Decision making under catastrophic risk

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Outline

Decision making under catastrophic risk.

- How to model catastrophic risk?
- Is safety-first principle rational?
- Robust decision making under catastrophic risk.

Estimation the catastrophe probability

- Generalized Chebyshev's inequalities and catastrophic risk estimation.

Catastrophic risk: estimation and decision making

Model:

- Uncertain outcome - (finite valued) random variable X.
- Catastrophe: $X \leq C$ for some $C \in \mathbb{R}$

Aims:

- Catastrophic risk estimation: find $\mathbb{P}[X \leq C]$.
- Decision making: Preference relation $\succeq$ on space of r.v.s

$\succeq$ is consistent with safety-first principle if

$$\mathbb{P}[X \leq C] < \mathbb{P}[Y \leq C] \quad \Rightarrow \quad X \succ Y$$

Safety-first principle: standard paradoxes

$\succeq$ is:

- Continuous if sets $\{Y : Y \succeq X\}$ and $\{Y : X \succeq Y\}$ - closed $\forall Y$
- Risk averse if $X \succeq Y$, whenever $Y \sim X + Z$ with $E[Z|X = x] = 0$, $\forall x$.
- (Equivalently, $X \succeq Y$, whenever $E[X] = E[Y]$ and $X \succeq_{SSD} Y$)

Paradoxes. Safety-first principle ($X \succeq Y \Leftrightarrow \mathbb{P}[X \leq C] \leq \mathbb{P}[Y \leq C]$):

- is not continuous;
- is not risk-averse.

Reason: It may be $X < C$. Wrong model?

Alternative model

- Random variable X takes values in $\mathbb{R} \cup \{-\infty\}$.
- Catastrophe: $X = -\infty$.
- Safety-first principle:

$$\mathbb{P}[X = -\infty] < \mathbb{P}[Y = -\infty] \quad \Rightarrow \quad X \succ Y$$

Theorem

(a) *$\succeq$ can be continuous, risk-averse, and consistent with the safety-first principle.*

(b) *Every risk-averse strictly monotone $\succeq$ is consistent with the safety-first principle.*

Is risk aversion "rational"?

Example

Person with capital 10^9.
Risk of imprisonment and confiscation 20%
A lawyer promises to reduce it to 19% for premium 10^7.
Should a person agree?

- Safety-first $\Rightarrow$ Yes.
- Risk aversion $\Rightarrow$ Yes.
- Common sense $\Rightarrow$ No.

Reason: 20% and 19% are practically indistinguishable!

Robust preference relation

$20\% \approx 19\%$, but $1\% \neq 0\%$.

Probabilities should be compared in relative sense!

$$\rho(\mathbb{P}, \mathbb{Q}) := \sup_{A \in \mathcal{M}} \frac{|\mathbb{Q}(A) - \mathbb{P}(A)|}{\mathbb{P}(A)}$$

$\rho(\mathbb{P}, \mathbb{Q}) \leq \epsilon \Rightarrow 1 - \epsilon \leq \mathbb{Q}(A)/\mathbb{P}(A) \leq 1 + \epsilon$ for any event A.

For any X, let

- $\widetilde{X}^{(\epsilon)}$ be an r.v. with the CDF $F_{\widetilde{X}^{(\epsilon)}}(x) = \mathbb{P}\left[\widetilde{X}^{(\epsilon)} \leq x\right] = \mathbb{Q}[X \leq x]$, where $\rho(\mathbb{P}, \mathbb{Q}) \leq \epsilon$;
- Δ_ϵ be an r.v. (error) such that $\sup |\Delta_\epsilon| \leq \epsilon$.

$\succeq$ is robust if $X \succ Y$ implies $\widetilde{X}^{(\epsilon)} + \Delta_\epsilon \succ \widetilde{Y}^{(\epsilon)}$ for some $\epsilon > 0$.

Sensitivity to catastrophic events

Definition

(Chichilnisky (2002)) $\succeq$ is insensitive to catastrophic events, if $X \succ Y$ implies that there exists $\epsilon > 0$ such that $X^{(\epsilon)} \succ Y^{(\epsilon)}$, where $X^{(\epsilon)}$ and $Y^{(\epsilon)}$ are any r.v.'s such that $\mathbb{P}\left[X = X^{(\epsilon)}\right] \geq 1 - \epsilon$ and $\mathbb{P}\left[Y = Y^{(\epsilon)}\right] \geq 1 - \epsilon$.

