition 1, for any $z$, $f_1(z, \eta)$ is decreasing in $\eta$ and $f_0(z, \eta)$ is increasing in $\eta$. Therefore, $z_0(\eta)$ is decreasing in $\eta$ and $z_1(\eta)$ is increasing in $\eta$. As a result, $f_{z_0(\eta)}[f_0(z, \eta) - f_1(z, \eta)]dz$ is increasing in $\eta$. Hence, as $\eta$ increases from $\eta$ to $\eta$, $f_{z_0(\eta)}[f_0(z, \eta) - f_1(z, \eta)]dz$ increases from 0 to $C = f_{z_0(\eta)}[f_0(\eta, \eta) - f_1(\eta, \eta)]dz$, which is greater than $C$ by our assumption. Hence, we conclude that there exists some $\eta^* \in (\eta, \eta)$ such that $f_{z_0(\eta^*)}[f_0(z, \eta^*) - f_1(z, \eta^*)]dz = C$. Denoting $\tilde{z}_0 = z_0(\eta^*)$ and $\tilde{z}_1 = z_1(\eta^*)$, we complete the proof of (i).

Next, let us prove (ii). We use a verification argument consisting of two steps.

Verification Argument: Step 1.

Let $v$ be such that (a) $v_1(0, z) = f_0(z, \eta^*)$ on $[0, z_i^*)$ and $v_1(0, z) = f_1(z, \eta^*)$ on $[z_i^*, \infty)$, (b) $v_1(1, z) = f_0(z, \eta^*)$ on $[0, z_0^*)$ and $v_1(0, z) = f_1(z, \eta^*)$ on $[z_0^*, \infty)$, and (c) $v(0, z) = v(1, z)$ on $z_0^*, \infty)$. Then, for any $y \in (0, 1)$, $v(y, z)$ is a twice continuously differentiable function on $[0, \infty)$ except at $z_0^*$ and $z_i^*$. Define $\mathcal{K} = \mathbb{R}^+ \setminus \{z_0^*, z_i^*\}$. It is easy to verify that this function $v(y, z)$ satisfies (14)–(18) and thus is a solution of Equation (12) when $z \in \mathcal{K}$. Then for any $z \in \mathcal{K}$ and $y \in (0, 1)$,

$$\frac{\sigma^2}{2}v(y, z) - \left(\beta \sqrt{\lambda + \kappa} \sqrt{\lambda y}\right) v_1(y, z) - g(v_1(y, z)) + h(z) + cy \geq \eta^*, \quad \text{(EC.2)}$$
and

$$0 \leq v(0, z) - v(1, z) \leq C, \quad (EC.3)$$

with \(v_z(y, 0) = 0\) and \(v_z(y, z)\) grows polynomially in \(z\) as \(z \to \infty\).

Note that based on our construction, although \(v_z(y, z)\) is not continuous on points \(z_0^*\) and \(z_1^*\), both of its left and right limits exist, since \(v_z(y, z)\) is absolutely continuous with respect to \(z\). Since the occupation time of a diffusion process at any single point is zero, we can arbitrarily assign value to \(v_z(y, z)\) at such points for calculus computation and the result will not be changed (see e.g. §3.6 of Karatzas and Shreve (1991)). Hence, in the rest of this section, we shall stipulate \(v_z(y, z)\) as its right limit, i.e.,

$$v_z(y, z) = \lim_{h \to 0^+} (v_z(y, z + h) - v_z(y, z))/h.$$

Then, by applying the generalized Itô’s formula for non-smooth functions and taking expectations, we get

$$
\begin{align*}
\mathbb{E}[v(Y(t), Z(t))] - v(Y(0), Z(0)) &= -\beta \sqrt{\lambda} \mathbb{E} \int_0^t v_z(Y(u), Z(u)) du - \mathbb{E} \int_0^t \dot{\theta}(u) v_z(Y(u), Z(u)) du \\
&\quad - \kappa \sqrt{\lambda} \mathbb{E} \left[ \int_0^t Y(u) v_z(Y(u), Z(u)) du \right] + \frac{\sigma^2}{2} \mathbb{E} \left[ \int_0^t (0, t, x) \rho(0, dx) + \int_0^t (1, t, x) \rho(1, dx) \right] \\
&\quad + \mu \mathbb{E} [v_z(Y(t), 0) L(t)] + \mathbb{E} \left[ \sum_{n \leq t} \Delta v(Y(u), Z(u)) \right],
\end{align*}
$$

where we used

$$\sigma \mathbb{E} \left[ \int_0^t v_z(Y(u), Z(u)) dB(u) \right] = 0,$$

which holds since \(v_z(y, z)\) has at most polynomial growth in \(z\) and by Lemma EC.12 part (ii)

$$\mathbb{E} \left[ \int_0^t (Z(u))^k du \right] < \infty,$$

for any \(k \in \mathbb{N}\). In Equation (EC.4), \(l(y, t, x)\) is local time of the part of process \(Z\) when \(Y = y\) by time \(t\) at level \(x\) and \(\rho(y, [\tilde{a}, \tilde{b}]) = v_z(y, \tilde{b}) - v_z(y, \tilde{a})\). Through integration by parts, we have

$$\frac{\sigma^2}{2} \left[ \int_{-\infty}^{\infty} l(0, t, x) \rho(0, dx) + \int_{-\infty}^{\infty} l(1, t, x) \rho(1, dx) \right] = \frac{\sigma^2}{2} \int_0^t \dot{v}_z(Y(u), Z(u)) du$$

almost surely. Thus, we can substitute the left-hand side by the right-hand side in the (EC.4). Since \(v_z(y, 0) = 0\), we have \(\mu \mathbb{E}[v_z(Y(t), 0) L(t)] = 0\). Moreover, the definition \(g(x) = \sup_{\theta} \{x \theta - \delta(\theta)\}\) and (EC.2) imply that

$$
\begin{align*}
&- \beta \sqrt{\lambda} \mathbb{E} \left[ \int_0^t v_z(Y(u), Z(u)) du \right] - \mathbb{E} \left[ \int_0^t \dot{\theta}(u) v_z(Y(u), Z(u)) du \right] \\
&\quad - \kappa \sqrt{\lambda} \mathbb{E} \left[ \int_0^t Y(u) v_z(Y(u), Z(u)) du \right] + \frac{\sigma^2}{2} \mathbb{E} \left[ \int_0^t v_z(Y(u), Z(u)) du \right] \\
&\geq - \beta \sqrt{\lambda} \mathbb{E} \left[ \int_0^t v_z(Y(u), Z(u)) du \right] - \mathbb{E} \left[ \int_0^t g(v_z(Y(u), Z(u))) du \right] - \mathbb{E} \left[ \int_0^t \delta(\dot{\theta}(u)) du \right] \\
&\quad - \kappa \sqrt{\lambda} \mathbb{E} \left[ \int_0^t Y(u) v_z(Y(u), Z(u)) du \right] + \frac{\sigma^2}{2} \mathbb{E} \left[ \int_0^t v_z(Y(u), Z(u)) du \right] \\
&\geq \eta^* t - \mathbb{E} \left[ \int_0^t \delta(\dot{\theta}(u)) du \right] - \mathbb{E} \left[ \int_0^t h(Z(u)) du \right] - \mathbb{E} \left[ \int_0^t Y(u) du \right].
\end{align*}
$$
and (EC.3) implies that
\[
\mathbb{E} \left[ \sum_{u \leq t} \Delta v(Y(u), Z(u)) \right] \geq - \mathbb{E} \left[ C \sum_{u \leq t} \Delta Y(u)^+ \right].
\]

Hence, we have
\[
\mathbb{E} [v(Y(t), Z(t))] - v(Y(0), Z(0)) \\
\geq \eta^* t - \mathbb{E} \left[ \int_0^t \delta(\tilde{\sigma}(u))du \right] - \mathbb{E} \left[ \int_0^t h(Z(u))du \right] - c\mathbb{E} \left[ \int_0^t Y(u)du \right] - \mathbb{E} \left[ C \sum_{u \leq t} \Delta Y(u)^+ \right]. \tag{EC.5}
\]

To proceed, we need the following Lemma.

**Lemma EC.12.** Regardless of the choice of \( Y \), we have (i) \( \limsup_{t \to \infty} \mathbb{E} [(Z(t))^k] < \infty \) for any \( k > 0 \); (ii) for any \( k, t > 0 \), \( \mathbb{E} \left[ \int_0^t (Z(s))^k ds \right] < \infty \).

