OPTIMIZATION OF CONDITIONAL VALUE-AT-RISK

RESEARCH REPORT #99-4

Stanislav Uryasev[1] and R. Tyrrell Rockafellar[2]

Center for Applied Optimization
Dept. of Industrial and Systems Engineering
University of Florida, Gainesville, FL 32611

Date: June 23, 1999

Correspondence should be addressed to: Stanislav Uryasev

Abstract

A new approach for optimization or hedging of a portfolio of finance instruments to reduce the risks of high losses is suggested and tested with several applications. As a measure of risk, the expected loss exceeding Value-at-Risk (VaR) is used. This measure is called Conditional VaR (CVaR), Mean Excess Loss, Mean Shortfall, or Tail VaR. CVaR is considered a more consistent measure of risk than VaR. Portfolios with low CVaR also have low VaR because CVaR is greater than VaR. The approach is based on a new representation of the performance function which allows simultaneous calculation of the VaR and minimization of the CVaR. It can be used in conjunction with analytical or scenario based optimization algorithms. If the number of scenarios is fixed, the problem is reduced to a Linear Programming or Nonsmooth Optimization Problem. These techniques allow to optimize portfolios with large numbers of instruments. The approach is tested with two examples: (1) portfolio optimization and comparison with the Minimum Variance approach; (2) hedging of an options portfolio. The suggested methodology can be used for optimizing of portfolios by investment companies, brokerage firms, mutual funds, and any business which evaluates risks. Although the approach is used for portfolio analyses, it is very general and can be applied to any financial or non-financial problems involving optimization of percentiles.

[1]University of Florida, Dept. of Industrial and Systems Engineering, PO Box 116595, 303 Weil Hall, Gainesville, FL 32611-6595, E-mail: uryasev@ise.ufl.edu, URL: http://www.ise.ufl.edu/uryasev

[2]University of Washington, Dept. of Applied Mathematics, 408 L Guggenheim Hall, Box 352420, Seattle, WA 98195-2420, E-mail: rtr@math.washington.edu, URL: www.amath.washington.edu/people/faculty/rockafellar/

1 Introduction

This paper develops a new approach for optimizing (reducing) the risks of high losses. As a measure of the risk we consider the expected value of loss exceeding β-Value-at-Risk (β-VaR). We call such measure of risk by Conditional Value-at-Risk (CVaR). Also, it is called Mean Excess Loss, Mean Shortfall, or Tail VaR. By definition, β-VaR is a value that with probability β the loss will not exceed β-VaR. See URL http://www.gloriamundi.org/ for description of various methodologies for modeling of VaR and related resources. Mostly, approaches for calculation of VaR are based on linear approximation of the portfolio risks and assume a joint (log)normal distribution of the underlying market parameters [21, 14, 15]. However, simulation base tools are used when the portfolio contains non-linear instruments, such as options [17, 5]. Although VaR is a very popular measure of risk, it has undesirable properties [1, 2] such as non-subadditivity and non-convexity. Sub-additivity means that the risk of having a portfolio consisting of two instruments is less or equal then sum of individual risks of these two instruments. Also, VaR is difficult to optimize when it is calculated using the scenarios approach. In this case, VaR is nonconvex and nonsmooth as a function of positions and it has multiple local extrema [17]. An alternative measure of extreme losses, with better properties, is β-CVaR [1, 9]. Although it has not become a standard in the finance industry, CVaR is likely to play a major role as it currently does in the insurance industry [10]. Also, CVaR is used in credit risk studies [5]. Risk measure CVaR is very closely related to VaR. The minimization of CVaR leads, also, to minimization of VaR because CVaR is greater than or equal to VaR. Moreover, for portfolios with a normal return/loss distribution, these two measures are equivalent (see Example 1 in this paper). The CVaR can be used in conjunction with VaR and is applicable to estimation of risks in various areas:

- it can be applied to linear and nonlinear derivatives (options, futures);
- it can be used for evaluation of market, credit, and operational risks;
- it can be applied to any corporation which is exposed to financial risks.

From a mathematical point of view β-VaR is the β-quantile of the loss distribution,i.e., probability that loss exceeds β-VaR equals 1-β. Accordingly, β-CVaR is the expected value of $(1-\beta)*100\%$ of highest losses. Usually, three values of coefficient β are considered, β=0.90, β=0.95 and β=0.99.

The approach considered in the paper can be relatively easily implemented and used for very large portfolios by investment companies, brokerage firms, mutual funds, and any business which evaluates risks of losses. It can be used in conjunction with analytical or simulation based techniques. The approach is based on a new algorithm which simultaneously calculates the VaR and minimizes the CVaR. Generally, this problem is a nonsmooth stochastic optimization problem which can be solved using general stochastic optimization approaches [18, 8, 20]. Also, further we show that for portfolio optimization and hedging, in case of finite number of scenarios, this problem can be solved using nonsmooth optimization or linear programming techniques.

2 Approach

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to R$ is the loss function which depends upon the control vector $\mathbf{x} \in \mathbb{R}^n$ and the random vector $\mathbf{y} \in \mathbb{R}^m$. We use bold face for the vectors to distinguish them from scalars. We consider that the random vector $\mathbf{y}$ has the probability distribution function $p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ which depends upon the parameter vector $\mathbf{x}$. Although existence of the density is not critical for the considered approach, we make this assumption to simplify the analysis. Denote by $\Psi(\mathbf{x}, a)$ the probability function

$$\Psi(\mathbf{x}, \alpha) = \int_{f(\mathbf{x},\mathbf{y}) \leq \alpha} p(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \ , \tag{1}$$

which by definition is the probability that the loss function $f(\mathbf{x}, \mathbf{y})$ does not exceed some threshold value α. The quantile function $\alpha(x, \beta)$ which is also called Value-at-Risk is defined as follows

$$\alpha(\mathbf{x}, \beta) = \min\{\alpha \in \mathbb{R} : \Psi(\mathbf{x}, \alpha) \geq \beta\} \ . \tag{2}$$

We consider the following CVaR performance function $\frac{1}{1-\beta}\Phi(\mathbf{x})$ which is the conditional expected value of the loss $f(\mathbf{x}, \mathbf{y})$ under the condition that it exceeds the quantile $\alpha(\mathbf{x}, \beta)$, where

$$\Phi(\mathbf{x}) = \int_{f(\mathbf{x},\mathbf{y}) \geq \alpha(\mathbf{x},\beta)} f(\mathbf{x}, \mathbf{y}) \, p(\mathbf{x}, \mathbf{y}) \, d\mathbf{y} \ . \tag{3}$$