A "good" $\succeq$ should be

- Continuous
- Robust
- But not insensitive to catastrophic events!

Ordinal utility theory

Expected utility theory:

$$X \succeq Y \quad \Leftrightarrow \quad 0 \leq E[u(X)] - E[u(Y)] = \int_0^1 \left(\int_{q_Y(t)}^{q_X(t)} v(x)dx \right) dt$$

where $v(x) = u'(x)$.

Generalization (ordinal utility theory):

$$X \succeq Y \quad \Leftrightarrow \quad 0 \leq \int_0^1 \left(\int_{q_Y(t)}^{q_X(t)} v(x,t)dx \right) dt$$

Analysis

(a) $v(x,t) \equiv 1$ - safety-first principle.

(b) $v(x,t)$ is independent of t - expected utility.

(c) $v(x,t)$ is independent of x - Yaari's dual utility theory.

(d) $v(x,t) = u(x)p(t)$ - generalizes Prospect theory.

Let $\varphi(t) = \int_{-\infty}^{0} v(x,t)\,dx$.

Theorem

(i) $\succeq$ *is robust* $\Leftrightarrow \int_{\epsilon}^{1-\epsilon} \varphi(t)\,dt < +\infty, \quad \forall \epsilon > 0.$

(ii) $\succeq$ *is sensitive to catastrophic events* $\Leftrightarrow$
$\int_{0}^{\epsilon} \varphi(t)\,dt = +\infty, \quad \forall \epsilon > 0.$

Example

$v(x,t) = (1+|x|)^{t^2-t-1}$ is strictly monotone, robust, and sensitive to catastrophic events.

Summary of Part I: decision making

- Modelling catastrophe as $X \leq C$, $C \in \mathbb{R}$ may not be appropriate.
- We suggest to model it as $X = -\infty$
- In this setting, every monotone risk averse $\succeq$ is "safety-first" one.
- However, one may replace risk aversion by robustness
- We derive family of $\succeq$ which are:
  - continuous;
  - monotone;
  - robust;
  - sensitive to catastrophic events.

How to estimate the probability of catastrophe

Back to standard setting:

- X - some random variable.
- Catastrophe occurs if $X \leq C$ for some C.

Problem:

- How to estimate $\mathbb{P}[X \leq C]$?
- Given i.i.d. sample $x_1, x_2, \ldots, x_n$ it may be that all $x_i > C$.
- However, we can estimate $E[X]$ and (for example) $\sigma(X)$.
- Option 1: Assume particular distribution (normal?) and find $\mathbb{P}[X \leq C]$.
- Option 2: No assumptions, and use (one sided) Chebyshev's inequality:

$$P[X \leq EX - a] \leq \frac{\sigma^2(X)}{\sigma^2(X) + a^2}$$

Options 1 and Options 2 are two extremes.

Chebyshev's inequalities for families of distributions

Options 1 and Options 2 are two extremes.
Let us assume that X belongs to some family F.
Can we then improve Chebyshev's inequality? Yes!
Examples:

- If X has log-concave cdf then:

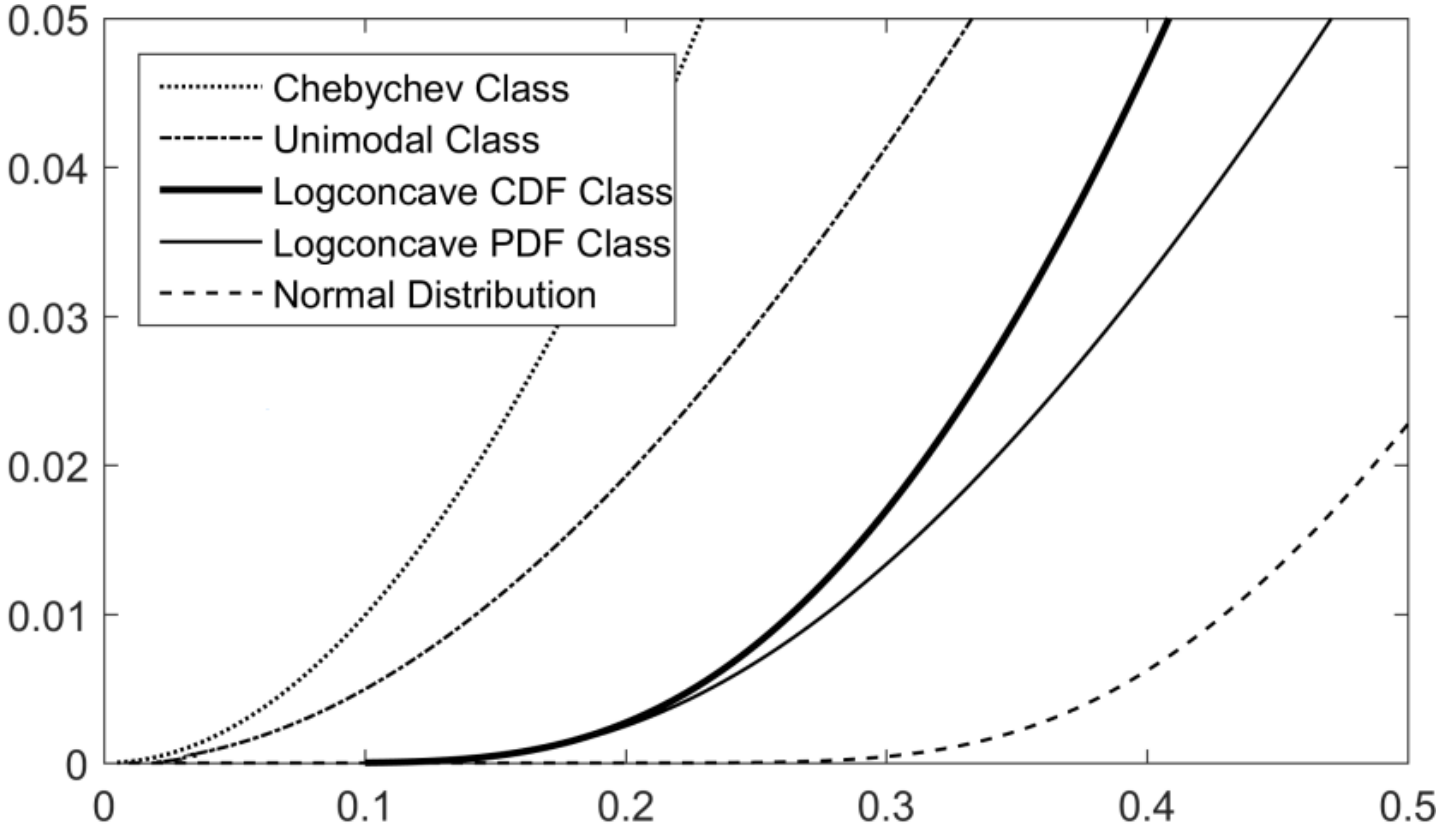
$$P[X \leq EX - a] \leq \beta, \quad \frac{\sqrt{1 + 2\beta \log \beta - \beta^2}}{\beta - \log \beta - 1} = \frac{\sigma(X)}{a}.$$

- For unimodal class (mode=mean)

$$P[X \leq EX - a] \leq \beta, \quad \frac{3a}{2}\sqrt{\frac{\beta}{(1-\beta)^3}} = \sigma(X).$$

Our methodology works for essentially any "natural" family F!
And for any "deviation measure" in place of σ!

Chebyshev's inequalities for families of distributions



Dependence of $\mathbb{P}[X \leq C]$ from $\frac{\sigma(X)}{EX-C}$[1].

[1] Figure from paper: Faridafshin, F., Grechuk, B., Naess, A., Calculating Exceedance Probabilities Using a Distributionally Robust Method. Structural Safety, 67, 2017, pp 132-141

Summary

Decision making under catastrophic risk.

- We suggest to model it as $X = -\infty$
- Formulate desirable properties of preference relation $\succeq$
- Derive family of $\succeq$ satisfying these properties.

Estimation the catastrophe probability

- Make assumptions about underlying distribution (assume that $X \in F$)
- Develop methodology to derive Chebyshev's inequalities for any family F
- Illustrate for log-concave and unimodal families.