To ease the flow of ideas, we defer the proof of this lemma to the end. Next, we divide both sides of (EC.5) by \( t \), send \( t \to \infty \), and then appeal to Lemma EC.12 part(i) plus the fact that \( v_z(y, z) \) has at most polynomial growth, to get that for any admissible strategy \( Y \),
\[
\limsup_{t \to \infty} \frac{1}{t} \mathbb{E} \left[ \int_0^t \delta(\tilde{\sigma}(u))du + \int_0^t h(Z(u))du + c \int_0^t Y(u)du + C \sum_{u \leq t} \Delta Y(u)^+ \right] \geq \eta^*.
\]

**Verification Argument: Step 2.** Next, we will show that for \( Y^*, Z^* \) and optimal pricing \( \tilde{\sigma}^* \), we have
\[
\limsup_{t \to \infty} \frac{1}{t} \mathbb{E} \left[ \int_0^t \delta(\tilde{\sigma}^*(u))du + \int_0^t h(Z^*(u))du + c \int_0^t Y^*(u)du + C \sum_{u \leq t} \Delta Y^*(u)^+ \right] = \eta^*.
\]

Following the same procedures as in **Step 1**, i.e., applying the generalized Itô’s formula for non-smooth function, making substitutions to the term with local time, and taking expectations, we get
\[
\mathbb{E}[v(Y^*(t), Z^*(t))] - v(Y(0), Z(0)) \\
= -\beta \mathbb{E} \left[ \int_0^t v_z(Y^*(u), Z^*(u))du \right] - \mathbb{E} \left[ \int_0^t \tilde{\sigma}^*(u)v_z(Y^*(u), Z^*(u))du \right] \\
- \kappa \mathbb{E} \left[ \int_0^t Y^*(u)v_z(Y^*(u), Z^*(u))du \right] + \frac{\sigma^2}{2} \mathbb{E} \left[ \int_0^t v_z(Y^*(u), Z^*(u))du \right] \\
+ \mathbb{E} \left[ \sum_{u \leq t} \Delta v(Y^*(u), Z^*(u)) \right] \\
= -\beta \mathbb{E} \left[ \int_0^t v_z(Y^*(u), Z^*(u))du \right] - \mathbb{E} \left[ \int_0^t g(v_z(Y^*(u), Z^*(u)))du \right] - \mathbb{E} \left[ \int_0^t \delta(\tilde{\sigma}^*(u))du \right] \\
- \kappa \mathbb{E} \left[ \int_0^t Y^*(u)v_z(Y^*(u), Z^*(u))du \right] + \frac{\sigma^2}{2} \mathbb{E} \left[ \int_0^t v_z(Y^*(u), Z^*(u))du \right] \\
+ \mathbb{E} \left[ \sum_{u \leq t} \Delta v(Y^*(u), Z^*(u)) \right],
\]
where we used the definition of the optimal pricing strategy \( \tilde{\sigma}^*(u) \) such that \( g(v_z(Y^*(u), Z^*(u))) = v_z(Y^*(u), Z^*(u))\tilde{\sigma}^*(u) - \delta(\tilde{\sigma}^*(u)) \) and
\[
\sigma\mathbb{E} \left[ \int_0^t v_z(Y^*(u), Z^*(u))dB(u) \right] = 0,
\]
which holds since \( v_z(y, z) \) has at most polynomial growth in \( z \) and by Lemma EC.12 part (ii)

\[
\mathbb{E} \left[ \int_0^t (Z^*(u))^k du \right] < \infty,
\]

for any \( k \in \mathbb{N} \), and \( \mu \mathbb{E}[v_z(Y^*(t), 0) L(t)] = 0 \) since \( v_z(0, 0) = 0 \). We can verify that

\[
\mathbb{E}[v(Y^*(t), Z^*(t)) - v(Y(0), Z(0))]
\]

\[
= \eta^* t - \mathbb{E} \left[ \int_0^t \delta(\theta^*(u)) du \right] - \mathbb{E} \left[ \int_0^t h(Z^*(u)) du \right] - c \mathbb{E} \left[ \int_0^t Y^*(u) du \right] - \mathbb{E} \left[ C \sum_{u \leq t} [\Delta Y^*(u)]^+ \right].
\]

If we divide both hand sides of the above equation by \( t \) and let \( t \) go to infinity, we obtain the desired result by appealing to Lemma EC.12 part (i) and the fact that \( v_z(y, z) \) has at most polynomial growth. This completes the proof of (ii).

Finally, let us prove (iii). Suppose that \( C \geq \bar{C} \). Let \( v_z(0, z) = f_0(z, \bar{\eta}) \) on \([0, \bar{z}_1]\) and \( v_z(0, z) = f_1(z, \bar{\eta}) \) on \([\bar{z}_1, \infty)\). Also, let \( v_z(1, z) = f_0(z, \bar{\eta}) \) on \([0, \bar{z}_0]\), and \( v_z(0, z) = f_1(z, \bar{\eta}) \) on \((\bar{z}_0, \infty)\). For any \( y \in \{0, 1\} \), \( v(y, z) \) is a twice continuously differentiable function on \([0, \infty)\) except at \( \bar{z}_0 \) and \( \bar{z}_1 \). By writing \( v_z(z) = v(y, z) \), we have

\[
\frac{\sigma^2}{2} v_0''(z) - \beta \sqrt{\lambda} v_0'(z) - g(v_0'(z)) + h(z) = \bar{\eta} \quad \text{for} \quad z \in [0, \bar{z}_1), \quad (EC.6)
\]

\[
\frac{\sigma^2}{2} v_0''(z) - (\beta + \kappa) \sqrt{\lambda} v_0'(z) - g(v_0'(z)) + h(z) + c = \bar{\eta} \quad \text{for} \quad z > \bar{z}_0. \quad (EC.7)
\]

In addition,

\[
v_0(z) = v_1(z) \quad \text{for} \quad 0 \leq z \leq \bar{z}_0, \quad \text{and} \quad v_0(z) = v_1(z) + \bar{C} \quad \text{for} \quad z \geq \bar{z}_1, \quad (EC.8)
\]

subject to the boundary conditions \( v_0'(0) = 0 \), and \( v_1'(z) \) grows polynomially in \( z \) as \( z \to \infty \), as well as a set of optimality conditions: \( v_0'(-\bar{z}_0) = v_1'(-\bar{z}_1) \) and \( v_0'(\bar{z}_0) = v_1'(\bar{z}_1) \). Then for any \( z \neq \bar{z}_0, \bar{z}_1 \) and \( y \in \{0, 1\} \), we have

\[
\frac{\sigma^2}{2} v_z(z, y) - \left( \beta \sqrt{\lambda} + \kappa \sqrt{\lambda} y \right) v_z(z, y) - g(v_z(z, y)) + h(z) + cy \geq \bar{\eta}. \quad (EC.9)
\]

and

\[
0 \leq v(0, z) - v(1, z) = \bar{C} \leq C, \quad (EC.10)
\]

with \( v_z(y, 0) = 0 \) and \( v_z(y, z) \) grows polynomially in \( z \) as \( z \to \infty \). By adapting the same argument as in Step 1 of the Verification Argument, we obtain

\[
\limsup_{t \to \infty} \frac{1}{t} \mathbb{E} \left[ \int_0^t \delta(\theta(u)) du + \int_0^t h(Z(u)) du + c \int_0^t Y(u) du + C \sum_{u \leq t} [\Delta Y(u)]^+ \right] \geq \bar{\eta} = \min(\eta_0, \eta_1).
\]

The above inequality becomes an equality by using the \( S_0 \) strategy \( (Y \equiv 0) \) or \( S_1 \) strategy \( (Y \equiv 1) \) depending on whether \( \eta_0 \leq \eta_1 \) or \( \eta_0 > \eta_1 \). This completes the proof of (iii). \( \square \)
**Proof of Theorem 2.** The proof of Theorem 2 departs from that of Theorem 1 mainly because the lower threshold for the approximating diffusion process $Z(t)$ is changed from 0 to $-b$. So the boundary conditions need to be modified accordingly. Despite this departure, the proofs of the two theorems are largely identical, except that we need to make two non-trivial modifications, one on Lemma EC.2 and the other on Lemma EC.12.

The following lemma extends Lemma EC.2 that is essential for proving Proposition 1.

**Lemma EC.13.** Let $L_0 := \{ \eta > 0 : \exists z \in (-b, z_{\eta, \infty}), f'_0(z, \eta) < 0 \}$. If $\eta \in L_0$, then $f_0(z, \eta)$ is quasi-concave and $\lim_{z \to z_{\eta, \infty}} f_0(z, \eta) = -\infty$. In addition, $L_0$ is nonempty.

The proof of Lemma EC.13 is much more sophisticated than the proof of Lemma EC.2 due to the fact that the waiting cost function $h(x)$ is no longer monotonically increasing everywhere. Indeed, $h(x)$ is strictly decreasing in negative $x$. We will provide the proof of Lemma EC.13 in § EC.2.