We consider that the decision vector $\mathbf{x}$ belongs to the feasible set $X \subset \mathbb{R}^n$ (for instance, this can be a set vectors with the mean return exceeding 10%). Further, we will show that the minimization of the excess loss function $\frac{1}{1-\beta}\Phi(\mathbf{x})$ on the feasible set $X \subset \mathbb{R}^n$ can be reduced to the minimization of the function

$$F(\mathbf{x}, \alpha) = (1 - \beta)\alpha + \int_{f(\mathbf{x},\mathbf{y}) \geq \alpha} (f(\mathbf{x}, \mathbf{y}) - \alpha) \, p(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}$$

on the set $X \times \mathbb{R}$. We will prove that

$$\min_{\alpha \in \mathbb{R}} F(\mathbf{x}, \alpha) \ = \ \Phi(\mathbf{x})$$

and the optimal solution α of this problem is VaR. Consequently,

$$\min_{\mathbf{x} \in X, \alpha \in \mathbb{R}} F(\mathbf{x}, \alpha) \ = \ \min_{\mathbf{x} \in X} \min_{\alpha \in \mathbb{R}} F(\mathbf{x}, \alpha) \ = \ \min_{\mathbf{x} \in X} F(\mathbf{x}, \alpha(\mathbf{x}, \beta)) \ = \ \min_{\mathbf{x} \in X} \Phi(\mathbf{x}) \ .$$

Thus, by minimizing function $F(\mathbf{x}, \alpha)$ we can simultaneously find the VaR and optimal CVaR. Under general conditions the function $F(\mathbf{x}, \alpha)$ is smooth [24]. The function $F(\mathbf{x}, \alpha)$ can be expressed equivalently as follows

$$F(\mathbf{x}, \alpha) = (1 - \beta)\alpha + \int_{\mathbf{y} \in \mathbb{R}^m} (f(\mathbf{x}, \mathbf{y}) - \alpha)^+ \, p(\mathbf{x}, \mathbf{y}) \; d\mathbf{y} \ , \tag{4}$$

where b^+ is the positive part of the number b, i.e., $b^+ = \max\{0, b\}$.

It can be verified that the function $F(\mathbf{x}, \alpha)$, given by equation (4), is convex with respect to (w.r.t.) α. Definition of the convex function can be found, for instance, in [22, 23]. Also, the function $F(\mathbf{x}, \alpha)$ is convex w.r.t. $\mathbf{x}$, if the function $f(\mathbf{x}, \mathbf{y})$ is convex w.r.t. $\mathbf{x}$ and density $p(\mathbf{x}, \mathbf{y})$ does not depend upon $\mathbf{x}$, i.e., $p(\mathbf{x}, \mathbf{y}) = p(\mathbf{y})$. To calculate the integral function $F(\mathbf{x}, \alpha)$ we can use various approaches. If the integral in formula (4) can be calculated or approximated analytically, then to optimize function $F(\mathbf{x}, \alpha)$ we can use nonlinear programming techniques. Also, we consider the case when this integral is approximated using scenarios $\mathbf{y}_j, \ j = 1, \ldots, J$, which are sampled with the density function $p(\mathbf{x}, \mathbf{y}) = p(\mathbf{y})$ which does not depend upon $\mathbf{x}$, i.e.,

$$\int_{\mathbf{y} \in \mathbb{R}^m} \big(f(\mathbf{x}, \mathbf{y}) - \alpha \big)^+ \, p(\mathbf{y}) \; d\mathbf{y} \ \approx \ J^{-1} \sum_{j=1}^{J} \big(f(\mathbf{x}, \mathbf{y}_j) - \alpha \big)^+ \ .$$

If the loss function $f(\mathbf{x}, \mathbf{y}_j)$ is convex, and the feasible set X is convex, then we should solve the following convex optimization problem

$$\tilde{F}(\mathbf{x}, \alpha) \ \stackrel{def}{=} \ (1 - \beta)\alpha \ + \ J^{-1} \sum_{j=1}^{J} \big(f(\mathbf{x}, \mathbf{y}_j) - \alpha \big)^+ \ \rightarrow \ \min_{\mathbf{x} \in X, \alpha \in \mathbb{R}} \tag{5}$$

By solving the last problem we find the optimal portfolio vector, $\mathbf{x}^*$, corresponding VaR,which equals to α^*, and the optimal CVaR, which equals to $\frac{1}{1-\beta} \tilde{F}(\mathbf{x}^*, \alpha^*)$. Moreover, if the loss function $f(\mathbf{x}, \mathbf{y}_j)$ is linear w.r.t. $\mathbf{x}$, and the set X is given by linear (in)equalities, then we can reduce optimization problem (5) to the linear programming problem

$$(1-\beta)\alpha \; + \; J^{-1} \sum_{j=1}^{J} z_j \quad \rightarrow \quad \min_{\mathbf{x}\in\mathbb{R}^n, \mathbf{z}\in\mathbb{R}^J, \alpha\in\mathbb{R}} \tag{6}$$

subject to constraints

$$\mathbf{x} \in X \tag{7}$$

$$z_j \;\geq\; f(\mathbf{x},\mathbf{y}_j) - \alpha \;, \quad z_j \;\geq\; 0 \;, \quad j = 1,\ldots,J, \tag{8}$$

where z_j, $j = 1,\ldots,J$ are dummy variables.

Also, if the loss function $f(\mathbf{x},\mathbf{y}) = |\tilde{f}(\mathbf{x},\mathbf{y})|$ is the module of a function $\tilde{f}(\mathbf{x},\mathbf{y})$ which is linear w.r.t. $\mathbf{x}$, and the set X is given by linear (in)equalities, then, similar to the previous case, we can reduce the optimization problem (5) to a linear programming problem.

3 Key Formal Results

This section formally justifies the statements presented in the previous section. We consider that the function $\Psi(\mathbf{x},\alpha)$ given by equation (1) is continuous w.r.t. α on $\mathbb{R}$ for all $\mathbf{x} \in X$. The probability function $\Psi(\mathbf{x},\alpha)$ monotonically increases w.r.t. α and takes values between 0 and 1. Consequently, for each $x \in X$ and β, belonging to the interval $0 < \beta < 1$, the set

$$A(\mathbf{x},\beta) \stackrel{\text{def}}{=} \{\alpha : \Psi(\mathbf{x},\alpha) = \beta)\} \tag{9}$$

is not empty and the quantile $\alpha(\mathbf{x},\beta)$ equals

$$\alpha(\mathbf{x},\beta) \;=\; \min\{\alpha : \alpha \in A(\mathbf{x},\beta)\} \;. \tag{10}$$

We start formal analysis with the proof that $A(\mathbf{x},\beta)$ is the set of minimal points of the function $F(\mathbf{x},\alpha)$ w.r.t. α. If there is only one minimum point, then it is the quantile $\alpha(\mathbf{x},\beta)$.

