ON RELATION BETWEEN EXPECTED REGRET AND CONDITIONAL VALUE-AT-RISK

RESEARCH REPORT # 2000-9

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Abstract

The paper compares portfolio optimization approaches with expected regret and Conditional Value-at-Risk (CVaR) utility functions. The expected regret is defined as an average portfolio underperformance comparing to a fixed target of some benchmark portfolio. By definition, CVaR is the mean of the worst x% portfolio losses in a specified time period. CVaR is also called Mean Excess Loss or Expected Shortfall. Recently, it was demonstrated that the optimization of CVaR can be performed using linear programming. We formally prove that a portfolio, which minimizes CVaR, can be obtained by doing a sensitivity analysis with respect to the threshold in the expected regret. An optimal portfolio in CVaR sense is also optimal in the expected regret sense for some threshold in the regret function. The inverse statement is also valid, i.e., if a portfolio minimizes the expected regret, this portfolio can be found by doing a sensitivity analysis with respect to the CVaR confidence level. A portfolio, optimal in expected regret sense, is also optimal in CVaR sense for some confidence level. The relation of the expected regret and CVaR minimization approaches is explained with a numerical example.

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1. Introduction

The modern portfolio optimization theory was originated by Markowitz (1952), who demonstrated that quadratic programming can be used for constructing efficient portfolios. Relatively recently, linear programming techniques, which have superior performance compared to quadratic programming, became popular in finance applications: mean absolute deviation approach, Konno and Yamazaki (1991), the regret optimization approach, Dembo and King (1992), Dembo and Rosen (1999), and the minimax approach, Young (1998). A reader interested in applications of optimization techniques in finance area can find many relevant papers in Ziemba and Mulvey (1998).

The *expected regret* optimization approach, probably one of the most popular portfolio optimization techniques, is used in various finance applications, such as, measuring underperformance of a portfolio. The expected regret is defined as an average portfolio underperformance compared to a fixed target or some benchmark portfolio. A similar concept was utilized by Carino and Ziemba (1998) in the Russell Yasuda Kasai financial planning model. In this application, several target thresholds were used and portfolio underperformance was penalized with different coefficients for various thresholds. High numerical efficiency of the approach is related to using state-of-the-art linear programming techniques.

Recently, it was demonstrated by Rockafellar and Uryasev (2000) that optimization of *Conditional Value-at-Risk* (CVaR) can be performed using linear programming. An overview of the approach can be found in Uryasev (2000). CVaR (also called Mean Excess Loss or Expected Shortfall) measures the expected losses exceeding Value-at-Risk (VaR). By definition, VaR with confidence level $\beta \in (0,1)$ is a threshold exceeded by the worst (1-β)*100% losses, while CVaR is the mean of the worst (1-β)*100% losses. A description of various methodologies for the modeling of VaR can be seen, along with related resources, at URL http://www.gloriamundi.org/. CVaR, which is a quite similar to the VaR measure of risk, has more attractive properties than VaR. CVaR is a more conservative and consistent measure of risk than VaR, see Artzner et. al (1997), Embrechts (1999), and Pflug (2000). Also, CVaR is convex with respect to portfolio positions, Rockafellar and Uryasev (2000). Several case studies showed that risk optimization with the CVaR performance function and constraints can be done with relatively small computational resources, see Rockafellar and Uryasev (2000), Palmquist, Uryasev, Krokhmal (1999), Andersson, Mausser, Rosen, Uryasev (2000). Also, performance measures similar to CVaR, such as the conditional expectation constraints and integrated chance constraints described in Prekopa (1995) have been successfully used in various engineering applications outside of finance context.

This paper compares portfolio optimization approaches with expected regret and CVaR utility functions. In order to make the comparison, regret is specified with ℓ^1-norm and it is assumed that expectations are calculated using a density function. We show that an optimal portfolio in CVaR sense is also optimal in the expected regret sense for some threshold in the regret function. The portfolio, which minimizes CVaR, can be obtained by doing a sensitivity analysis with respect to the threshold in the expected regret. An inverse statement is also valid, i.e., a portfolio, optimal in expected regret sense, is also optimal in CVaR sense for some confidence level. If a portfolio minimizes the expected regret, it can be found by doing a sensitivity analysis with respect to CVaR confidence level. We formally prove the statement on the relation of the expected regret and CVaR minimization approaches and explain statements with an example.