Second, to show Theorem 2, since the approximating diffusion process $Z(t)$ can take negative values, we need the following modification of Lemma EC.12:

**Lemma EC.14.** Regardless of the choice of $Y$, we have (i) $\limsup_{t \to \infty} \mathbb{E} [ |Z(t)|^k ] < \infty$ for any $k > 0$; (ii) for any $k, t > 0$, $\mathbb{E} [ f'_0 |Z(s)|^k ds ] < \infty$.

The proof of Lemma EC.14 is more involved than the proof of Lemma EC.12 due to the fact that the $Z$ process can take negative values. We will provide the proof of Lemma EC.14 in § EC.2.

The rest of the proof of Theorem 2 follows similarly from the proof of Theorem 1.

**Proof of Theorem 3.** Proof of Theorem 3 is nearly identical to the proof of Theorem 1 and is thus omitted here.

### EC.2 Proofs of Auxiliary Lemmas

**Proof of Lemma EC.1.** Since $g(\cdot)$ is a locally Lipschitz continuous function (by our hypothesis), there exists a continuous function $f_0(\cdot, \eta)$ satisfying (19) on $[0, z_{\eta, \infty})$ for each $\eta \in \mathbb{R}$. If $\eta_1 < \eta_2$, then $f'_0(0, \eta_1) < f'_0(0, \eta_2)$. Hence, there exists some $z > 0$ such that $f_0(z, \eta_1) < f_0(z, \eta_2)$. Then the lemma follows by applying the Comparison Theorem of ordinary differential equations (see, e.g., the Basic Comparison Theorem (McNabb 1986)).

**Proof of Lemma EC.2.** Assume $\eta \in L_0$ and $f_0(z, \eta)$ exists. To prove $f_0(z, \eta)$ is quasi-concave, we need to show there exists only one local maximum point $z_0$ such that $f_0(z_0, \eta) \geq f_0(z, \eta)$ for all $z \in [0, z_{\eta, \infty})$ and $\eta \in L_0$.

Let $z_0 := \inf \{ z > 0 : f'_0(z, \eta) < 0 \}$. Then $f'_0(z_0, \eta) = 0$, and $f_0(z_0, \eta) \geq f_0(z, \eta)$ for all $z \in (0, z_0)$. In addition, there exists some $\varepsilon > 0$ such that $f'_0(z, \eta) < 0$ and $f_0(z, \eta) < f_0(z_0, \eta)$ on $(z_0, z_0 + \varepsilon)$. We claim for all $z \in (z_0, z_{\eta, \infty})$, $f'_0(z, \eta) \leq 0$ and $f_0(z, \eta) < f_0(z_0, \eta)$. Assume by contradiction that this is not true, then there
exists some \( z_1 > z_0 \) such that \( f'_0(z_1, \eta) > 0 \) and \( f_0(z_1, \eta) < f_0(z_0, \eta) \). We can deduce there exists \( \bar{z}_0 \in (z_0, z_1) \) such that \( f_0(\bar{z}_0, \eta) = f_0(z_1, \eta) \). \( f'_0(\bar{z}_0, \eta) \leq 0 \). We have

\[
\frac{\sigma^2}{2} (f'_0(\bar{z}_0, \eta) - f'_0(z_1, \eta)) = -(h(\bar{z}_0) - h(z_1)),
\]

which is a contradiction, because the left-hand side is strictly negative but the right-hand side is non-negative.

Next, we prove \( \lim_{z \to z_{0}^{-}} f_0(z, \eta) = -\infty \). If \( z_{0}^{-} < \infty \), due to \( f'_0(z, \eta) \leq 0 \) for all \( z \in (z_0, z_{0}^{-}) \), \( \lim_{z \to z_{0}^{-}} f_0(z, \eta) = -\infty \). If, on the other hand, \( z_{0}^{-} = \infty \), assume \( \lim_{z \to \infty} f_0(z, \eta) = l > -\infty \). We have \( \lim_{z \to \infty} f'_0(z, \eta) = 0 \), so \( \lim_{z \to \infty} h(z) = \eta + \beta \sqrt{\lambda} - g(l) \), which is a contradiction since the left-hand side is \( \infty \) but the right-hand side is finite.

Finally, to prove \( \mathcal{L}_0 \neq \emptyset \), we first notice that for all \( \eta < 0 \), \( f'_0(0, \eta) < 0 \). Based on the above arguments, we have \( f_0(z, \eta) < 0 \) for all \( z \in (0, z_{0}^{-}) \) and \( \lim_{z \to z_{0}^{-}} f_0(z, \eta) = -\infty \). By the continuity of \( f_0(z, \eta) \) with respect to \( \eta \), we have \( f_0(z, 0) = \lim_{\eta \to 0} f_0(z, \eta) \leq 0 \) and \( f'_0(z, 0) = \lim_{\eta \to 0} f'_0(z, \eta) \leq 0 \). But since \( f_0(z, 0) \equiv 0 \) is not possible, there must exist some \( z_1 > 0 \) such that \( f_0(z_1, 0) < 0 \). Using the continuity of \( f_0(z, \eta) \) with respect to \( \eta \) again, we know that there exists some \( \eta_1 > 0 \) such that \( f_0(z_1, \eta_1) < 0 \). Hence, there exists some \( z \in (0, z_1) \) such that \( f'_0(z, \eta_1) < 0 \), so \( \eta_1 \in \mathcal{L}_0 \).

Proof of Lemma EC.3. Assume by contradiction that \( \mathcal{U}_0 = \emptyset \). Then for all \( \eta > 0 \), by Lemma EC.2 \( \lim_{z \to z_{0}^{-}} f_0(z, \eta) = -\infty \).

To proceed, we first claim the set defined as \( \tilde{U}(z, y) := \{ \eta : f_0(z, \eta) \geq y \} \) is nonempty for any \( z \geq 1, y \geq 0 \). Since \( g(x) \geq 0 \), we have

\[
f_0(z, \eta) \geq \frac{2}{\sigma^2} \left( \eta + \beta \sqrt{\lambda} f_0(x, \eta) - h(x) \right), \\
f_0(z, \eta) \geq \frac{2}{\sigma^2} e^{-\frac{2\sqrt{x}}{\sigma^2}} \int_0^z e^{-\frac{2\sqrt{x}}{\sigma^2}} (\eta - h(x)) \, dx.
\]

So, given \( z \), the right-hand side could be larger than any \( y \) if \( \eta \) is sufficiently large. Let \( \eta(z, y) := \inf \tilde{U}(z, y) \).

By the super-linear growth assumption of function \( g(\cdot) \), we have \( g(x) > lx \) for any \( l \in \mathbb{R} \) when \( x \) is sufficiently large. In particular, if we let \( l = -\beta \sqrt{\lambda} + 1 \), then there exists some \( \bar{x} \) such that \( g(x) > (-\beta \sqrt{\lambda} + 1)x \) for all \( x \geq \bar{x} \). Let \( \eta = \eta(1, \bar{x}) \), then we have \( g(f_0(1, \eta)) + \beta \sqrt{\lambda} f_0(1, \eta) > f_0(1, \eta) \).

Hence, we can deduce

\[
f'_0(1, \eta) > \frac{2}{\sigma^2} (f_0(1, \eta) - h(1) + \eta),
\]

which suggests there exists \( \bar{z}(\bar{z} \text{ could be } \infty) \) such that \( f_0(\bar{z}, \eta) \geq \bar{x} \) on \( (1, \bar{z}) \). We also have \( f'_0(\bar{z}, \eta) > \frac{2}{\sigma^2} (f_0(\bar{z}, \eta) - h(\bar{z}) + \eta) \) on \( (1, \bar{z}) \). Next, for any \( z \in (1, \bar{z}) \), we have

\[
f_0(z, \eta) \geq f_0(1, \eta) + \frac{2}{\sigma^2} e^{\frac{2\sqrt{x}}{\sigma^2}} \int_1^z e^{-\frac{2\sqrt{x}}{\sigma^2}} (-h(x) + \eta) \, dx \\
= f_0(1, \eta) + \eta \left( e^{\frac{2\sqrt{x}}{\sigma^2} - 1} \right) - \frac{2}{\sigma^2} \int_1^z e^{-\frac{2\sqrt{x}}{\sigma^2}} h(x) \, dx.
\]
By the polynomial growth of $h(\cdot)$, we have $h(x) \leq Ax^k + B$ for some constants $A, B \in \mathbb{R}^+$ and some integer $k > 0$ for all $x > 0$. Then, we have

