Abstract

A retiree with a savings account balance, but without a pension is confronted with an im-
portant investment decision that has to satisfy two conflicting objectives. Without a pension
the function of the savings is to provide post-employment income to the retiree. At the same
time, most retirees will want to leave an estate to their heirs. Guaranteed income can be
acquired by investing in an annuity. However, that decision necessarily takes funds away from
investment alternatives that might grow the estate. The decision is made even more com-
plicated because one does not know how long one will live. A long life expectancy would
suggest more annuities, and short life expectancy will immediately promote more risky invest-
ments. However there are very mixed opinions about either strategy. A framework has been
developed to assess consequences and the trade-offs of alternative investment strategies. We
propose a stochastic programming model to frame this complicated problem. The objective is
to maximize expected terminal net worth (the estate) subject to CVaR constraints on target
income shortfalls. Objective is calculated using probabilities of scenarios of returns of invested
instruments and mortality probabilities. The CVaR constraints are applied each year of the
portfolio investment horizon. We consider that the investment strategy is running for the
whole investment horizon and the CVaR constraints should be satisfied for each year. We
use kernel functions to building position adjustment functions in investment strategies. These
adjustments nonlinearly depend upon on asset returns in previous years.

Keywords: kernel, radial basis function, CVaR, dimensionality reduction, mortality tables,
portfolio selection, retirement, annuity, stocks

1. Introduction

The problem of selecting optimal portfolios for retirement has unique features that is not
addressed by more commonly used portfolio selection models used in trading. One of the
distinct features of retirement portfolios is that it needs to incorporate mortality tables. The
planning horizon depends on the age of investor, or more specifically, on the conditional life
expectancy of the investor. Another important feature is to guarantee, in some sense, that
the individual will be able to withdraw some amount of money every year from a portfolio
by selling some predefined amount of assets without injecting external funds. The model that
is developed in this paper assumes that the investor wishes to maximize the terminal wealth
while maintaining predefined cash outflows each year. The mortality tables are used to weight the portfolio value in each year in the objective function.

2. Notations

- \( N := \) number of assets available for investment,
- \( S := \) number of scenarios,
- \( T := \) portfolio investment horizon,
- \( r_{i,t}^s := \) growth rate of asset \( i \in \{1, \ldots, N\} \) during period \( t \in \{1, \ldots, T\} \) in scenario \( s \in \{1, \ldots, S\} \), the vector form of returns is denoted as \( \mathbf{r}_t^s = (r_{1,t}^s, \ldots, r_{N,t}^s) \),
- \( \rho_t^s := \) inflation rate at time \( t \) in scenario \( s \),
- \( d_t^s := \) discount factor at time \( t \) in scenario \( s \); discounting is done using inflation rate \( \rho_t^s \), \( d_t^s = 1/(1 + \rho_t^s)^t \),
- \( p_t := \) probability that a person will die at time \( t \),
- \( y_{i,t} := \) vector of control variables for investment adjustment function.
- \( f(\mathbf{r}_t^s, y_{i,t}) := \) investment adjustment function. This function controls how much investment is made in each scenario \( s \) in asset \( i \) at the end of period \( t \),
- \( G(\mathbf{y}_{i,t}) := \) regularization function of control parameters,
- \( K_{i,j,s}(t) := \) kernel functions (nonlinear mapping to the feature space),
- \( x_{i,t}^s := \) investment amount to \( i \)-th asset at time \( t \) in scenario \( s \),
- \( x_i := \) investments to \( i \)-th asset at time \( t = 0 \),
- \( u_{i,t}^s := \) adjustment for asset \( i \) at the beginning of period \( t \) in scenario \( s \),
- \( u_{i,t} := \) adjustment for asset \( i \) at period \( t \) calculated with information available at \( t = 0 \),
- \( L_{i,t} := \) lower bound on position in asset \( i \) at time \( t \) as a fraction of portfolio value \( (L_{i,t} \in [0, 1]) \),
- \( U_{i,t} := \) upper bound on position in asset \( i \) at time \( t \) as a fraction of portfolio value \( (U_{i,t} \in [0, 1]) \),
- \( V_0 := \) value of the portfolio at time \( t = 0 \) (initial investment),
- \( z := \) investment in an annuity,
- \( A_t^s := \) coefficient converting investment in annuity to payment at time \( t \) in scenario \( s \),
- \( l_t := \) amount of money that the portfolio holder is planning to withdraw as each time \( t \) before paying tax,
- \( \alpha_t := \) confidence level of CVaR at time \( t \), \( \alpha_t \in [0, 1) \),
- \( \zeta := \) free variable in CVaR formula.
- \( \lambda := \) regularization coefficient.
3. Model Formulation