2. Comparison of mean regret and CVaR

2.1 Assumptions and notations

Let $f(\mathbf{x},\mathbf{y})$ be a loss function associated with the decision vector **x** and the random vector **y**. Vector **x** can be interpreted in various ways; for instance, it is a portfolio consisting of *n* instruments with positions belonging to a feasible set $X \subseteq \Re^n$. The random vector $\mathbf{y} \in \Re^m$ accounts for uncertainties in the loss function. To simplify formal analysis, it is supposed that the vector **y** is drawn from a joint density function $p(\mathbf{y})$. For each **x**, $f(\mathbf{x},\mathbf{y})$ is a random variable, since it is a function of the random vector **y**. The cumulative distribution function of $f(\mathbf{x},\mathbf{y})$ for a given **x** is denoted as

$$\Psi(\mathbf{x},\alpha) \hat{=} \int_{f(\mathbf{x},\mathbf{y})\le\alpha} p(\mathbf{y})\ d\mathbf{y}\ .$$

The function $\Psi(\mathbf{x},\alpha)$ is nondecreasing with respect to (w.r.t.) parameter α. It is assumed that $\Psi(\mathbf{x},\alpha)$ is continuous w.r.t. α. This assumption as well as the assumption about existence of density $p(\mathbf{y})$ is made for simplicity of the analysis. Without it, there are mathematical complications, even in the definition of CVaR, which would need more explanation. In some common situations, the required continuity follows from properties of loss $f(\mathbf{x},\mathbf{y})$ and the density $p(\mathbf{y})$; see Uryasev (1995). Let us denote by $G_\alpha(\mathbf{x})$ the ℓ^1-norm regret function

$$G_\alpha(\mathbf{x})\ \hat{=} \int_{\mathbf{y}\in\Re^m} [f(\mathbf{x},\mathbf{y})-\alpha]^+ p(\mathbf{y})d\mathbf{y}\ ,$$

where the integrand may be interpreted as a measure of underperformance of the portfolio w.r.t. a given benchmark α ; and the positive part operator, $(u)^+$, is defined as $max\{0,u\}$. VaR with a confidence level $\beta \in$ (0,1) is defined as a minimal value exceeded with probability 1-β,

$$\alpha_\beta(\mathbf{x})\ \hat{=} \min\{\alpha \in \Re : \Psi(\mathbf{x},\alpha) \ge \beta\}\ . \qquad (1)$$

CVaR is defined for a given confidence level β as the conditional expectation of losses exceeding VaR

$$\phi_\beta(\mathbf{x}) \,\hat{=}\, (1-\beta)^{-1} \int_{f(\mathbf{x},\mathbf{y}) \ge \alpha_\beta(x)} f(\mathbf{x},\mathbf{y}) p(\mathbf{y}) d\mathbf{y} \quad . \qquad (2)$$

2.2 Formal statements

The minimum regret problem on the feasible set X is stated as follows

$$\min_{\mathbf{x}\in X} \quad G_\alpha(\mathbf{x}) \quad . \qquad (3)$$

Similar, we formulate the CVaR minimization problem

$$\min_{\mathbf{x}\in X} \quad \phi_\beta(\mathbf{x}) \quad . \qquad (4)$$

Rockafellar and Uryasev (2000) showed that β-CVaR can be characterized in terms of the function

$$F_\beta(\mathbf{x},\alpha) \,\hat{=}\, \alpha + (1-\beta)^{-1} G_\alpha(\mathbf{x}) \quad . \qquad (5)$$

They showed that the β-CVaR of the losses associated with $\mathbf{x} \in X$ can be determined from

$$\phi_\beta(\mathbf{x}) = \min_{\alpha\in\Re} \quad F_\beta(\mathbf{x},\alpha) \qquad (6)$$

and the corresponding β-VaR equals

$$\alpha_\beta(\mathbf{x}) \;=\; \inf \quad A_\beta(\mathbf{x}) \quad , \qquad (7)$$

where

$$\mathrm{A}_\beta(\mathbf{x}) = \quad \operatorname{argmin}_{\alpha\in\Re} \quad F_\beta(\mathbf{x},\alpha). \qquad (8)$$