$$f_0(z, \eta) > f_0(1, \eta) + \eta \left( e^{\hat{\eta}^z - 1} - 1 \right) - \frac{2}{\sigma^2} e^{\hat{\eta}^z} \int_1^z e^{-\hat{\eta}^x} (Ax^k + B) \, dx$$

$$> f_0(1, \eta) + \eta \left( e^{\hat{\eta}^z - 1} - 1 \right) - \frac{2}{\sigma^2} e^{\hat{\eta}^z} \int_1^\infty e^{-\hat{\eta}^x} (Ax^k + B) \, dx$$

$$= f_0(1, \eta) + \left( \frac{\eta}{e^{\hat{\eta}^z} - M} \right) e^{\hat{\eta}^z} - \eta,$$

where $M$ is some constant. Then there exists some $\eta$ such that $\frac{\eta}{e^{\hat{\eta}^z} - M} > 0$ and $\left( \frac{\eta}{e^{\hat{\eta}^z} - M} \right) e^{\hat{\eta}^z} - \eta > 0$. Denoting an arbitrary such $\eta$ satisfying the inequality as $\bar{\eta}$, and let $\eta = \max \{ \bar{\eta}, \bar{\eta}(\bar{1}, \bar{z}) \}$, we have $f_0(z, \eta) > f_0(1, \eta)$ on $(1, \bar{z})$. Also, $f_0(z, \eta)$ is greater than a strictly increasing function on $(1, \bar{z})$. By the continuity of $f_0$, this suggests $f_0(z, \eta) > f_0(1, \eta)$ for all $1 \leq z \leq z_{\eta, \infty}$. However, this contradicts to $\lim_{z \to z_{\eta, \infty}} f_0(z, \eta) = -\infty$. Hence we conclude that $\eta \in \mathcal{U}_0$. \hfill \Box

**Proof of Lemma EC.4.** By contradiction, if $\eta_0 \in L_0$, then $\lim_{z \to z_{\eta_0, \infty}} f_0(z, \eta_0) = -\infty$. Hence, there exists some $z_0 > 0$ such that $f_0(z_0, \eta_0) < 0$. By the continuity of $f_0(z, \eta)$ with respect to $\eta$, there exists $\varepsilon > 0$ such that $f_0(z_0, \eta_0 + \varepsilon) < 0$, which contradicts the definition of $\eta_0$. Therefore, $\eta_0 \in \mathcal{U}_0$. We can also deduce that $\eta_0 = \inf \mathcal{U}_0$.

Next, by definition of $\mathcal{U}_0$, $f_0(z, \eta_0) \geq 0$ for all $z \in (0, z_{\eta_0, \infty})$. We prove $z_{\eta_0, \infty} = \infty$. Assume by contradiction $z_{\eta_0, \infty} < \infty$. Then for any number $a \in \mathbb{R}$, there exists some $z$ such that $f_0(z, \eta_0) > a$. By the continuity of $f_0(z, \eta)$ with respect to $\eta$, there exists $\varepsilon > 0$ such that $f_0(z, \eta - \varepsilon) > a$. Next, by the similar arguments as in the proof of Lemma EC.3, we can deduce that $f_0(z, \eta - \varepsilon)$ is greater than some strictly increasing function, which suggests $\eta - \varepsilon \in \mathcal{U}_0$. However, this contradicts to the definition of $\mathcal{U}_0$, so we conclude $z_{\eta_0, \infty} = \infty$.

**Proof of Lemma EC.5.** For clarity, we will assume in this proof that $\beta = 0$. The case for $\beta \neq 0$ can be proven in a similar manner, albeit with cumbersome notation. To start, for some $\bar{z} > 0$, let $a > 0$ be such that

$$(2/\sigma^2)g'(\bar{z})(h(z) + a) - 2h'(z) > 0. \quad \text{(EC.11)}$$

Note that the existence of such a constant $a$ is ensured by Assumption 1. Define $\varphi_\eta(z) := g(f_0(z, \eta)) - 2h(z) - a$, where the value of $a$ will be defined more clearly in (EC.12). Now, let $\hat{z}_\eta$ be such that $f_0(\hat{z}_\eta, \eta) = \sup_{z \geq a} f_0(z, \eta)$. Since $f_0(\hat{z}_\eta, \eta) = 0$, we know that $g(f_0(\hat{z}_\eta, \eta)) - h(\hat{z}_\eta) = -\eta < 0$. Hence, $\varphi_\eta(\hat{z}_\eta) < -h(\hat{z}_\eta) - a \leq 0$. Next, we argue that $\varphi_\eta(z) \leq 0$ for all $z \in [0, \hat{z}_\eta]$. Suppose this is not true. Then there must exist some $\bar{z}_\eta$ and $\tilde{z}_\eta$ with $0 < \bar{z}_\eta < \bar{z}_\eta < \tilde{z}_\eta$ such that $\varphi_\eta(\bar{z}_\eta) = \varphi_\eta(\tilde{z}_\eta) = 0$, and $\varphi_\eta(z) > 0$ for all $z \in (\bar{z}_\eta, \tilde{z}_\eta)$. In particular, the foregoing implies that $g(f_0(z, \eta)) > 2h(z) + a$, and so

$$\frac{\sigma^2}{2} f_0(z, \eta) = g(f_0(z, \eta)) - h(z) + \eta = g(f_0(z, \eta)) - 2h(z) + h(z) + \eta > h(z) + a$$

for every $z \in (\bar{z}_\eta, \tilde{z}_\eta)$. We note that the above arguments hold for any $a > 0$. 

On the other hand, by the definition of $\varphi_n$ and (EC.11), we can find some $a$ such that

$$\varphi'_n(z) = g'(f_0(z, \eta))f'_0(z, \eta) - 2h'(z) > (2/\sigma^2)g'(g^{-1}(2h(\eta)))h(z) + a - 2h'(z) > 0. \quad \text{(EC.12)}$$

This, however, is a contradiction due to our hypothesis which holds that $\varphi_n(\zeta_n) = \varphi_n(\xi_n) = 0$. As a result, $\varphi_n(z) \leq 0$ for all $z \in [0, \zeta_n]$, from which the desired result follows.

The proofs of Proposition EC.1 and the related Lemmas EC.6-EC.10 follow the similar arguments as the proofs of Lemmas EC.1-EC.5, so they are omitted.

**Proof of Lemma EC.11.** We first prove if $\alpha_1 > \alpha_2$, then $\xi_{\alpha_1}(z, \eta(\alpha_1)) > \xi_{\alpha_2}(z, \eta(\alpha_2))$. Suppose by way of contradiction that this is not true. Then there exists $\xi$ such that $\xi_{\alpha_1}(\xi, \eta(\alpha_1)) = \xi_{\alpha_2}(\xi, \eta(\alpha_2))$ and $\xi_{\alpha_1}'(\xi, \eta(\alpha_1)) \leq \xi_{\alpha_2}'(\xi, \eta(\alpha_2))$. Then we can deduce that

$$\frac{\sigma^2}{2}(\xi_{\alpha_1}'(\xi, \eta(\alpha_1)) - \xi_{\alpha_2}'(\xi, \eta(\alpha_2))) = \eta(\alpha_1) - \eta(\alpha_2) \leq 0.$$

By the uniqueness of solutions to ODE, $\eta(\alpha_1) < \eta(\alpha_2)$, so that there exists some $z > \xi$ such that $\xi_{\alpha_1}(z, \eta(\alpha_1)) < \xi_{\alpha_2}(z, \eta(\alpha_2))$. Next, using the Basic Comparison Theorem (McNabb 1986) as in the proof of Lemma EC.1, we know that $\xi_{\alpha_1}(z, \eta(\alpha_1)) < \xi_{\alpha_2}(z, \eta(\alpha_2))$ must hold for all $z \in (\xi, \infty)$. However, by the continuity of $\xi_{\alpha}(z, \eta)$ with respect to $\eta$, there exists some $\epsilon > 0$ such that $\xi_{\alpha_2}(z, \eta(\alpha_2) - \epsilon) > \xi_{\alpha_1}(z, \eta(\alpha_1))$ for all $z > \xi$, which suggests $\xi_{\alpha_1}(z, \eta(\alpha_2) - \epsilon)$ grows to infinity as $z$ grows large. But this contradicts to the definition $\eta(\alpha) = \inf \mathcal{U}_\alpha$. Hence, $\xi_{\alpha_1}(z, \eta(\alpha_1)) > \xi_{\alpha_2}(z, \eta(\alpha_2))$ on $(0, \infty)$.