Theorem 3.1 *Let the function $\Psi(\mathbf{x},\alpha)$ be continuous w.r.t. α on $\mathbb{R}$ for all $\mathbf{x} \in X$, the function $F(\mathbf{x},\alpha)$ be differentiable w.r.t. α on $\mathbb{R}$ for all $\mathbf{x} \in X$, then*

$$A(\mathbf{x},\beta) \;=\; \{\alpha : F(\mathbf{x},\alpha) = \min_{\alpha\in\mathbb{R}} F(\mathbf{x},\alpha)\} \;, \tag{11}$$

and

$$\alpha(\mathbf{x},\beta) \;=\; \min\{\alpha : \alpha \in A(\mathbf{x},\beta)\} \;. \tag{12}$$

Proof of Theorem 3.1. Convexity of the function $f(\mathbf{x},\mathbf{y}) - \alpha)^+$ w.r.t. α implies convexity of the function $F(\mathbf{x},\alpha)$ w.r.t. α. The function $F(\mathbf{x},\alpha)$ is convex and differentiable w.r.t. α. Therefore, minimum points of the function $F(\mathbf{x},\alpha)$ w.r.t. α can be found by equating the partial derivative $\nabla_\alpha F(\mathbf{x},\alpha)$ to zero. Partial derivative w.r.t α equals

$$\nabla_\alpha F(\mathbf{x},\alpha) \;=\; \nabla_\alpha \left((1-\beta)\alpha + \int\limits_{\mathbf{y}\in\mathbb{R}^m} (f(\mathbf{x},\mathbf{y}) - \alpha)^+ \, p(\mathbf{x},\mathbf{y}) \, d\mathbf{y} \right)$$

$$= \; (1-\beta) + \nabla_\alpha \int\limits_{\mathbf{y}\in\mathbb{R}^m} (f(\mathbf{x},\mathbf{y}) - \alpha)^+ \, p(\mathbf{x},\mathbf{y}) \, d\mathbf{y} \;=\; (1-\beta) + \int\limits_{f(\mathbf{x},\mathbf{y})\geq\alpha} \nabla_\alpha (f(\mathbf{x},\mathbf{y}) - \alpha) \, p(\mathbf{x},\mathbf{y}) \, d\mathbf{y}$$

$$= \; (1-\beta) - \int\limits_{f(\mathbf{x},\mathbf{y})\geq\alpha} p(\mathbf{x},\mathbf{y}) \, d\mathbf{y} \;=\; (1-\beta) - (1 - \Psi(\mathbf{x},\alpha)) \;=\; \Psi(\mathbf{x},\alpha) - \beta \, .$$

This means that the partial derivative $\nabla_\alpha F(\mathbf{x},\alpha)$ equals zero if and only if

$$\Psi(\mathbf{x},\alpha) - \beta = 0$$

which proves (11).

The function $\Psi(\mathbf{x},\alpha)$ defined by equation (1) is continuous and monotone w.r.t. α. It takes all values in the open interval $]0,1[$. Therefore, for each $\mathbf{x} \in X$ and $0 < \beta < 1$ the equation

$$\Psi(\mathbf{x},\alpha) = \beta$$

has at least one solution w.r.t. α. By definition of the quantile, it is the lowest value which satisfies the last equation, i.e. we have (12). $\diamond$

Theorem 3.2 *If all conditional of Theorem 3.1 are valid, then for any* $\mathbf{x} \in X$ *the function* $F(\mathbf{x},\alpha)$ *is constant for* $\alpha \in A(\mathbf{x},\beta)$ *and*

$$\Phi(\mathbf{x}) = F(\mathbf{x},\alpha) \, , \quad \alpha \in A(\mathbf{x},\beta) \, . \tag{13}$$

Before proving Theorem 3.2 let us point out that Theorems 3.1 and 3.2 imply that minimization of the function $\Phi(\mathbf{x})$ on the set X can be reduced to the minimization of the function $F(\mathbf{x},\alpha)$ on the set $X \times \mathbb{R}$. Indeed,

$$\min_{\mathbf{x}\in X,\alpha\in\mathbb{R}} F(\mathbf{x},\alpha) \;=\; \min_{\mathbf{x}\in X} \min_{\alpha\in\mathbb{R}} F(\mathbf{x},\alpha) \;=\; \min_{\mathbf{x}\in X} F(\mathbf{x},\alpha(\mathbf{x},\beta)) \;=\; \min_{\mathbf{x}\in X} \Phi(\mathbf{x}) \, .$$

Suppose, using optimization methods, we found the optimal vector $(\mathbf{x}^*, \alpha^*)$, i.e.,

$$F(\mathbf{x}^*, \alpha^*) = \min_{\mathbf{x}\in X, \alpha\in \mathbb{R}} F(\mathbf{x}, \alpha) \ .$$

Then, $\mathbf{x}^*$ is a minimum point of the function $\Phi(\mathbf{x})$ on the set X. The quantile $\alpha(\mathbf{x}, \beta)$ can be found using the line search, i.e.,

$$\alpha(\mathbf{x}, \beta) \ = \ \text{argmin}\{\alpha : F(\mathbf{x}^*, \alpha) = F(\mathbf{x}^*, \alpha^*)\} \ .$$

Proof of Theorem 3.2. Theorem 3.1 implies that the function $F(\mathbf{x}, \alpha)$ is constant on the set $A(\mathbf{x}, \beta)$. Since the quantile $\alpha(\mathbf{x}, \beta) \in A(\mathbf{x}, \beta)$ to prove (13) it is enough to show that

$$\Phi(\mathbf{x}) = F(\mathbf{x}, \alpha(\mathbf{x}, \beta)) \ .$$

By definition of the function $F(\mathbf{x}, \alpha)$ we have

$$F(\mathbf{x}, \alpha(\mathbf{x}, \beta)) \ = \ (1-\beta)\alpha(\mathbf{x}, \beta) + \int\limits_{f(\mathbf{x},\mathbf{y})\geq \alpha(\mathbf{x},\beta)} (f(\mathbf{x}, \mathbf{y}) - \alpha(\mathbf{x}, \beta))\, p(\mathbf{x}, \mathbf{y}) \ d\mathbf{y} \ . \qquad (14)$$

Equations (9) and (10) imply that

$$\beta \ = \ \Psi(\mathbf{x}, \alpha(\mathbf{x}, \beta)) \ .$$

Consequently, the first term in the right handside of equation (14) equals

$$(1-\beta)\, \alpha(\mathbf{x}, \beta) \ = \ \alpha(\mathbf{x}, \beta)\, (1 - \Psi(\mathbf{x}, \alpha(\mathbf{x}, \beta)))$$

$$= \ \alpha(\mathbf{x}, \beta) \left(1 - \int\limits_{\alpha(\mathbf{x},\beta)\leq f(\mathbf{x},\mathbf{y})} p(\mathbf{x}, \mathbf{y}) \ d\mathbf{y} \right) \ = \ \alpha(\mathbf{x}, \beta) \int\limits_{f(\mathbf{x},\mathbf{y})\geq \alpha(\mathbf{x},\beta)} p(\mathbf{x}, \mathbf{y}) \ d\mathbf{y} \ .$$

The second term in the right handside of equation (14) equals

$$\int\limits_{f(\mathbf{x},\mathbf{y})\geq \alpha(\mathbf{x},\beta)} (f(\mathbf{x}, \mathbf{y}) - \alpha(\mathbf{x}, \beta))\, p(\mathbf{x}, \mathbf{y}) \ d\mathbf{y}$$

$$= \int\limits_{f(\mathbf{x},\mathbf{y})\geq \alpha(\mathbf{x},\beta)} f(\mathbf{x}, \mathbf{y})\, p(\mathbf{x}, \mathbf{y}) \ d\mathbf{y} \ - \ \alpha(\mathbf{x}, \beta) \int\limits_{f(\mathbf{x},\mathbf{y})\geq \alpha(\mathbf{x},\beta)} p(\mathbf{x}, \mathbf{y}) \ d\mathbf{y} \ .$$