Therefore, CVaR minimization problem (4) may be equivalently solved using the function $F_\beta(\mathbf{x},\alpha)$,

$$\min_{\mathbf{x}\in X} \quad \phi_\beta(\mathbf{x}) \;=\; \min_{(\mathbf{x},\alpha)\in X\times\Re} \quad F_\beta(\mathbf{x},\alpha) \; . \qquad (9)$$

Let us denote by $S_\beta^* \,\hat{=}\, \operatorname{argmin}_{(\mathbf{x},\alpha)\in X\times\Re} F_\beta(\mathbf{x},\alpha)$ a solution set of the second optimization problem in (9), a solution set of the CVaR optimization problem by $X_\beta^{*C} \,\hat{=}\, \operatorname{argmin}_{\mathbf{x}\in X}\phi_\beta(\mathbf{x})$, a solution set of the minimum regret problem by $X_\alpha^{*R} \,\hat{=}\, \operatorname{argmin}_{\mathbf{x}\in X} G_\alpha(\mathbf{x})$, and a projection of S_β^* on α line by $A_\beta \,\hat{=}\, \{\alpha : \textit{there exist } \mathbf{x} \textit{ such that } (\mathbf{x},\alpha) \in S_\beta^*\}$.

For a two dimension example (n = 2), the relations between the defined sets are illustrated with the following Figure 1. It is considered that

$$X \,\hat{=}\, \{x_1 + x_2 = 1,\; x_1 \ge 0,\; x_2 \ge 0\}.$$

Figure 1 displays X , S_β^* (the shaded region), and the solution set S_β^* belonging to $X \times \Re$. It is shown that A_β is a projection of S_β^* on α line, and X_β^{*C} is a projection of S_β^* on X. Also, it is shown that the solution set $X_{\bar{\alpha}}^{*R}$ of the minimum regret problem (3) for some $\bar{\alpha} \in A_\beta$ is contained in X_β^{*C} .