Next, we prove $\eta(\alpha_1) < \eta(\alpha_2)$ holds for all $\alpha_1 > \alpha_2$. Assume by contradiction this is not true, then there exists some $\alpha_1 > \alpha_2$ such that $\eta(\alpha_1) \geq \eta(\alpha_2)$. Using the Basic Comparison Theorem (McNabb 1986) and the continuity of $\xi_{\alpha}(z, \eta)$ with respect to $\eta$ again, we know that there exists $\epsilon > 0$ such that $\xi_{\alpha_1}(z, \eta(\alpha_1) - \epsilon)$ grows to infinity, which leads to the contradiction. Hence, $\eta(\alpha)$ is a decreasing function.

Next, we show $\eta(\cdot)$ is a continuous mapping. Consider an increasing sequence $\{\alpha_n\}$ with $\alpha$ being the limit. For ease of notation, we write $(\xi_{\alpha_n}(z, \eta(\alpha_n)), \eta(\alpha_n))$ as $(\xi_n(z), \eta_n)$. We aim to show that $\eta_n \to \eta(\alpha)$ as $n \to \infty$. The aforementioned arguments show that $\xi_{\alpha_n}(z, \eta(\alpha_n)) \leq \xi_{\alpha}(z, \eta(\alpha))$ for each fixed $z$. Hence $\xi_n(z) := \lim_{n \to \infty} \xi_{\alpha_n}(z)$ is well defined for each fixed $z$. Integrating (20), we have

$$\frac{\sigma^2}{2}(\xi_n(z) - \alpha_n) = \int_0^z \left[(\beta + \kappa)\sqrt{\lambda}\xi_n(x) + g(\xi_n(x)) - h(x) - c + \eta_n\right] dx.$$

Sending $n \to \infty$, we have

$$\frac{\sigma^2}{2}(\xi_n(z) - \alpha) = \int_0^z \left[(\beta + \kappa)\sqrt{\lambda}\xi_n(x) + g(\xi_n(x)) - h(x) - c + \eta\right] dx.$$

By the uniqueness of the solution, we have $\xi_n(z) = \xi_{\alpha}(z, \eta(\alpha))$. Similar arguments also apply to a decreasing sequence of $\{\alpha_n\}$ with $\alpha$ being the limit. Hence, we have proved $\eta(\alpha)$ is a continuous mapping. The last part of the lemma can be established by using a routine proof-by-contradiction argument, which is elementary and thus omitted.
**Proof of Lemma EC.12.** We first prove part (i). The infinitesimal generator for \((Y(t), Z(t))\) process can be written as

\[
L f(y, z) = \begin{cases} 
\frac{\sigma^2}{2} f_z(y, z) - \beta \sqrt{\lambda} f_y(y, z) - \kappa \sqrt{\lambda} y f_z(y, z) - \hat{\theta}^*(y, z) f_z(y, z), & \text{if } z > 0, \\
\theta f_y(y, 0), & \text{if } z = 0, 
\end{cases} \tag{EC.13}
\]

where \(\theta\) is some constant (the derivation of the generator can be found on page 5 in Varadhan (2011)), and \(\hat{\theta}^*(y, z) = \arg \max_x \{v_z(y, z) \theta - \delta(\theta)\}\).

Define the Lyapunov function as \(f(y, z) = z^k + 1\). We have \(f(y, 0) = 1\). For any \(k \geq 2\), we can compute that

\[
L(z^k + 1) = \frac{\sigma^2}{2} k(k - 1) z^{k-2} - \beta \sqrt{\lambda} k z^{k-1} - \kappa \sqrt{\lambda} y z^{k-1} - \hat{\theta}^*(y, z) k z^{k-1} \tag{EC.14}
\]

for any \(z > 0\).

Let \(F(z) := z^{k-1} - z^{k-2}\). Then \(F'(z) = (k - 1) z^{k-2} - (k - 2) z^{k-3} \geq 0\) if and only if \(z \geq \frac{k - 2}{k - 1}\). So, \(z^{k-2} \leq z^{k-1} - C_1\) where \(C_1 = F\left(\frac{k - 2}{k - 1}\right)\). Then we have

\[
L(z^k + 1) \leq \left(\frac{\sigma^2}{2} (k - 1) - \beta \sqrt{\lambda} \hat{\theta}^*(y, z)\right) k z^{k-1} - \frac{\sigma^2}{2} k(k - 1) C_1 \\
= \left(\frac{\sigma^2}{2} (k - 1) - \beta \sqrt{\lambda} \hat{\theta}^*(y, z)\right) k(z^k)^{\frac{k-1}{k}} - \frac{\sigma^2}{2} k(k - 1) C_1.
\]

If \(\frac{\sigma^2}{2} (k - 1) - \beta \sqrt{\lambda} < 0\), \(L(z^k + 1) \leq \left(\frac{\sigma^2}{2} (k - 1) - \beta \sqrt{\lambda}\right) k(z^k)^{\frac{k-1}{k}} - \frac{\sigma^2}{2} k(k - 1) C_1\). Let \(\phi(x) := x^{\frac{k-1}{k}}\), then \(\phi\) is a strictly concave function with \(\phi(0) = 0\) and \(\phi(x)\) increases to infinity as \(x \to \infty\). So, \(\phi(z^k + 1) - \phi(z^k) = \int_{z^k}^{z^k + 1} \phi'(t) dt\) strictly decreases such that \(\phi(z^k + 1) - \phi(z^k) \leq \phi(1) - \phi(0) = 1\). So, we have

\[
L(z^k + 1) \leq k \left(\frac{\sigma^2}{2} (k - 1) - \beta \sqrt{\lambda}\right) \left((z^k + 1)^{\frac{k-1}{k}} - 1\right) - \frac{\sigma^2}{2} k(k - 1) C_1.
\]

By Theorem 4.1 on page 16 of Hairer (2021), we have \(\limsup_{t \to \infty} \mathbb{E}[(|Z(t)|^k + 1)^{\frac{k-1}{k}}] < \infty\). This implies \(\limsup_{t \to \infty} \mathbb{E}[|Z(t)|^{k-1}] < \infty\). Since \(k \geq 2\) can be any large number, we have the desired result.

Next, suppose \(\frac{\sigma^2}{2} (k - 1) - \beta \sqrt{\lambda} \geq 0\). By our assumption on the super linear growth of function \(g\), we know that \(\hat{\theta}^*(y, z)\) grows to infinity as \(z \to \infty\). Therefore, for any constant \(\tau \in \mathbb{R}\), we can find a \(\tilde{z}\) such that \(\hat{\theta}^*(y, z) \geq \min(\hat{\theta}^*(0, z), \hat{\theta}^*(1, z)) \geq \tau\) for all \(z \geq \tilde{z}\). Let \(\tau = \frac{\sigma^2}{2} (k - 1) - \beta \sqrt{\lambda} + 1\). Hence,

\[
\hat{\theta}^*(y, z) \geq \left(\frac{\sigma^2}{2} (k - 1) - \beta \sqrt{\lambda} + 1\right) 1(z \geq \tilde{z}),
\]

where \(1(\cdot)\) is the indicator function. So, we have

\[
L(z^k + 1) \leq \begin{cases} 
\left(\frac{\sigma^2}{2} (k - 1) - \beta \sqrt{\lambda}\right) k(z^k)^{\frac{k-1}{k}} - \frac{\sigma^2}{2} k(k - 1) C_1, & \text{if } z < \tilde{z}, \\
-k(z^k)^{\frac{k-1}{k}} - \frac{\sigma^2}{2} k(k - 1) C_1, & \text{if } z \geq \tilde{z},
\end{cases}
\]

which implies

\[
L(z^k + 1) \leq -k(z^k)^{\frac{k-1}{k}} - \frac{\sigma^2}{2} k(k - 1) C_1 + \left(\frac{\sigma^2}{2} (k - 1) - \beta \sqrt{\lambda} + 1\right) k(z^k)^{\frac{k-1}{k}}. \tag{EC.15}
\]
Use the above arguments and Theorem 4.1 of Hairer (2021) again, we have the desired result. So the proof of part (i) is complete.