Therefore,

$$F(\mathbf{x}, \alpha(\mathbf{x}, \beta)) \ = \int\limits_{f(\mathbf{x},\mathbf{y})\geq \alpha(\mathbf{x},\beta)} f(\mathbf{x}, \mathbf{y})\, p(\mathbf{x}, \mathbf{y}) \ d\mathbf{y} \ = \ \Phi(\mathbf{x}) \ ,$$

and statement (13) of the theorem is proved. $\diamond$

4 Application Examples

4.1 Minimum CVaR, Minimum VaR, and Minimum Variance Approaches for the Portfolio Optimization

This example considers a portfolio optimization problem with normally distributed portfolio return/loss. The calculations for this example were conducted by Carlos Testuri as part of the project on the Stochastic Optimization Course at the University of Florida. We showed that in the case of the normally distributed loss, the Minimum CVaR, the Minimum VaR (see, for instance, [11, 17]), and Minimum Variance Approaches [16] are equivalent, i.e., the same portfolio is optimal for all three criteria. Since the Minimum Variance approach can be reduced to a quadratic programming problem, a well studied topic, we use it as a benchmark for our algorithm.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be positions of a portfolio and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are sample instrument returns. Portfolio loss, $f(\mathbf{x}, \mathbf{y}) = -\mathbf{y}^T\mathbf{x}$, is defined as the negative return. We consider optimization problems with the following three performance functions: 1) variance of the portfolio denoted by $\sigma^2(\mathbf{x})$; 2) portfolio β-VaR($\mathbf{x}$) which is defined by equation (2); and 3) portfolio β-CVaR$(\mathbf{x}) = \frac{1}{1-\beta}\Phi(\mathbf{x})$, where $\Phi(\mathbf{x})$ is defined by equation (3). Let us impose the following constraint on the mean loss

$$\mu(\mathbf{x}) \stackrel{\text{def}}{=} \mathbb{E}[f(\mathbf{x}, \mathbf{y})] \leq -R \, . \tag{15}$$

Also, we have constraints on positions of instruments

$$\sum_{j=1}^{n} x_j = 1 \, , \tag{16}$$

$$x_j \geq 0 \, , \quad j = 1, \ldots, n \, . \tag{17}$$

We consider three problems:

Problem 1

$$\sigma^2(\mathbf{x}) \rightarrow \min_{\mathbf{x} \in \mathbb{R}^n} \tag{18}$$

subject to constraints (15), (16) and (17).

Problem 2

$$\beta\text{−VaR}(\mathbf{x}) \;\; \rightarrow \;\; \min_{\mathbf{x}\in\mathbb{R}^n} \tag{19}$$

subject to constraints (15), (16) and (17).

Problem 3

$$\beta\text{−CVaR}(\mathbf{x}) \;\; \rightarrow \;\; \min_{\mathbf{x}\in\mathbb{R}^n} \tag{20}$$

subject to constraints (15), (16) and (17).

Proposition 4.1 *Suppose that portfolio loss $f(\mathbf{x}, \mathbf{y})$ is normally distributed. If $\beta > 0.5$ and constraint (15) is active at optimal point in Problems 1,2, and 3, then these problems are equivalent: they have the same unique optimal position vector $\mathbf{x}^*$.*

Proof. It can be proved that

$$\beta\text{−VaR}(\mathbf{x}) \; = \; \mu(\mathbf{x}) + a(\beta)\,\sigma(\mathbf{x})\,, \qquad a(\beta) = \sqrt{2}\,\text{erf}^{-1}(2\beta - 1)\;, \tag{21}$$

and

$$\beta\text{−CVaR}(\mathbf{x}) \; = \; \mu(\mathbf{x}) + a_1(\beta)\,\sigma(\mathbf{x})\,, \qquad a_1(\beta) = \Big(\sqrt{2\pi}\,\exp\!\big(\text{erf}^{-1}(2\beta - 1)\big)^2(1 - \beta)\Big)^{-1}\;, \tag{22}$$

where $\exp(z)$ denotes the exponential function and $\text{erf}^{-1}(z)$ denotes the inverse function of the error function

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,\text{d}t\;.$$

If for each of Problems 1,2, and, 3, constraint (15) is active at the optimal point, then constraint (15) can be replaced by

$$\mu(\mathbf{x}) \; = \; -R\;. \tag{23}$$

Then, Problems 1,2, and, 3 can be rewritten using this constraint as follows

Problem 1a

$$\sigma^2(\mathbf{x}) \quad \rightarrow \quad \min_{\mathbf{x} \in \mathbb{R}^n} \tag{24}$$

subject to constraints (23), (16) and (17).

Problem 2a

$$-R + a(\beta)\, \sigma(\mathbf{x}) \quad \rightarrow \quad \min_{\mathbf{x} \in \mathbb{R}^n} \tag{25}$$

subject to constraints (23), (16) and (17).

Problem 3a

$$-R + a_1(\beta)\, \sigma(\mathbf{x}) \quad \rightarrow \quad \min_{\mathbf{x} \in \mathbb{R}^n} \tag{26}$$

subject to constraints (23), (16) and (17).

Evidently, problems 1a, 2a, and 3a are equivalent which proves the proposition. $\diamond$

This illustrative example compares the Minimum CVaR, the Minimum VaR, and the Markowitz Minimum Variance approaches for a small portfolio consisting of three instruments: S&P 500, a portfolio of small cap stocks , and a portfolio of long term U.S. government bonds. We consider the case with a normally distributed loss/return and an active constraint (15). Therefore the Minimum Variance, VaR, and CVaR approaches are equivalent in this case. First, we solved problems (18) using a quadratic programming algorithm. Then, we calculated with formulas (21), (22) the optimal values of VaR, and CVaR using the constraint on mean loss and optimal standard deviation. Although for the portfolio with normally distributed return, the Minimum CVaR approach does not produce a new result (comparing to the Minimum Variance or Minimum VaR approaches), we consider it here to explain the approach with a simple, familiar to the reader setup. Also, the Minimum CVaR approach can be implemented with the linear programming techniques. This may significantly speedup calculations for the portfolios with large number of instruments. To some extent, the technique for finding minimum CVaR approach is similar to the Mean Absolute Deviation Model [12] and the Regret Optimization [6, 7] which also use linear programming algorithms.

Denote by $\mathbf{m}$ the mean monthly returns and by $\mathbf{V}$ the covariance matrix of the returns for the three funds, see Table 1 and Table 2, accordingly. With the Markowitz Minimum Variance approach, the optimal proportions x_1, x_2, x_3 of instruments in the portfolio are selected by minimizing portfolio variance subject to constraint on mean portfolio loss. Let the required portfolio return equals $R = 0.011$. The optimal portfolio vector $\mathbf{x} = (x_1, x_2, x_3)$ is a solution of the following quadratic optimization problem:

$$C(x) \;=\; \mathbf{x}^T\mathbf{V}\mathbf{x} \;\;\rightarrow\;\; \min_{\mathbf{x}\in\mathbb{R}^3} \tag{27}$$

subject to

$$\mu(\mathbf{x}) \;=\; -\mathbf{m}^T\mathbf{x} \leq -R\,, \tag{28}$$

$$\sum_{j=1}^{3} x_j \;=\; 1\,, \tag{29}$$

$$x_j \geq 0\,, \quad j = 1, 2, 3\,. \tag{30}$$

The optimal solution of this problem equals $\mathbf{x}^* = (0.452013, 0.115573, 0.432414)$ and the minimal variance equals $\sigma^2(\mathbf{x}^*) = (\sigma^*)^2 = 0.00378529$. With (21), β-VaR of the optimal portfolio for $\beta = 0.99$, $\beta = 0.95$, and $\beta = 0.90$ equals 0.99-VaR$= -R + 2.32635\,\sigma^* = 0.132128$, 0.95-VaR$= -R + 1.64485\,\sigma^* = 0.0902$, and 0.9-VaR$= -R + 1.28155\,\sigma^* = 0.06785$, respectively. Similar, with formula (22), we calculate β-CVaR of the portfolio. The optimal position vector, VaR, and CVaR with the Markowitz approach are presented in Tables 3 and 4, accordingly.