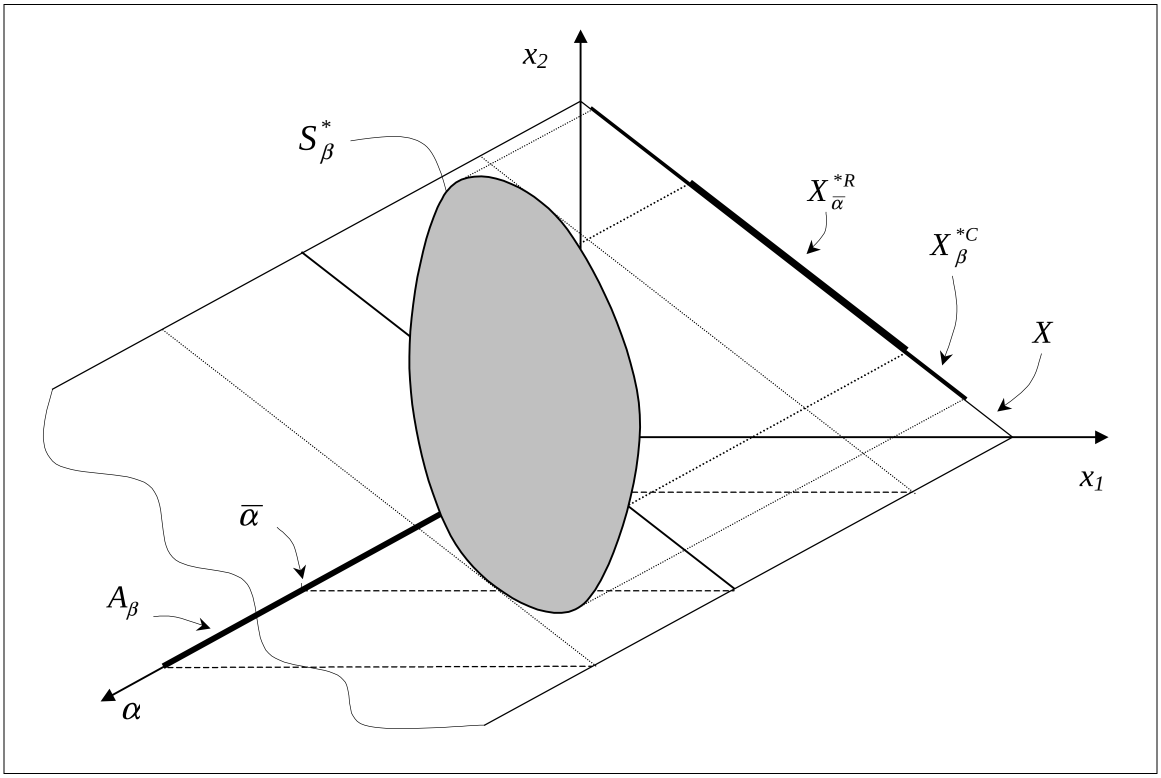

Figure 1. Two dimension example illustrating basic definitions.

Further, we formulate a theorem stating that for each CVaR optimization problem (4), there is a regret optimization problem (3) having the same set of portfolio solutions.

Theorem 1. (**CVaR ⇒ Regret**) *For any* $\beta \in (0,1)$ *and* $\mathbf{x}^* \in X_\beta^{*C}$ *there exists a pair* $(\mathbf{x}^*, \alpha^*) \in S_\beta^*$ *such that* $\mathbf{x}^* \in X_{\alpha^*}^{*R}$.

Proof. Equality (9) implies (see Theorem 2 in Rockafellar and Uryasev (2000)) that for $\mathbf{x}^* \in X_\beta^{*C}$ there exists $(\mathbf{x}^*, \alpha^*) \in S_\beta^*$; therefore, we need to prove only that $\mathbf{x}^* \in X_{\alpha^*}^{*R}$.

Indeed, if

$$(x^*, \alpha^*) \in \operatorname{argmin}_{(\mathbf{x},\alpha) \in X \times \Re} F_\beta(\mathbf{x}, \alpha) \ ,$$

then

$$x^* \in \operatorname{argmin}_{\mathbf{x}\in X} F_\beta(\mathbf{x},\alpha^*) \;=\; \operatorname{argmin}_{\mathbf{x}\in X} \left\{\alpha^* + (1-\beta)^{-1} G_{\alpha^*}(\mathbf{x})\right\}$$

$$= \quad \operatorname{argmin}_{\mathbf{x}\in X} G_{\alpha^*}(\mathbf{x}) \;=\; X_{\alpha^*}^{*R}.$$

The theorem is proved.

The next theorem proves an inverse statement to Theorem 1, i.e., for each regret optimization problem (3) there exists an equivalent CVaR optimization problem (4).

Theorem 2. (**Regret $\Rightarrow$ CVaR**) *For any* $\alpha \in \Re$ *and* $\mathbf{x}^* \in X_\alpha^{*R}$ *there exists a unique* $\beta \in (0,1)$ *such that* $\alpha \in A_\beta(\mathbf{x}^*)$, $(\mathbf{x}^*,\alpha) \in S_\beta^*$, and $\mathbf{x}^* \in X_\beta^{*C}$.

Proof. Since $\Psi(\mathbf{x}^*,\alpha)$ is continuous with respect to α, there exists $\beta \in (0,1)$ such that $\Psi(\mathbf{x}^*,\alpha) = \beta$. The derivative of the function $F_\beta(\mathbf{x}^*,\alpha)$ with respect to α equals (see proof of Theorem 1 in Rockafellar and Uryasev (2000))

$$\frac{\partial}{\partial\alpha} F_\beta(\mathbf{x}^*,\alpha) = (1-\beta)^{-1}\left(\Psi(\mathbf{x}^*,\alpha) - \beta\right) = 0\,.$$

The function $F_\beta(\mathbf{x}^*,\alpha)$ is convex w.r.t. α (see Theorem 2 in Rockafellar and Uryasev (2000)). Therefore,

$$\alpha \in A_\beta(\mathbf{x}^*) = \operatorname{argmin}_{\tau\in\Re} F_\beta(\mathbf{x}^*,\tau)\,, \tag{10}$$

and the first statement of the theorem is proved.