Next, to prove part (ii), we apply Theorem 16.2 and Remark 16.1 in the notes of Varadhan (2011). For $f(z) = z^k$ with $k > 1$, it has polynomial growth and we have $f'(0) = 0$. Therefore, $f(Z(t)) - f(Z(0)) - \int_0^t L Z(s) \, ds$ is a martingale so that $E \int_0^t L Z(s) \, ds = E[f(Z(t))] - E[f(Z(0))]$. We can see that $L(z^k) = L(z^k + 1)$. From our previous calculations in Equation (EC.15) in part (i), we get $L(z^k) \leq -az^{k-1} + b$ for some $a, b > 0$. This gives

$$E[Z(t)^k] - E[Z(0)^k] = E \left[ \int_0^t L Z(s)^k \, ds \right] \leq -E \left[ \int_0^t a Z(s)^{k-1} \, ds \right] + bt,$$

so that

$$E \left[ \int_0^t a Z(s)^{k-1} \, ds \right] \leq bt + E[Z(0)^k] - E[Z(t)^k] \leq bt + E[Z(0)^k]. \quad \text{(EC.16)}$$

Since it holds for any $k > 1$, the proof of part (ii) is done.

\[ \square \]

**Proof of Lemma EC.13.** Assume $\eta \in \mathcal{L}_0$ and $f_0(z, \eta)$ exists. To prove $f_0(z, \eta)$ is quasi-concave, we need to show there exists only one local maximum point $z_0$ such that $f_0(z_0, \eta) \geq f_0(z, \eta)$ for all $z \in [-b, \eta, \infty)$ and $\eta \in \mathcal{L}_0$. Let $z_0 := \inf \{ z > -b : f_0'(z, \eta) < 0 \}$. Then $f_0'(z_0, \eta) = 0$, and $f_0(z_0, \eta) \geq f_0(z, \eta)$ for all $z \in (-b, z_0)$. Let $z_2$ be the smallest value of $z$ such that $z > z_0$ and $z_2$ is a local minimum of $f_0(z, \eta)$; if there exists no $z$ satisfying these conditions then it is immediate that $f_0(z, \eta)$ is quasi-concave.

Case 1: Suppose $z_2 > 0$. Then since $z_2$ is a local minimum it follows that there exist $z_0, z_1$ such that $z_2 > z_0 > 0$, $z_1 > z_2$, $f_0(z_0, \eta) = f_0(z_1, \eta)$, $f_0'(z_0, \eta) < 0$, and $f_0'(z_1, \eta) > 0$. We have

$$\frac{\sigma^2}{2} (f_0(z, \eta) - f_0'(z_1, \eta)) = - (h(z_0) - h(z_1)),$$

which is a contradiction because the left-hand side is strictly negative but the right-hand side is strictly positive.

Case 2: Suppose $z_2 \leq 0$. The tangent line to $f_0(z, \eta)$ at the point $z = z_2$ meets $f_0(z, \eta)$ at some point $z = z_3$ where $-b \leq z_3 < z_0$. To verify that $z_3$ exists, since $f_0(-b, \eta) = 0$, we need only check that $f_0(z, \eta)$ does not have a root in the interval $(-b, z_2)$; for if $f_0(c, \eta) = 0$ for $c \in (-b, z_2)$ then using the facts that $h(z)$ is strictly decreasing in $z$ for all $z < 0$ and that the slope of $f_0(z, \eta)$ is positive at $z = -b$ and negative at $z = c$ it follows that (19) cannot hold at both $z = -b$ and $z = c$, a contradiction. It is possible that $f_0(z, \eta)$ has a root at $z = z_2$; this situation corresponds to $z_3 = -b$. We have thus verified that $z = z_3$ exists, where $-b \leq z_3 < z_0$.

Writing (19) for $z = z_3$ and $z = z_2$ and subtracting we get the equation below where the left hand side is positive and the right hand side is strictly negative (since $h(z)$ is strictly decreasing in $(-b, 0)$), yielding a contradiction:

$$\frac{\sigma^2}{2} (f_0'(z_3, \eta) - f_0'(z_2, \eta)) = h(z_2) - h(z_3).$$
This completes the proof of quasi-concavity. Next, we prove \( \lim_{z \to \eta} f_0(z, \eta) = -\infty \). If \( z_{\eta, \infty} < \infty \), due to \( f_0'(z, \eta) \leq 0 \) for all \( z \in (z_0, z_{\eta, \infty}) \), \( \lim_{z \to \eta} f_0(z, \eta) = -\infty \). If, on the other hand, \( z_{\eta, \infty} = \infty \), assume \( \lim_{z \to \infty} f_0(z, \eta) = l > -\infty \). We have \( \lim_{z \to \infty} f_0'(z, \eta) = 0 \), so \( \lim_{z \to \infty} h(z) = \eta + \beta \sqrt{k_l} - g(l) \), which is a contradiction since the left-hand side is \( \infty \) but the right-hand side is finite.

Finally, to prove \( L_0 \neq \emptyset \), we first notice that for all \( \eta < 0 \), \( f_0'(-b, \eta) < 0 \). Based on the above arguments, we have \( f_0(z, \eta) < 0 \), \( f_0'(z, \eta) \leq 0 \) for all \( z \in (-b, z_{\eta, \infty}) \) and \( \lim_{z \to \infty} f_0(z, \eta) = -\infty \). By the continuity of \( f_0(z, \eta) \) with respect to \( \eta \), we have \( f_0(z, 0) = \lim_{\eta \to 0} f_0(z, \eta) \leq 0 \) and \( f_0'(z, 0) = \lim_{\eta \to 0} f_0'(z, \eta) \leq 0 \). But since \( f_0(z, 0) \equiv 0 \) is not possible, there must exist some \( z_1 > 0 \) such that \( f_0(z_1, 0) < 0 \). Using the continuity of \( f_0(z, \eta) \) with respect to \( \eta \) again, we know that there exists some \( \eta_1 > 0 \) such that \( f_0(z_1, \eta_1) < 0 \). Hence, there exists some \( z \in (-b, z_1) \) such that \( f_0'(z, \eta_1) < 0 \), so \( \eta_1 \in L_0 \).

**Proof of Lemma EC.14.** We first prove part (i). The infinitesimal generator for \((Y_t, Z(t))\) process can be written as

\[
L f(y, z) = \begin{cases} \frac{\sigma^2}{2} f_z(y, z) - \beta \sqrt{k_l} f_z(y, z) - \kappa \sqrt{k} y f_z(y, z) - \theta^*(y, z) f_z(y, z) , & \text{if } z > -b , \\ t f_z(y, -b) , & \text{if } z = -b , \end{cases}
\]

where \( t \) is some constant (the derivation of the generator can be found on page 5 in Varadhan (2011)), and \( \hat{\theta}^*(y, z) = \arg\max \{ z, (y, z) \hat{\theta} - \delta(\hat{\theta}) \} \).

Define the Lyapunov function as \( f(y, z) = (z + b)^k + 1 \). We have \( f(y, -b) = 1 \). For any \( k \geq 2 \), we can compute that

\[
L((z + b)^k + 1) = \frac{\sigma^2}{2} k(k-1)(z+b)^{k-2} - \beta \sqrt{k_l} (z+b)^{k-1} - \kappa \sqrt{k} y (z+b)^{k-1} - \hat{\theta}^*(y, z) (z+b)^{k-1}
\]

for any \( z > -b \).

Let \( F(z) := (z+b)^{k-1} - (z+b)^{k-2} \). Then \( F'(z) = (k-1)(z+b)^{k-2} - (k-2)(z+b)^{k-3} \geq 0 \) if and only if \( z \geq \frac{k-2}{k-1} - b \). Therefore, \((z+b)^{k-2} \leq (z+b)^{k-1} - C_1 \) where \( C_1 = F\left(\frac{k-2}{k-1} - b\right) \). Then we have

\[
L((z+b)^k + 1) \leq \left( \frac{\sigma^2}{2} (k-1) - \beta \sqrt{k_l} - \hat{\theta}^*(y, z) \right) (z+b)^{k-1} - \frac{\sigma^2}{2} k(k-1) C_1
\]

If \( \frac{\sigma^2}{2} (k-1) - \beta \sqrt{k_l} < 0 \), then

\[
L((z+b)^k + 1) \leq \left( \frac{\sigma^2}{2} (k-1) - \beta \sqrt{k_l} \right) ((z+b)^k + 1)^{\frac{1}{k+1}} - \frac{\sigma^2}{2} k(k-1) C_1.
\]

Let \( \phi(x) := x^{\frac{1}{k+1}} \), then \( \phi \) is a strictly concave function in \( x > 0 \) with \( \phi(0) = 0 \) and \( \phi(x) \) increases to infinity as \( x \to \infty \). Thus, \( \phi((z+b)^k + 1) - \phi((z+b)^k) \leq \phi(1) - \phi(0) = 1 \). Therefore, we have

\[
L((z+b)^k + 1) \leq k \left( \frac{\sigma^2}{2} (k-1) - \beta \sqrt{k_l} \right) \left( ((z+b)^k + 1)^{\frac{1}{k+1}} - 1 \right) - \frac{\sigma^2}{2} k(k-1) C_1.
\]
By Theorem 4.1 on page 16 of Hairer (2021), we have \( \limsup_{t \to \infty} E[(Z(t) + b)^k + 1]^{\frac{k+1}{k}} < \infty \). Since 
\( E[(Z(t) + b)^k + 1]^{\frac{k+1}{k}} 1_{b \leq Z(t) \leq 0} \in [0, (b^k + 1)^{\frac{k+1}{k}}] \), we have
\[
\limsup_{t \to \infty} E[(Z(t) + b)^k + 1]^{\frac{k+1}{k}} 1_{Z(t) > 0} < \infty.
\]
This implies
\[
\limsup_{t \to \infty} E[|Z(t)|^{k-1}] = \limsup_{t \to \infty} \left(E[|Z(t)|^{k-1} 1_{Z(t) > 0}] + E[|Z(t)|^{k-1} 1_{b \leq Z(t) \leq 0}]\right) 
\leq \limsup_{t \to \infty} \left(E[(Z(t) + b)^k + 1]^{\frac{k+1}{k}} 1_{Z(t) > 0} + b^{k-1}\right),
\]
where \( k \geq 2 \). Since \( k \geq 2 \) can be any large number, we have the desired result.