With the minimum CVaR approach, we minimized the function $F(\mathbf{x}, \alpha)$ which is given by equation (4) under constraints (28), (29), and (30). The return vector $\mathbf{y}$ is distributed with the density $p(\mathbf{y})$, which is the density of the multinormal distribution $\mathcal{N}(\mathbf{m}, \mathbf{V})$ in $\mathbb{R}^3$. The sample loss of the portfolio equals $f(\mathbf{x}, \mathbf{y}) = -\mathbf{y}^T\mathbf{x}$. To minimize CVaR we solved the following problem

$$F(\mathbf{x}, \alpha) = (1-\beta)\alpha \;+ \int\limits_{\mathbf{y}\in\mathbb{R}^3} \left(-\mathbf{y}^T\mathbf{x} - \alpha\right)^+ p(\mathbf{y})\; d\mathbf{y} \;\;\rightarrow\;\; \min_{x\in\mathbb{R}^3}\,, \tag{31}$$

subject to constraints (28),(29), and (30).

To calculate the integral in problem (31) we can use various approaches. Here, we considered that this integral is approximated using points $\mathbf{y}_j,\ j = 1, \ldots, J$ in the space $\mathbb{R}^3$ which were sampled with the density function $p(\mathbf{y})$, i.e.,

$$\int\limits_{\mathbf{y}\in\mathbb{R}^3} \left(-\mathbf{y}^T\mathbf{x} - \alpha\right)^+ p(\mathbf{y})\; d\mathbf{y} \;\approx\; J^{-1} \sum_{j=1}^{J} \left(-\mathbf{y}_j^T\mathbf{x} - \alpha\right)^+. \tag{32}$$

Instrument	Mean Return
S&P	0.0101110
Gov Bond	0.0043532
Small Cap	0.0137058

Table 1: Portfolio Mean Return

	S& P	Gov Bond	Small Cap
S&P	0.00324625	0.00022983	0.00420395
Gov Bond	0.00022983	0.00049937	0.00019247
Small Cap	0.00420395	0.00019247	0.00764097

Table 2: Portfolio Covariance Matrix

S&P	Gov Bond	Small Cap
0.452013	0.115573	0.432414

Table 3: Optimal Portfolio with the Minimum Variance Approach

	$\beta = 0.9$	$\beta = 0.95$	$\beta = 0.99$
VaR	0.067847	0.090200	0.132128
CVaR	0.096975	0.115908	0.152977

Table 4: Optimal VaR and CVaR with the Minimum Variance Approach

β	Smpls #	S&P	Gov Bond	Small Cap	VaR	VaR Dif(%)	CVaR	CVaR Dif(%)	Iter	Time (min)
0.9	1000	0.35250	0.15382	0.49368	0.06795	0.154	0.09962	2.73	1157	0.0
0.9	3000	0.55726	0.07512	0.36762	0.06537	3.645	0.09511	-1.92	636	0.0
0.9	5000	0.42914	0.12436	0.44649	0.06662	1.809	0.09824	1.30	860	0.1
0.9	10000	0.48215	0.10399	0.41386	0.06622	2.398	0.09503	-2.00	2290	0.3
0.9	20000	0.45951	0.11269	0.42780	0.06629	-2.299	0.09602	-0.98	8704	1.5
0.95	1000	0.53717	0.08284	0.37999	0.09224	2.259	0.11516	-0.64	156	0.0
0.95	3000	0.54875	0.07839	0.37286	0.09428	4.524	0.11888	2.56	652	0.0
0.95	5000	0.57986	0.06643	0.35371	0.09175	1.715	0.11659	0.59	388	0.1
0.95	10000	0.47102	0.10827	0.42072	0.08927	-1.03	0.11467	-1.00	1451	0.2
0.95	20000	0.49038	0.10082	0.40879	0.09136	1.284	0.11719	1.11	2643	0.7
0.99	1000	0.41844	0.12848	0.45308	0.13454	1.829	0.14513	-5.12	340	0.0
0.99	3000	0.6196	0.05116	0.32924	0.12791	-3.187	0.14855	-2.89	1058	0.0
0.99	5000	0.63926	0.04360	0.31714	0.13176	-0.278	0.15122	-1.14	909	0.1
0.99	10000	0.45203	0.11556	0.43240	0.12881	-2.51	0.14791	-3.31	680	0.1
0.99	20000	0.45766	0.11340	0.42894	0.13153	-0.451	0.15334	0.24	3083	0.9

Table 5: The portfolio, VaR, and CVaR with Minimum CVaR Approach (β Value, Number of Samples in a Simulation Run, Three Positions, Calculated VaR, Difference Between Calculated and Exact VaR, Calculated CVaR, Difference Between Calculated and Exact VaR, Number of Iterations of CPLEX, Processor Time: Pentium II, 300MHz). Monte Carlo Simulations Are Conducted With *Pseudo-Random Numbers*.

β	Smpls #	S&P	Gov Bond	Small Cap	VaR	VaR Dif(%)	CVaR	CVaR Dif(%)	Iter	Time (min)
0.9	1000	0.43709	0.12131	0.44160	0.06914	1.90	0.09531	-1.71	429	0.0
0.9	3000	0.45425	0.11471	0.43104	0.06762	-0.34	0.09658	-0.41	523	0.0
0.9	5000	0.44698	0.11751	0.43551	0.06784	-0.02	0.09664	-0.35	837	0.1
0.9	10000	0.45461	0.11457	0.43081	0.06806	0.32	0.09695	-0.02	1900	0.3
0.9	20000	0.46076	0.11221	0.42703	0.06790	0.08	0.09692	-0.06	4818	0.6
0.95	1000	0.43881	0.12065	0.44054	0.09001	-0.21	0.11249	-2.95	978	0.0
0.95	3000	0.43881	0.12065	0.44054	0.09001	-0.21	0.11511	-0.69	407	0.0
0.95	5000	0.46084	0.11218	0.42698	0.09036	0.18	0.11516	-0.64	570	0.1
0.95	10000	0.45723	0.11357	0.42920	0.09016	-0.05	0.11577	-0.12	1345	0.2
0.95	20000	0.45489	0.11447	0.43064	0.09023	0.03	0.11577	-0.12	1851	0.7
0.99	1000	0.52255	0.08846	0.38899	0.12490	-5.47	0.14048	-8.17	998	0.0
0.99	3000	0.43030	0.12392	0.44578	0.12801	-3.12	0.15085	-1.39	419	0.0
0.99	5000	0.45462	0.11457	0.43081	0.13073	-1.06	0.14999	-1.95	676	0.1
0.99	10000	0.39156	0.13881	0.46963	0.13288	0.57	0.15208	-0.59	1065	0.2
0.99	20000	0.46065	0.11225	0.42710	0.13198	-0.11	0.15211	-0.57	1317	0.5