This section develops a general model for a retirement portfolio selection. We consider a portfolio including stocks, bonds, and an annuity. The annuity pays inflation adjusted amount \( A_t \) at each period \( t \). Given initial investments in the assets \( x_i \), the dynamics of investments in stocks and bonds are as follows:

\[
\begin{align*}
    x_{i,1}^s &= (1 + r_{i,1}^s)x_i, \\
    x_{i,t}^s &= (1 + r_{i,t}^s)(x_{i,t-1}^s + u_{i,t}^s).
\end{align*}
\]

Variables \( u_{i,t} \) and \( u_{i,t}^s \) control how much is invested at the beginning of each period \( t \) in each asset \( i \) and in each scenario \( s \). The adjustment \( u_{i,t}^s \) equals

\[
    u_{i,t}^s = u_{i,t} + f_t(r_{t}^s, y_{i,t}),
\]

where

\[
    f_t(r_{t}^s, y_{i,t}) \text{ is a control function that depends on history of returns for each asset up to period } t \text{ and control variables } y_{i,t} \text{ that determine the shape of the function.}
\]

The objective is to maximize the expected discounted terminal value of a portfolio, weighted by the probability of death for each time \( t \), under some constraints on risk. The portfolio value at \( t \) is discounted to time 0 using inflation as the discount rate. In order to avoid over-fitting the data, we included the regularization term \( G(y_{i,t}) \) in the objective function. The objective function is

\[
    -\frac{1}{S} \sum_{t=1}^T \sum_{s=1}^S p_t d_t^s \sum_{i=1}^N x_{i,t}^s + \lambda \sum_{i=1}^N \sum_{t=1}^T G(y_{i,t}).
\]

The function \( G \) is a nonnegative convex function.

Because \( G(h_{i,t}) \) is a convex function by assumption then (4) is also a convex function (it is the sum of linear and convex functions).

Let \( X \) be some random variable. We measure risk of \( X \) using CVaR\(_\alpha\) defined as

\[
    \text{CVaR}_\alpha(X) = \min_\zeta \left( \zeta + \frac{1}{1-\alpha} E[X - \zeta]^+ \right) \quad \text{for } \alpha \in [0, 1),
\]

where \( [x]^+ = \max(x, 0) \) and \( \zeta \in \mathbb{R} \). For a fixed number \( S \) of equally probable scenarios and corresponding random variable realizations \( X_s \) the CVaR\(_\alpha\) equals

\[
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\]

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4. SPECIFIC FORMULATION

\[ \text{CVaR}_\alpha(X) = \min_{\zeta} \left( \zeta + \frac{1}{S(1-\alpha)} \sum_{s=1}^{S} [X_s - \zeta]^+ \right) \text{ for } \alpha \in [0,1). \]

For a comprehensive analysis of the CVaR\(_\alpha(X)\) risk measure see Rockafellar and Uryasev [1, 2].

We impose CVaR constraint on the shortage of funds with confidence level \(\alpha_t\) at time \(t\)

\[
\min_{\zeta_t} \left\{ \zeta_t + \frac{1}{S(1-\alpha_t)} \sum_{s=1}^{S} \left[ \sum_{i=1}^{N} u_{i,t+1}^{s} - A_i^{s} z - \zeta_t \right]^+ \right\} \leq -l_t \tag{5}
\]

Because CVaR constraint is more conservative than VaR constraint, (5) ensures that for more than 100(1 - \(\alpha_t\))% of the scenarios the investor is able to withdraw amount \(l_t\) from the portfolio.