Also, since $\mathbf{x}^* \in X_\alpha^{*R}$, we

$$\mathbf{x}^* \in \operatorname{argmin}_{\mathbf{x}\in X} G_\alpha(\mathbf{x}) = \operatorname{argmin}_{\mathbf{x}\in X} \left\{\alpha + (1-\beta)^{-1} G_\alpha(\mathbf{x})\right\} =$$

$$= \operatorname{argmin}_{\mathbf{x}\in X} F_\alpha(\mathbf{x},\alpha)\;. \tag{11}$$

Inclusions (10) and (11) imply that $(\mathbf{x}^*,\alpha) \in \operatorname{argmin}_{(\mathbf{x},\tau)\in X\times\Re} F_\beta(\mathbf{x},\tau) = S_\beta^*$, and the second statement of the theorem is proved. Finally, the last statement $\mathbf{x}^* \in X_\beta^{*C}$ follows from Theorem 2 in Rockafellar and Uryasev (2000).

The following corollary considers the case when $S_\beta^* = X_\beta^{*C} \times A_\beta$. For instance, this is valid when the set A_β consists only from one VaR point $\alpha_\beta(\mathbf{x})$.

Corollary. *If in addition to conditions of Theorem 2,* $S_\beta^* = X_\beta^{*C} \times A_\beta$, *then* $X_\beta^{*C} = X_\alpha^{*R}$ *for any* $\alpha^* \in A_\beta$.

Proof. The statement follows from (11).

3. Numerical example

In this section, we illustrate the formal statements with a numerical example demonstrating equivalence of the expected regret and CVaR approaches. For numerical calculations, we used the implementation framework described by Rockafellar and Uryasev (2000).

We considered a portfolio of n = 1792 listed stocks. The decision vector **x** consists of positions of stocks in the portfolio. Components of the vector $\mathbf{y} \in \Re^m$ are random returns of instruments and m=n. Distribution of the vector $\mathbf{y} \in \Re^m$ is modeled by s = 156 historical weekly returns, $\{\mathbf{y}^1, ..., \mathbf{y}^s\}$. The feasible set X is a convex polytope given by the following constraints:

$x_i \ge 0, \quad i = 1,...,n$ (no short positions),

$\sum_{i=1}^n x_i = 1$ (normalization constraint)

$s^{-1} \sum_{j=1}^s \mathbf{x}^T \mathbf{y}^j \ge R$ (lower bound on expected return).

The loss function for a scenario j is given by $f(\mathbf{x}, \mathbf{y}^j) = \mathbf{x}^T(-\mathbf{y}^j)$, which is the negative return on portfolio **x**. The corresponding approximations to $G_\alpha(\mathbf{x})$ and $F_\beta(\mathbf{x}, \alpha)$ are

$$\tilde{G}_\alpha(\mathbf{x}) \;\hat{=}\; \frac{1}{s} \sum_{j=1}^s \left[-\mathbf{x}^T \mathbf{y}^j - \alpha\right]^+ ,$$

and

$$\tilde{F}_\beta(\mathbf{x}, \alpha) \;\hat{=}\; \alpha + (1-\beta)^{-1} \tilde{G}_\alpha(\mathbf{x}) .$$

Then, the problems of minimizing $\tilde{G}_\alpha(\mathbf{x})$ on X and minimizing $\tilde{F}_\beta(\mathbf{x}, \alpha)$ on $X \times \Re$ are convex programming problems, which can be reduced to linear programming using additional variables.

Distribution characteristics over the population of instruments of the instrument returns are shown in Table 1.