Table 6: The portfolio, VaR, and CVaR with Minimum CVaR Approach (β Value, Number of Samples in a Simulation Run, Three Positions, Calculated VaR, Difference Between Calculated and Exact VaR, Calculated CVaR, Difference Between Calculated and Exact VaR, Number of Iterations of CPLEX, Processor Time: Pentium II, 300MHz). Simulations Are Conducted With *Quasi-Random Sobel Sequences.*

With approximation (32), we reduced problem (31) to the linear programming problem using dummy variables $z_j,\ j = 1, \ldots, J$,

$$\tilde{F}(\mathbf{x}, \alpha) \stackrel{def}{=} (1-\beta)\alpha \; + \; J^{-1} \sum_{j=1}^{J} z_j \; , \;\; \rightarrow \;\; \min_{\mathbf{x},\alpha} \tag{33}$$

subject to constraints

$$z_j \;\geq\; -\mathbf{y}_j^T \mathbf{x} - \alpha \; , \;\; j = 1, \ldots, J, \tag{34}$$

$$z_j \;\geq\; 0 \; , \;\; j = 1, \ldots, J, \tag{35}$$

and (28),(29),(30).

By solving the last problem we found the optimal portfolio vector, $\mathbf{x}^*$, corresponding VaR,which equals to α^*, and the optimal CVaR, which equals to $\frac{1}{1-\beta}\tilde{F}(\mathbf{x}^*, \alpha^*)$.

We solved linear optimization problem (33) with constraints (34), (35), (28), (29), and (30) using the CPLEX linear programming solver on Pentium-II, 300 MHz. To approximate the integral we considered two types of "random" numbers: the pseudo-random sequence of numbers (conventional Monte-Carlo approach) and the Sobol quasi-random sequence ([19], page 310). See, similar applications of the quasi-random sequences in [3, 4, 13]. The calculation results for the conventional Monte-Carlo approach (pseudo-random sequence of numbers) are given in the Table 5. Comparing calculation results for the Minimum CVaR approach with pseudo-random sampling (Table 5) and the Minimum Variance approach (Tables 3, 4) wee see that VaR and CVaR with these two approaches differ only few percents (depending upon the number of samples). However, portfolio positions converge quite slowly to the true values which are presented in Tables 3, 4. The slow convergence of the positions is related to the sampling errors of the Monte-Carlo simulations. Also, at the optimal point, the variance, VaR, and CVaR have low sensitivities to the changes in the portfolio positions. Therefore, changes in the positions around minimum point have insignificant influence on the variance, VaR, and CVaR. We have done similar calculations using the Sobel quasi-random sequence for approximating the integral, see Table 6. In this case, the convergence to true values of VaR, CVaR, and optimal portfolio positions is relatively fast (compared to traditional Monte Carlo). For instance, when the number of samples is larger than 10000, the difference between calculated VaR, CVaR, and true value of VaR, and CVaR is less than 1%. Although with the Minimum CVaR approach VaR is not included directly in the performance function, this approach also minimizes the VaR because CVaR $\geq$ VaR. The numerical experiments confirmed Proposition (4.1) stating that for a portfolio with normally distributed loss the Minimum CVaR and the Minimum VaR approaches are equivalent.

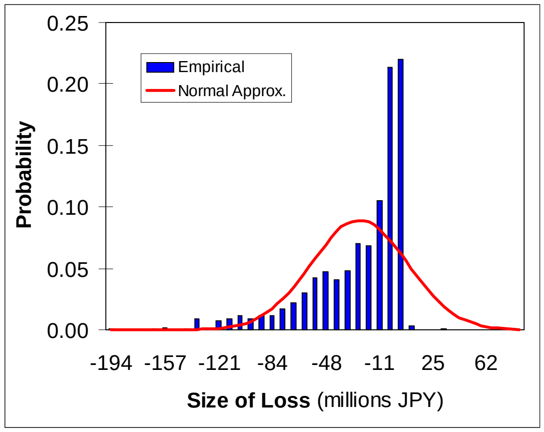

Figure 1: Distribution of losses for the NIKKEI portfolio with best normal approximation, (1,000 scenarios), reproduced from [17].

4.2 Optimal Hedging: NIKKEI Portfolio

This example (see, [17]) was provided by Dr. Helmut Mausser from Algorithmics Inc. for testing the Minimum CVaR approach. The paper [17] considers two approaches for hedging of the portfolio: parametric and simulation VaR techniques. With these approaches, the best hedge is calculated by one-instrument minimization of VaR. Here, we show that one-instrument minimum CVaR hedges are very close to the one-instrument minimum VaR hedges. Also, we show that with Minimum CVaR approach, hedging can be conducted by changing multiple positions. Similar to the previous example, the hedging can be reduced to the linear program by introducing additional variables (one additional variable for each scenario). Especially, linear programming techniques are beneficial for large dimension problems (>1000 instruments). However, here we demonstrate that for portfolios with relatively small number of instruments nonsmooth optimization approaches can compete with linear programming techniques. Comparing to the linear programming, the dimension of the nonsmooth optimization problem does not change when the number of scenarios is increased. Therefore, the nonsmooth optimization techniques may have some advantages for small and medium dimensions (<1000 instruments) and very large number of scenarios.

Table 7 shows a portfolio that implements a butterfly spread on the NIKKEI index, as of July 1, 1997. In addition to common shares of Komatsu and Mitsubishi, the portfolio includes several

European call and put options on these equities. This portfolio makes extensive use of options to achieve the desired payoff profile. Figure 1 (reproduced from [17]) shows the distribution of one-day losses over the set of 1,000 Monte Carlo scenarios. It indicates that the normal distribution fits the data poorly. Therefore, Minimum CVaR and Minimum Variance approaches, for this case, may lead to quite different optimal solutions.

Denote by $\mathbf{m}$ the vector of initial prices per share for the instruments in the portfolio. Denote by n the total number of instruments in the portfolio. We divided instruments in the portfolio into two groups: n_1 is the number of instruments in the first group and n_2 is in the second one, i.e., $n_1 + n_2 = n$. We minimized CVaR by changing positions of the second group. Denote by $\mathbf{z}$ the positions vector of the first group and by $\mathbf{x}$ the position vector of the second group. Also, we denoted by $\mathbf{y}$ the vector of prices of the instruments the next day after the current one. With these notations, the initial value of the portfolio equals $\mathbf{m}^T(\mathbf{z}, \mathbf{x})$ and the value of the portfolio in one day equals $\mathbf{y}^T(\mathbf{z}, \mathbf{x})$.Then, the sample loss of the portfolio equals $f(\mathbf{x}, \mathbf{y}) = \mathbf{m}^T(\mathbf{z}, \mathbf{x}) - \mathbf{y}^T(\mathbf{z}, \mathbf{x}) = (\mathbf{m} - \mathbf{y})^T(\mathbf{z}, \mathbf{x})$. We imposed the following constraints on the position changes

$$-|x_i^0| \ \leq \ x_i \leq \ |x_i^0| \ , \quad i = 1 \ldots, n_2 \ , \tag{36}$$

where x_i^0 is the initial position of the instrument i in the portfolio. Initial positions of the instruments are presented in the 5-th column of Table 7. The negative sign means a short position. Constraints (36) restrict the increase of absolute values of the positions, however signs of the positions can be changed. With the Minimum CVaR approach, we should solve the following problem

$$F(\mathbf{x}, \alpha) = (1 - \beta)\alpha \ + \int\limits_{\mathbf{y} \in \mathbb{R}^n} \Big((\mathbf{m} - \mathbf{y})^T(\mathbf{z}, \mathbf{x}) - \alpha \Big)^+ \, p(\mathbf{y}) \ d\mathbf{y} \ \ \rightarrow \ \min_{\mathbf{x} \in \mathbb{R}^{n_2}} \tag{37}$$

subject to constraints (36).