The objective is to maximize regularized portfolio value (4) while satisfying the constraints (1) to (3) and (5). Finally we arrive to the following optimization problem

\[
\min_{u_{i,t}, u_{i,t}^{s}, y_{i,t}, x_{i,t}, x_{i,t}^{s}, z, \zeta_t} \quad -\frac{1}{S} \sum_{t=1}^{T} \sum_{s=1}^{S} p_t d_t^s \sum_{i=1}^{N} x_{i,t}^{s} + \lambda \sum_{i=1}^{N} \sum_{t=2}^{T} G(y_{i,t}) \tag{6}
\]

\[
s.t.
\]

\[
\zeta_t + \frac{1}{S(1-\alpha_t)} \sum_{s=1}^{S} \left[ \sum_{i=1}^{N} u_{i,t+1}^{s} - A_i^{s} z \right]^+ \leq -l_t \quad t = 1, \ldots, T - 1
\]

\[
x_{i,1}^{s} = (1 + r_{i,1}^{s}) x_i \quad i = 1, \ldots, N; \quad s = 1, \ldots, S
\]

\[
x_{i,t}^{s} = (1 + r_{i,t}^{s})(x_{i,t-1}^{s} + u_{i,t}^{s}) \quad i = 1, \ldots, N; \quad t = 2, \ldots, T; \quad s = 1, \ldots, S
\]

\[
\sum_{i=1}^{N} x_i = V_0 - z
\]

\[
L_{i,1} \sum_{j=1}^{N} x_j \leq x_i \leq U_{i,1} \sum_{j=1}^{N} x_j \quad i = 1, \ldots, N
\]

\[
L_{i,t} \sum_{j=1}^{N} (x_{j,t-1}^{s} + u_{j,t}^{s}) \leq x_{i,t}^{s} + u_{i,t}^{s} \leq U_{i,t} \sum_{j=1}^{N} (x_{j,t}^{s} + u_{j,t}^{s}) \quad i = 1, \ldots, N;
\]

\[
t = 2, \ldots, T; \quad s = 1, \ldots, S
\]

\[
u_{i,t}^{s} = u_{i,t} + f_t(r_{i,t}^{s}, y_{i,t}) \quad i = 1, \ldots, N; \quad t = 2, \ldots, T; \quad s = 1, \ldots, S
\]

\[
z \geq 0
\]

\[
x_i \geq 0 \quad i = 1, \ldots, N
\]

\[
x_{i,t}^{s} \geq 0 \quad i = 1, \ldots, N; \quad t = 1, \ldots, T; \quad s = 1, \ldots, S
\]

4. Specific Formulation

In this section we choose a specific form for the functions \(G(y_{i,t})\) and \(f(r_{i,t}^{s}, y_{i,t})\). This model is similar to Takano and Gotoh (2014) [3]. In this formulation, let \(K_{i,j,s}(t)\) be the kernel
5. **Sample Paths of Returns, Inflation, and Mortality Tables**

Initial data are the historical yearly inflation rates and returns of stock and bond indexes over some time period $\hat{T}$. Let $\hat{r}_{i,\hat{t}}$ be the historical return of asset $i$ at some past time $\hat{t} \in \{1, \ldots, \hat{T}\}$ (where $\hat{t}$ is a year number), the stock and bond index return data is represented as $\hat{T} \times N$ matrix

$$
R = \begin{bmatrix}
\hat{r}_{1,1} & \hat{r}_{2,1} & \cdots & \hat{r}_{N,1} \\
\hat{r}_{1,2} & \hat{r}_{2,2} & \cdots & \hat{r}_{N,2} \\
\cdots & \cdots & \cdots & \cdots \\
\hat{r}_{1,\hat{T}} & \hat{r}_{2,\hat{T}} & \cdots & \hat{r}_{N,\hat{T}}
\end{bmatrix}
$$

We generate the sample paths (scenarios) with the historical simulation method (also known as Bootstrap method). Each such path is the random time series that represent a future dynamics of return of stock, bond and inflation.

We use the mortality table of USA (Table 1) for year 2017. Mortality table give probability that a person who is $x$ years old will die within a year, more specifically $\hat{p}(x) = \mathbb{P}(\text{age of death} < x + 1 \mid \text{age} = x)$. We calculate the probability that a person dies at the given age conditional that he/she is 65 years old, or $\bar{p}(x) = \mathbb{P}(\text{age of death} = x \mid \text{age} = 65)$. The formula for $\bar{p}(x)$ is following

$$
\bar{p}(x) = \hat{p}(i) \prod_{i=65}^{x} (1 - \hat{p}(i - 1))
$$

Figure 1 shows the the function $\bar{p}(x)$. 
Table 1: USA Mortality table. This table shows the conditional probability that a person with given age dies within next year, for Male and Female US citizens. $P(\text{age of death} < x + 1 \mid \text{age} = x)$
6. Case Study