Table 1. Distribution characteristics for the population of instruments of average instrument returns in 156 weeks (minimum of instrument returns, maximum of instrument returns, mean of instrument returns, standard deviation of instrument returns, skewness of instrument returns, kurtosis of instrument returns)

Minimum	Maximum	Mean	Std. Dev.	Skewness	Kurtosis
-0.30684	0.327555	0.003048	0.078163	-0.259866	9.568777

For the considered portfolio of stocks, we conducted numerical experiments comparing minimum regret and minimum CVaR approaches. We specified a set of minimum regret problems by making a grid in parameter $\overline{\alpha}$ (50 $\overline{\alpha}$ values). Further, for each value $\overline{\alpha}$, a sensitivity analysis with respect to β was performed to find a matching CVaR problem with $min\,|\alpha - \overline{\alpha}|$. The target weekly return was set to R = 0.003. The 50 $\overline{\alpha}$ values produced β ranging approximately between 0.75 and 0.98 . The minimum regret and CVaR models were solved using linear programming (see description of the linear programming formulation of CVaR problem in Rockafellar and Uryasev (2000)). The search procedure ($\alpha^* = \text{argmin}\,|\alpha - \overline{\alpha}|$) and linear programming were implemented with GAMS, Brooke at al (1992) and solved with the CPLEX's mathematical programming library, ILOG (1997), in a personal computer with a Pentium-II 300 MHz processor and 128 MB memory. Table 2 depicts the summary of numerical experiments. The numerical experiments indicated that the expected regret solution can be found quite precisely by doing a sensitivity analysis with respect to β in the CVaR minimization approach. The relative differences of portfolio norms with two considered approaches were less that 1%. Also, there is a close correspondence of α^* and $\overline{\alpha}$ values: the relative difference is less than 5% or the absolute difference is less than 0.0002. Figure 2 depicts a summary of numerical comparisons of minimum regret and CVaR approaches.

3. Conclusion

This paper demonstrated that the minimum expected regret and the minimum CVaR approaches are closely related. For the case with ℓ^1-norm and constant target value, a portfolio optimal in expected regret sense can be obtained by a sensitivity analysis w.r.t. the confidence parameter in the minimum CVaR approach. Also, the inverse statement is valid, i.e., a portfolio optimal in CVaR sense can be obtained by a sensitivity analysis w.r.t. the target value in the minimum expected regret approach. Numerical experiments confirmed the formal mathematical statements. Also, numerical experiments demonstrated that both approaches, minimum regret and minimum CVaR, can be very efficiently implemented using small computational resources, such as a Pentium 300 MHz computer.

Table 2. Comparison of the minimum expected regret and CVaR solutions. Regret heading: $\overline{\alpha}$ value, solution size = number of non-zero components in the solution, solution norm; CVaR heading: β value, α solution value, solution size, and solution norm; Comparison heading: relative alpha difference and relative solution norm difference.