We approximated the integral in the last sum by sampling points $\mathbf{y}_j, \ j = 1, \ldots, J$ with the density function $p(\mathbf{y})$

$$\int\limits_{\mathbf{y} \in \mathbb{R}^n} \Big((\mathbf{m} - \mathbf{y})^T(\mathbf{z}, \mathbf{x}) - \alpha \Big)^+ \, p(\mathbf{y}) \ d\mathbf{y} \ \approx \ J^{-1} \sum_{j=1}^{J} \Big((\mathbf{m} - \mathbf{y}_j)^T(\mathbf{z}, \mathbf{x}) - \alpha \Big)^+ \ .$$

Therefore, we reduced (37) to the following nonsmooth optimization problem

$$\tilde{F}(\mathbf{x},\alpha) \stackrel{def}{=} (1-\beta)\alpha \;+\; J^{-1}\sum_{j=1}^{J}\Big((\mathbf{m}-\mathbf{y}_j)^T(\mathbf{z},\mathbf{x})-\alpha\Big)^+ \;\rightarrow\; \min_{\mathbf{x},\alpha} \tag{38}$$

subject to constraints (36).

By optimizing function (38) we found the optimal portfolio vector, $\mathbf{x}^*$, corresponding VaR,which equals to α^*, and the optimal CVaR, which equals to $\frac{1}{1-\beta}\tilde{F}(\mathbf{x}^*,\alpha^*)$. The function $\tilde{F}(\mathbf{x},\alpha)$ is convex (definition of the convex function and the generalized gradient can be found, for instance, in [22, 23]). The generalized gradient of the function $\tilde{F}(\mathbf{x},\alpha)$ can be calculated with formulas

$$\nabla_\alpha \tilde{F}(\mathbf{x},\alpha) \;=\; (1-\beta) \;-\; J^{-1}\sum_{j=1}^{J} I_{\{(\mathbf{m}-\mathbf{y}_j)^T(\mathbf{z},\mathbf{x})-\alpha\geq 0\}}\;, \tag{39}$$

$$\nabla_{\mathbf{x}} \tilde{F}(\mathbf{x},\alpha) \;=\; \left[J^{-1}\sum_{j=1}^{J} I_{\{(\mathbf{m}-\mathbf{y}_j)^T(\mathbf{z},\mathbf{x})-\alpha\geq 0\}}\right]\mathbf{m}_{\mathbf{x}}\;, \tag{40}$$

where $\mathbf{m}_{\mathbf{x}}$ is the part of the vector $\mathbf{m}$ corresponding to the vector $\mathbf{x}$, and $I_{\{.\}}$ is the indicator function, i.e.,

$$I_{\{(\mathbf{m}-\mathbf{y}_j)^T(\mathbf{z},\mathbf{x})-\alpha\geq 0\}} \;=\; \begin{cases} 1\,, & \text{if}\,, \quad (\mathbf{m}-\mathbf{y}_j)^T(\mathbf{z},\mathbf{x})-\alpha \geq 0\,; \\ 0\,, & \text{otherwise}\,. \end{cases}$$

To solve optimization problem (38) we used the Variable Metric Algorithm for nonsmooth optimization problems [26]. We considered the case with $\beta = 0.95$. First, we optimized CVaR w.r.t. each position $i = 1,\ldots,n$. Initial values of VaR and CVaR equal 657816 and 2.02206E+06, accordingly. Calculation results with one dimension hedging are presented in Table 8. These optimal hedges are close to the hedges which were obtained in [17] by minimizing the VaR. Since CVaR≥VaR, minimization of CVaR also leads to minimization of VaR. In this case, the optimization problem has dimension two: the first variable is VaR and the second is the one dimension hedge. The algorithm with variable metric needed less than 100 iterations to find 6 correct digits in the performance function and variables. For testing purposes, we used MATHEMATICA version of the variable metric code on Pentium II, 450MHz (the FORTRAN and MATHEMATICA versions of the code are available at http://www.ise.ufl.edu/uryasev). The constraints were accounted for using nonsmooth penalty functions. Each run took less than one minute of computer time. The calculation time can be significantly improved using the algorithm implemented with FORTRAN or C, however such computational studies were beyond the scope of this paper. Further, we calculated optimal hedges for the last four positions in the portfolio. The calculation

Instrument	Type	Day to Maturity	Strike Price (10^3 JPY)	Position (10^3)	Value (10^3 JPY)
Mitsubishi EC 6mo 860	Call	184	860	11.5	563,340
Mitsubishi Corp	Equity	n/a	n/a	2.0	1,720,00
Mitsubishi Cjul29 800	Call	7	800	-16.0	-967,280
Mitsubishi Csep30 836	Call	70	836	8.0	382,070
Mitsubishi Psep30 800	Put	70	800	40.0	2,418,012
Komatsu Ltd	Equity	n/a	n/a	2.5	2,100,000
Komatsu Cjul29 900	Call	7	900	-28.0	-11,593
Komatsu Cjun2 670	Call	316	670	22.5	5,150,461
Komatsu Cjun2 760	Call	316	760	7.5	1,020,110
Komatsu Paug31 760	Put	40	760	-10.0	-68,919
Komatsu Paug31 830	Put	40	830	10.0	187,167

Table 7: NIKKEI Portfolio, reproduced from [17].

results with $\beta = 0.95$ are presented in Table 9. The optimization did not change positions of Komatsu Cjun2 670 and Komatsu Paug31 760. Positions of Komatsu Cjun2 760 and Komatsu Paug31 830 changed signs and values. Comparing to individual minimization with respect to these two positions, we see that joint minimization considerably improves the VaR and CVaR. In this case, VaR with optimized four positions equals -1.40E+06, and CVaR equals 37334.6 which is lower than best one-dimension hedge with VaR=-1.20E+06 and CVaR=363556 (see line nine in Table 8). Six correct digits in the performance function and the positions were obtained after 400-800 iterations of the variable metric algorithm [26], depending upon the initial parameters. It took about 4-8 minutes of the computer time with MATHEMATICA version of the variable metric code on Pentium II, 450MHz. Comparing to the previous example where we used linear programming techniques, dimension of the nonsmooth optimization problem does not change with increase in the number of scenarios. This may give some computational advantages for the problems with very large number of scenarios.