This case study considers the following investment scenario:

- Age of person at retirement := 65
- Investment horizon := 35 years
- Sex := male
- Available investment amount := 500,000
- Yearly inflation rate := 3%
- Yearly rate of return on annuity := 3%
- Cash outflows := $50,000 at the end of each year
- CVaR confidence level (\(\alpha\)) := 90% for all periods
- \(\lambda := 1\)

The model developed in Section 4 is used to select a retirement portfolio. The model is fitted on 100 scenarios of stock movements. This case study assumes that the inflation rate and annuity rates are fixed over the investment horizon. We used PSG version 2.3 to solve the resulting quadratic optimization problem.

Table 2 shows the average investment in assets over time, the average is taken over scenarios. The numbers in the brackets are percentages of total portfolio. Figure 2 shows the average stock index portfolio value over time, the average is taken over all scenarios. The average value of the stock index portfolio grows from $500,000 at time \(t=0\) to $10,934,376 at time \(t=35\), this is equivalent to 7.07% yearly growth.

<table>
<thead>
<tr>
<th>Asset</th>
<th>t=0</th>
<th>t=5</th>
<th>t=10</th>
<th>t=15</th>
<th>t=20</th>
<th>t=25</th>
<th>t=30</th>
<th>t=35</th>
</tr>
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<tbody>
<tr>
<td>FI-MUNI</td>
<td>0(0)</td>
<td>109(32)</td>
<td>152(18)</td>
<td>171(19)</td>
<td>459(23)</td>
<td>486(20)</td>
<td>851(20)</td>
<td>1525(24)</td>
</tr>
<tr>
<td>INVGRD</td>
<td>0(0)</td>
<td>235(23)</td>
<td>16(2)</td>
<td>14(1)</td>
<td>19(1)</td>
<td>146(4)</td>
<td>22(0)</td>
<td>30(0)</td>
</tr>
<tr>
<td>USEQ-SM</td>
<td>0(0)</td>
<td>4(0)</td>
<td>16(1)</td>
<td>26(0)</td>
<td>50(0)</td>
<td>94(0)</td>
<td>182(0)</td>
<td>372(0)</td>
</tr>
<tr>
<td>USEQ-SMVAL</td>
<td>0(0)</td>
<td>12(1)</td>
<td>37(1)</td>
<td>67(1)</td>
<td>118(1)</td>
<td>235(0)</td>
<td>445(0)</td>
<td>874(0)</td>
</tr>
<tr>
<td>USEQ-SMGRTTH</td>
<td>0(0)</td>
<td>65(7)</td>
<td>252(18)</td>
<td>206(11)</td>
<td>339(12)</td>
<td>655(14)</td>
<td>1222(16)</td>
<td>2274(16)</td>
</tr>
<tr>
<td>USEQ-LG</td>
<td>0(0)</td>
<td>30(5)</td>
<td>7(1)</td>
<td>6(1)</td>
<td>7(0)</td>
<td>9(0)</td>
<td>6(0)</td>
<td>7(0)</td>
</tr>
<tr>
<td>USEQ-MID</td>
<td>0(0)</td>
<td>33(3)</td>
<td>90(2)</td>
<td>144(1)</td>
<td>245(1)</td>
<td>483(1)</td>
<td>883(1)</td>
<td>1680(1)</td>
</tr>
<tr>
<td>USEQ-LGVAL</td>
<td>0(0)</td>
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<td>26(3)</td>
<td>42(3)</td>
<td>86(4)</td>
<td>121(3)</td>
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</tr>
<tr>
<td>USEQ-LGGRTH</td>
<td>500(100)</td>
<td>126(23)</td>
<td>241(54)</td>
<td>569(63)</td>
<td>565(58)</td>
<td>924(58)</td>
<td>1810(60)</td>
<td>3462(56)</td>
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<td>11(1)</td>
<td>20(0)</td>
<td>35(0)</td>
<td>71(0)</td>
<td>152(0)</td>
<td>289(0)</td>
</tr>
</tbody>
</table>
Figure 2: Average value of the portfolio across time. Average is taken over scenarios.
Figure 3: Average investment in assets across time. Average is taken over scenarios. On the right side there are labels for the 4 largest investments at the end of year 35.
Figure 4: Average investment in assets across time. Each chart corresponds to individual assets. Average is taken over scenarios.
References