Regret			CVaR				Comparison	
$\overline{\alpha}$	Sol. size	$\left\Vert \mathbf{x}_{\overline{\alpha}}^{*R} \right\Vert_2$	β	α^*	Sol. size	$\left\Vert \mathbf{x}_{\beta}^{*C} \right\Vert_2$	$\frac{\alpha^* - \overline{\alpha}}{\lvert\overline{\alpha}\rvert}$	$\frac{\left\Vert \mathbf{x}_{\beta}^{*C} - \mathbf{x}_{\overline{\alpha}}^{*R} \right\Vert_2}{\left\Vert \mathbf{x}_{\overline{\alpha}}^{*R} \right\Vert_2}$
-0.0011884	54	0.546686	0.7363	-0.0011891	53	0.546485	-0.056%	0.083%
-0.0011033	53	0.545907	0.7393	-0.0011066	52	0.546145	-0.298%	0.212%
-0.0010182	51	0.541216	0.7435	-0.0010102	50	0.541307	0.780%	0.327%
-0.0009330	49	0.541990	0.7589	-0.0009313	49	0.542439	0.182%	0.121%
-0.0008479	47	0.543070	0.7728	-0.0008423	46	0.542642	0.664%	0.147%
-0.0007628	48	0.542182	0.7754	-0.0007661	47	0.542092	-0.433%	0.107%
-0.0006777	48	0.540022	0.7781	-0.0006798	47	0.539613	-0.310%	0.110%
-0.0005925	49	0.549551	0.7803	-0.0006175	48	0.548881	-4.206%	0.881%
-0.0005074	47	0.551212	0.7864	-0.0005093	46	0.551060	-0.365%	0.047%
-0.0004223	47	0.555201	0.7871	-0.0004244	47	0.555241	-0.512%	0.032%
-0.0003372	48	0.556890	0.7901	-0.0003281	48	0.557170	2.702%	0.118%
-0.0002520	48	0.560760	0.7930	-0.0002636	47	0.559461	-4.579%	0.630%
-0.0001669	49	0.565039	0.7958	-0.0001496	49	0.566291	10.34%	0.599%
-0.0000818	49	0.569552	0.7996	-0.0000861	49	0.569453	-5.282%	0.062%
0.0000033	49	0.570694	0.7999	0.0000104	48	0.570732	211.0%	0.130%
0.0000885	49	0.573080	0.8013	0.0000956	48	0.573292	8.015%	0.137%
0.0001736	47	0.571145	0.8073	0.0001614	46	0.571929	-6.994%	0.332%
0.0002587	47	0.570329	0.8164	0.0002591	46	0.570296	0.162%	0.037%
0.0003438	46	0.570130	0.8213	0.0003288	46	0.570573	-4.380%	0.269%
0.0004290	46	0.568194	0.8230	0.0004290	46	0.568193	0.019%	0.002%
0.0005141	45	0.572725	0.8267	0.0005320	45	0.574391	3.492%	0.646%
0.0005992	46	0.572726	0.8268	0.0006008	46	0.572689	0.262%	0.042%
0.0006843	48	0.566529	0.8286	0.0006815	48	0.566545	-0.412%	0.045%
0.0007695	46	0.566841	0.8335	0.0007707	45	0.566861	0.168%	0.023%
0.0008546	44	0.567263	0.8358	0.0008558	43	0.567181	0.138%	0.026%
0.0009397	43	0.572998	0.8430	0.0009403	43	0.573061	0.060%	0.023%
0.0010248	43	0.569597	0.8584	0.0010167	42	0.569538	-0.796%	0.139%
0.0011100	44	0.570064	0.8586	0.0011250	44	0.570147	1.359%	0.245%
0.0011951	43	0.573454	0.8672	0.0011681	43	0.572051	-2.256%	0.469%

0.0012802	43	0.570857	0.8713	0.0013021	43	0.568413	1.708%	0.695%
0.0013653	44	0.566464	0.8784	0.0013723	44	0.566491	0.513%	0.181%
0.0014504	42	0.566328	0.8846	0.0014407	42	0.566368	-0.670%	0.109%
0.0015356	45	0.566256	0.8898	0.0015535	45	0.566520	1.170%	0.194%
0.0016207	43	0.563752	0.8965	0.0016232	42	0.563500	0.155%	0.124%
0.0017058	42	0.555593	0.9004	0.0017006	41	0.556432	-0.307%	0.212%
0.0017909	41	0.547912	0.9014	0.0017907	41	0.547814	-0.014%	0.037%
0.0018761	42	0.544711	0.9017	0.0018901	41	0.544074	0.750%	0.283%
0.0019612	42	0.536406	0.9063	0.0019404	41	0.536946	-1.060%	0.532%
0.0020463	42	0.532997	0.9074	0.0020672	42	0.532065	1.021%	0.342%
0.0021314	42	0.529311	0.9076	0.0021650	41	0.527889	1.576%	0.560%
0.0022166	41	0.525621	0.9101	0.0022211	41	0.525570	0.205%	0.131%
0.0023017	40	0.520152	0.9127	0.0022944	39	0.520477	-0.318%	0.149%
0.0023868	40	0.516449	0.9127	0.0024482	40	0.513859	2.571%	1.255%
0.0024719	41	0.515979	0.9134	0.0024780	41	0.516526	0.245%	0.249%
0.0025571	39	0.509568	0.9141	0.0025592	38	0.509469	0.083%	0.042%
0.0026422	38	0.511696	0.9214	0.0026316	37	0.511280	-0.402%	0.479%
0.0027273	37	0.514012	0.9238	0.0027237	37	0.514044	-0.132%	0.135%
0.0028124	37	0.519918	0.9258	0.0028086	36	0.519317	-0.135%	0.198%
0.0028976	38	0.514950	0.9434	0.0028908	37	0.514028	-0.232%	0.436%
0.0029827	38	0.515775	0.9453	0.0029683	37	0.517958	-0.483%	0.691%
0.0030678	38	0.503796	0.9688	0.0030676	37	0.503834	-0.005%	0.023%