Next, suppose \( \frac{\sigma^2}{2}(k-1) - \beta \sqrt{k} \geq 0 \). By our assumption on the super linear growth of function \( g \), we know that \( \vartheta^*(y, z) \) grows to infinity as \( z \to \infty \). Therefore, for any constant \( \tau \in \mathbb{R} \), we can find a \( \bar{z} \) such that \( \vartheta^*(y, z) \geq \min(\vartheta^*(0, z), \vartheta^*(1, z)) \geq \tau \) for all \( z \geq \bar{z} \). Let \( \tau = \frac{\sigma^2}{2}(k-1) - \beta \sqrt{k} + 1 \). Hence,
\[
\vartheta^*(y, z) \geq \left( \frac{\sigma^2}{2}(k-1) - \beta \sqrt{k} + 1 \right) \mathbb{1}(z \geq \bar{z}),
\]
where \( \mathbb{1}(\cdot) \) is the indicator function. So, we have
\[
\mathcal{L}((z + b)^k + 1) \leq \begin{cases} \left( \frac{\sigma^2}{2}(k-1) - \beta \sqrt{k} \right) k((z + b)^k)^{\frac{k-1}{k}} - \frac{\sigma^2}{2} k(k-1) C_1, & \text{if } z < \bar{z}, \\ -k((z + b)^k)^{\frac{k-1}{k}} - \frac{\sigma^2}{2} k(k-1) C_1, & \text{if } z \geq \bar{z}, \end{cases}
\]
which implies
\[
\mathcal{L}((z + b)^k + 1) \leq -k((z + b)^k)^{\frac{k-1}{k}} - \frac{\sigma^2}{2} k(k-1) C_1 + \left( \frac{\sigma^2}{2}(k-1) - \beta \sqrt{k} + 1 \right) k(z^k)^{\frac{k-1}{k}}. \quad (\text{EC.19})
\]
Use the above arguments and Theorem 4.1 of Hairer (2021) again, we have the desired result. Thus the proof of part (i) is complete.

Next, to prove part (ii), we apply Theorem 16.2 and Remark 16.1 in the notes of Varadhan (2011). For \( f(z) = (z + b)^k \) with \( k > 1 \), it has polynomial growth and we have \( f'(-b) = 0 \). Therefore, \( f(Z(t)) - f(Z(0)) - \int_0^t \mathcal{L}(Z(s)) ds \) is a martingale so that \( E \int_0^t \mathcal{L}(Z(s)) ds = E[f(Z(t))] - E[f(Z(0))] \). We can see that \( \mathcal{L}((z + b)^k) = \mathcal{L}((z + b)^k + 1) \). From our previous calculations in Equation (EC.19) in part (i), we get
\[
\mathcal{L}((z + b)^k) \leq -a(z + b)^{k-1} + \tilde{b} \text{ for some } a, \tilde{b} > 0.
\]
This gives
\[
E[(Z(t) + b)^k] - E[(Z(0) + b)^k] = E \left[ \int_0^t \mathcal{L}(Z(s) + b)^k ds \right] \leq -E \left[ \int_0^t a(Z(s) + b)^{k-1} ds \right] + \tilde{b} t,
\]
so that
\[
E \left[ \int_0^t \tilde{a}(Z(s) + b)^{k-1} ds \right] \leq \tilde{b} t + E[(Z(0) + b)^k] - E[(Z(t) + b)^k] \leq \tilde{b} t + E[(Z(0) + b)^k], \quad (\text{EC.20})
\]
and therefore

\[
E \left[ \int_0^t \tilde{a}(Z(s))^k \, ds \right] = E \left[ \int_0^t \left( \tilde{a}(Z(s))^k 1_{b \leq Z(s) \leq d} + \tilde{a}(Z(s))^{k-1} \right) \, ds \right]
\]

\[
\leq \tilde{a} \tilde{b} t^{k-1} + E \left[ \int_0^t \tilde{a}(Z(s) + b)^{k-1} \, ds \right] \leq \tilde{a} \tilde{b} t^{k-1} + \tilde{b} t + E[(Z(0) + b)^k].
\]

Since it holds for any \( k > 1 \), the proof of part (ii) is done.

\[ \square \]

EC.3 Comment on the Scaling Conditions and Baseline Quantities/Functions

We first argue that, under the scaling conditions specified in (4), all terms in the objective function given by (3) have an order of \( \sqrt{\lambda} \) as \( \bar{\lambda} \to \infty \). The third and fourth terms are clearly of order \( \sqrt{\lambda} \) by the scaling conditions on \( c \) and \( C \), namely, \( c = \sqrt{\lambda} \hat{c}, C = \sqrt{\lambda} \hat{C} \), plus the fact that \( Y(t) \) takes values in \( \{0, 1\} \). The queue length in heavy traffic is known to be of order \( \sqrt{\lambda} \). By the scaling condition for \( h \), we know that the second term in the objective is also of order \( \sqrt{\lambda} \). To show that the first term is also of order \( \sqrt{\lambda} \), we use Taylor’s series expansion of \( \pi \) at \( \bar{\lambda} \) and get

\[ \delta(\vartheta) \approx -\pi''(\bar{\lambda}) \frac{\vartheta^2}{2}, \]

where \( \pi'(\bar{\lambda}) = 0 \) based on the definition of \( \bar{\lambda} \). Substituting the scaling condition for \( \Xi^{-1} \) back into \( \pi \), we get

\[ \pi''(\bar{\lambda}) = \frac{2}{\sqrt{\lambda}} (\hat{\Xi}^{-1})'(1) + \frac{1}{\sqrt{\lambda}} (\hat{\Xi}^{-1})''(1). \]

Since \( \vartheta = \sqrt{\lambda} \theta \), we know that \( \delta(\vartheta) \) is of order \( \sqrt{\lambda} \), and therefore the first term is also of order \( \sqrt{\lambda} \). In conclusion, all terms in the objective function are of the same order of \( \sqrt{\lambda} \). A similar analysis can be applied to the multi-product setting under the specified scaling conditions.

Next, we show that for \( \bar{\lambda} \) to be the revenue-maximizing demand rate, the baseline quantities/functions \( \hat{\Xi}^{-1} \) and \( \hat{q} \) must satisfy a certain relationship. The first-order optimality condition states that \( \pi'(\bar{\lambda}) = 0 \). Taking into account the scaling conditions \( \Xi^{-1}(\cdot) = \sqrt{\lambda} \hat{\Xi}^{-1}(\cdot/\bar{\lambda}) \) and \( q = \sqrt{\lambda} \hat{q} \), this condition implies that \( \hat{q} = \hat{\Xi}^{-1}(1) + (\hat{\Xi}^{-1})'(1) \).

EC.4 An Exact MDP Solution to Multi-Product Systems

EC.4.1 State Descriptions and Transition Rates

To apply the MDP framework, we need further restrictions on assumptions about the processing times of primary and secondary servers. Let the processing time of class \( k \) jobs on the primary server be exponentially distributed with rate \( \mu_k \) and the processing time on the secondary server be exponentially distributed with rate \( \gamma_k \).
We next define the state variables for the exact continuous-time MDP solution framework under a multi-product system. Let \((M_1(t), Q(t), M_2(t), Y(t))\) be the state descriptor, where \(Q(t)\) is the \(K\)-dimensional vector representing the number of jobs waiting in buffers and \(Y(t) = 0\) or \(1\) is a scalar denoting the secondary server is either on or off. The two new defined scalars \(M_1(t)\) and \(M_2(t)\) represent whether or not the primary and secondary servers are busy and, if so, which class of jobs they are working on. To be more specific, \(M_1(t) \in \{0, 1, \ldots, K\}\), where \(M_1(t) = 0\) indicates the primary server is idle while \(M_1(t) = k > 0\) represents the server is busy processing the class-\(k\) jobs. \(M_2(t)\) is defined in a similar way.