Instrument	Best Hedge	VaR	CVaR
Mitsubishi EC 6mo 860	7337.53	-205927	1.18304E+06
Mitsubishi Corp	-926.073	-1.18E+06	551892
Mitsubishi Cjul29 800	-18978.6	-1.17E+06	553696
Mitsubishi Csep30 836	4381.22	-1.15E+06	549022
Mitsubishi Psep30 800	43637.1	-1.15E+06	542168
Komatsu Ltd	-196.167	-1.18E+06	551892
Komatsu Cjul29 900	-124939	-1.20E+06	593078
Komatsu Cjun2 670	19964.9	-1.22E+06	385698
Komatsu Cjun2 760	4745.20	-1.20E+06	363556
Komatsu Paug31 760	31426.3	-1.12E+06	538662
Komatsu Paug31 830	19356.3	-1.15E+06	536500

Table 8: Best Hedge, VaR, and CVaR with Minimum CVaR Approach (One-Instrument Minization, $\beta = 0.95$).

Instrument	Position in Portfolio	Best Hedge
Komatsu Cjun2 670	22500	22500
Komatsu Cjun2 760	7500.0	-528.66
Komatsu Paug31 760	-10000	-10000
Komatsu Paug31 830	10000	-10000

Table 9: Initial Positions and Best Hedges with Minimum CVaR Approach (Simultaneous Optimization of Four Instruments, $\beta = 0.95$), VaR with Best Hedges Equals -1.40E+06, and CVaR Equals 37334.6.

5 Concluding Remarks

The paper considered a new approach for simultaneous calculation of VaR and optimization of CVaR for a general class of problems. We showed that CVaR can be efficiently minimized using Linear Programming and Nonsmooth Optimization techniques. Although, formally, the method minimizes only CVaR, it also minimizes VaR because CVaR majorates VaR. Moreover, in the case of a portfolio with a normally distributed return/loss the Minimum CVaR and the Minimum VaR approaches are equivalent. We have demonstrated with two examples that the approach provides valid results. These examples have relatively low dimensions and are considered here for illustrative purpose. We conducted numerical experiments for larger problems. However they will be published separately in a paper on comparison of numerical aspects of various Linear Programming techniques for portfolio optimization. There is a lot of room for improvement and refinement of the suggested approach. For instance, the assumption that there is a joint density of instrument returns can be relaxed. The approach can be extended on optimization problems with Value-at-Risk constraints. Linear Programming and Nonsmooth Optimization approaches which utilize the special structure of the Minimum CVaR approach can be developed. Additional research need to be conducted on various theoretical and numerical aspects of the methodology.

Acknowledgments

Authors are grateful to Carlos Testuri who conducted numerical experiments for the example on comparison of the Minimum CVaR and the Minimum Variance Approaches for the Portfolio Optimization. Also, we want to thank Dr. Helmut Mausser from Algorithmics Inc. for the fruitful discussions and providing data for conducting numerical experiments with the NIKKEI portfolio of options.

References

[1] Artzner, P., Delbaen F., Eber, J.M., and D. Heath (1997): Thinking Coherently. *Risk,* 10, November, 68-71.

[2] Artzner, P., Delbaen F., Eber, J.M., and D. Heath (1999): Coherent Measures of Risk. Mathematical Finance, June.

[3] Birge, J.R. (1995): Quasi-Monte Carlo Methods Approaches to Option Pricing. *Technical report 94-19.*, Department of Industrial and Operations Engineering, The University of Michigan, 15p.

[4] Boyle, P.P., Broadie, M., and P. Glasserman (1997): Monte Carlo Methods for Security Pricing *Journal of Economic Dynamics and Control.* Vol. 21 (8-9), 1267 – 1321.

[5] Bucay, N. and D. Rosen (1999): Credit Risk of an International Bond Portfolio: a Case Study. *ALGO Research Quarterly.* Vol.2, 1, 9–29.

[6] Dembo, R.S. (1995): Optimal Portfolio Replication. *Algorithmics Technical paper series.* 95-01.

[7] Dembo, R.S. and A.J. King (1992): Tracking Models and the Optimal Regret Distribution in Asset Allocation. *Applied Stochastic Models and Data Analysis.* Vol. 8, 151-157.

[8] Ermoliev, Yu. (1983): Stochastic Quasi-Gradient Methods and Their Applications to System Optimization. *Stochastics*, **4**, 1–36.

[9] Embrechts, P. (1999): Extreme Value Theory as a Risk Management Tool. *North American Actuarial Journal*, 3(2), April 1999.

[10] Embrechts, P., Kluppelberg, S., and T. Mikosch (1997): *Extremal Events in Finance and Insurance.* Springer Verlag.

[11] Kast, R., Luciano, E., and L. Peccati (1998): *VaR and Optimization.* 2nd International Workshop on Preferences and Decisions, Trento, July 1-3 1998.

[12] Konno, H. and H. Yamazaki (1991): Mean Absolute Deviation Portfolio Optimization Model and Its Application to Tokyo Stock Market. *Management Science.* **37**, 519-531.

[13] Kreinin, A., Merkoulovitch, L., Rosen, D., and Z. Michael (1998): Measuring Portfolio Risk Using Quasi Monte Carlo Methods. *ALGO Research Quarterly.* Vol.1, 1, 17–25.

[14] Litterman, R. (1997): Hot Spots and Hedges (I). *Risk.* 10 (3), 42-45.

[15] Litterman, R. (1997): Hot Spots and Hedges (II). Risk, 10 (5), 38-42.

[16] Markowitz, H.M. (1952): Portfolio Selection. *Journal of finance.* Vol.7, 1, 77–91.

[17] Mauser, H. and D. Rosen (1991): Beyond VaR: From Measuring Risk to Managing Risk. *ALGO Research Quarterly.* Vol.1, 2, 5–20.

[18] *Numerical Techniques for Stochastic Optimization*, (1988): (Eds.) Yu. Ermoliev and R. J-B Wets, Springer Series in Computational Mathematics, 10.

[19] Press, W.H., Teukolsky, S.A, Vetterling, W.T., and B.P. Flannery (1992): *Numerical Recipes in C.* Cambridge University Press.

[20] Prekopa, A. (1995): *Stochastic Programming.* Kluwer Academic Publishers, Dordrecht, Boston.

[21] RiskMetrix^{TM}. (1996): *Technical Document*, 4-th Edition, J.P.Morgan, December 1996.

[22] Rockafellar, R.T. (1970): *Convex Analysis.* Princeton Mathematics, Vol. 28, Princeton Univ. Press.

[23] Shor, N.Z. (1985): *Minimization Methods for Non–Differentiable Functions.* Springer-Verlag.

[24] Uryasev, S. (1995): Derivatives of Probability Functions and Some Applications. *Annals of Operations Research*, V56, 287-311.

[25] Uryasev, S. (1980): Stepsize Control for Direct Stochastic Programming Methods. *Kibernetika* (Kiev), **6**, 85–87 (in Russian). *Cybernetics*, v.16, **6**, (1980), 886–890 (in English).

[26] Uryasev, S. (1991): New Variable-Metric Algorithms for Nondifferential Optimization Problems. *J. of Optim. Theory and Applic.* Vol. 71, No. 2, 359-388.