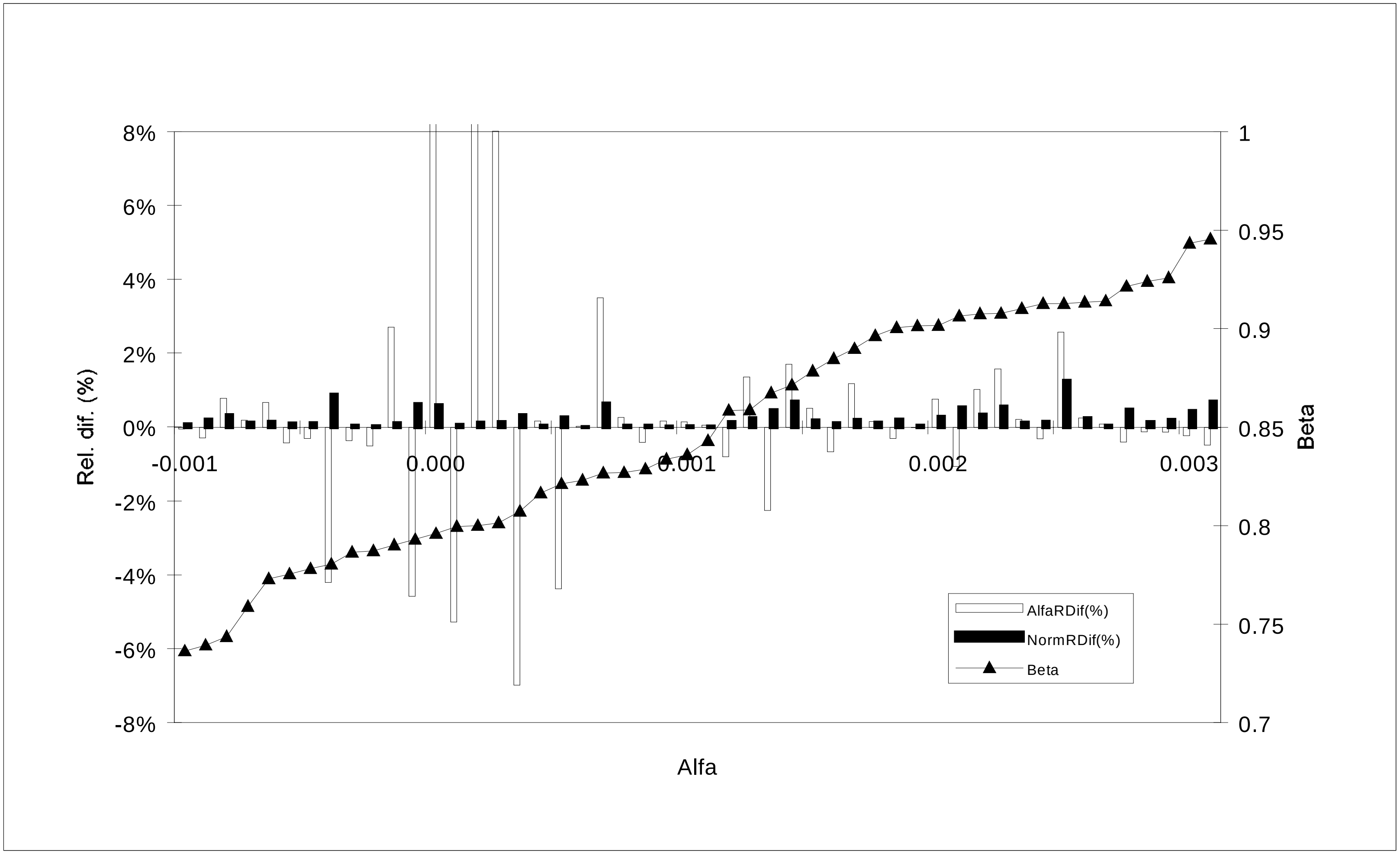

Figure 2 Alpha and solution norm relative differences between minimum regret and CVaR for different $\overline{\alpha}$ values and associated β.

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