Next, we describe the state transitions under different actions. There are three control levers at the system manager’s discretion: pricing, scheduling, and activating or deactivating the surge capacity. Pricing control can be represented by \(\lambda\), the arrival rates. Let \(m_1(m_2) \in \{0, 1, \ldots, K\}\) be the priority rule of accepting the class-\(m_1(m_2)\) jobs to the primary server (secondary server), if there exists some class-\(m_1(m_2)\) jobs waiting in the queue for \(m_1, m_2 > 0\). \(0\) indicates to make the server idle, and under the non-idling policy, this action only happens when \(Q_k(t) = 0\) for all \(k\). Henceforth, we use the tuple \((\lambda, m_1, m_2)\) to represent the control action, without changing the status of surge capacity.

We assume the system is non-preemptive, such that both the primary and secondary servers have to complete processing the current in-service jobs before accepting new ones. In addition, we assume jobs are routed to the primary server if both primary and secondary servers are idle. We next write down the transition rate from state \(X\) to \(X'\) under the control \((\lambda, m_1, m_2)\) as \(P_{(\lambda, m_1, m_2)}(X, X')\), where \(X = (M_1, Q, M_2, Y)\) and \(X' = (M_1', Q', M_2', Y')\). Let \(e_k\) be the \(K\)-dimensional vector with the \(k\)-th component being 1 and all other components being 0. Also, let \(1_{\{\cdot\}}\) be the indicator function. We divide the discussion into two scenarios: \(Y = 0\) or \(Y = 1\).

When \(Y = 0\), we have for \(k = 1, \ldots, K\), the state transitions due to arrival events are

\[
P_{(\lambda, m_1, m_2)}((M_1, Q, M_2, 0), (M_1, Q + e_k, M_2, 0)) = \lambda_k, \quad \text{if} \quad M_1 \neq 0,
\]
\[
P_{(\lambda, m_1, m_2)}((0, Q, M_2, 0), (k, Q, M_2', 0)) = \lambda_k, \quad \text{if} \quad M_1 = 0.
\]

The transition rates due to service completions when \(Y = 0\) are

\[
P_{(\lambda, m_1, m_2)}((M_1, Q, M_2, 0), (m_1, Q - e_{m_1} 1_{\{m_1 \neq 0\}}, M_2, 0)) = \mu_{M_1} 1_{\{M_1 \neq 0\}},
\]
\[
P_{(\lambda, m_1, m_2)}((M_1, Q, M_2, 0), (M_1, Q, 0, 0)) = \gamma_{M_2} 1_{\{M_2 \neq 0\}}.
\]

When \(Y = 1\), we have for \(k = 1, \ldots, K\), the state transitions due to arrival events are

\[
P_{(\lambda, m_1, m_2)}((0, 0, M_2, 1), (k, 0, M_2, 1)) = \lambda_k, \quad \text{if} \quad M_1 = 0,
\]
\[
P_{(\lambda, m_1, m_2)}((M_1, 0, 0, 1), (M_1, 0, k, 1)) = \lambda_k, \quad \text{if} \quad M_1 \neq 0 \quad \text{but} \quad M_2 = 0,
\]
\[
P_{(\lambda, m_1, m_2)}((M_1, Q, M_2, 1), (M_1, Q + e_k, M_2, 1)) = \lambda_k, \quad \text{if} \quad M_1 \neq 0 \quad \text{and} \quad M_2 \neq 0.
\]
For the service completion, when \( Y = 1 \), we have

\[
P(\lambda m_1, m_2)( (M_1, Q, M_2, 1), (m_1, Q - e_{m_1} 1_{m_1 \neq 0}, M_2, 1) ) = \mu_{M_1} 1_{M_1 \neq 0},
\]

\[
P(\lambda m_1, m_2)( (M_1, Q, M_2, 1), (M_1, Q - e_{m_2} 1_{m_2 \neq 0}, M_2, 1) ) = \gamma_{M_2} 1_{M_2 \neq 0}.
\]

In addition to the transition rates discussed above in the case that surge capacity status is kept unchanged, we also need to specify the immediate system change when the surge capacity is activated or deactivated. This immediate system change can be characterized by a map \( T_m \). Specifically, we have

\[
T_m((M_1, Q, M_2, 0)) = (M_1, Q, M_2, 1), \quad \text{if} \quad M_2 \neq 0,
\]

\[
T_m((M_1, Q, M_2, 0)) = (M_1, Q - e_m 1_{m \neq 0}, M, 1), \quad \text{if} \quad M_2 = 0,
\]

\[
T_m((M_1, Q, M_2, 1)) = (M_1, Q + e_{M_2} 1_{M_2 \neq 0}, 0, 0).
\]

**EC.4.2 Uniformization and Optimality Conditions**

Directly applying the continuous-time MDP framework is not amenable for computation. Hence, we apply the uniformization technique to convert the continuous-time MDP to its embedded discrete-time MDP. To this end, let us make a further assumption on the arrival rates such that \( 0 \leq \sum_{k=1}^{K} \lambda_k(t) \leq \Lambda \) for all \( t \geq 0 \). This assumption is not restrictive since \( \Lambda \) can be regarded as the external job arrival rates. Next, we can define a constant

\[
\bar{\alpha} := \Lambda + \max_k \mu_k + \max_k \gamma_k.
\]

We are now ready to write down the Bellman equation of the embedded discrete-time MDP, based on the defined state descriptors and transition rates. Let \( V \) be the relative value function and \( \eta \) be the long-run average cost. Also, let \( P(\lambda m_1, m_2)( (M_1, Q, M_2, 0), \cdot ) \) denote the row of transition rate matrix. We have

\[
V((M_1, Q, M_2, 0)) + \frac{\eta}{\bar{\alpha}} = \max_{\lambda m_1, m_2} \left\{ \max_{\lambda m_1, m_2} \frac{1}{\bar{\alpha}} \left( \langle P(\lambda m_1, m_2)((M_1, Q, M_2, 0), \cdot ), V \rangle + \pi(\lambda') - \sum_k h_k(Q_k + 1_{(M_1 = k)} + 1_{(M_2 = k)}) \right) \right.
\]

\[
+ \left( 1 - \frac{\|P(\lambda m_1, m_2)((M_1, Q, M_2, 0), \cdot )\|_1}{\bar{\alpha}} \right) V((M_1, Q, M_2, 0)) \right\},
\]

\[
-C + \max_{\lambda' m_1', m_2'} \left\{ \frac{1}{\bar{\alpha}} \left( \langle P(\lambda' m_1', m_2') (T_m(M_1, Q, M_2, 0), \cdot ), V \rangle + \pi(\lambda') - c \right.ight.
\]

\[
- \sum_k h_k(T_m(M_1, Q, M_2, 0)(2, k) + 1_{(M_1 = k)} + 1_{(M_2 = k)}) \right.
\]

\[
+ \left. \left( 1 - \frac{\|P(\lambda' m_1', m_2') (T_m(M_1, Q, M_2, 0), \cdot )\|_1}{\bar{\alpha}} \right) V(T_m(M_1, Q, M_2, 0)) \right\}
\]
and

\[
V((M_1, Q, M_2, 1)) + \frac{n}{\bar{\alpha}} = \\
\max_{\lambda, m_1, m_2} \left[ \frac{1}{\bar{\alpha}} \left( \langle P(\lambda_{m_1, m_2}), (M_1, Q, M_2, 1), \cdot, V \rangle + \pi(\lambda) - \sum_k h_k(Q_k + 1_{\{M_1 = k\}} + 1_{\{M_2 = k\}} - c) \right) \\
+ \left( 1 - \frac{\|P(\lambda_{m_1, m_2})((M_1, Q, M_2, 0), \cdot, V)\|_1}{\bar{\alpha}} \right) V((M_1, Q, M_2, 1)) \right],
\]

\[
\max_{\lambda', m'_1, m'_2, m} \left[ \frac{1}{\bar{\alpha}} \left( \langle P(\lambda', m'_1, m'_2), (T_m(M_1, Q, M_2, 1), \cdot, V) + \pi(\lambda') \right) \\
- \sum_k h_k(T_m(M_1, Q, M_2, 1)(2, k) + 1_{\{M_1 = k\}} + 1_{\{M_2 = k\}}) \\
+ \left( 1 - \frac{\|P(\lambda', m'_1, m'_2)(T_m(M_1, Q, M_2, 1), \cdot, V)\|_1}{\bar{\alpha}} \right) V(T_m(M_1, Q, M_2, 1)) \right]
\]

where \(\|\cdot\|_1\) denotes the \(L^1\)-norm. The optimal policies can then be derived by solving the Bellman equations.