Application of Buffered Probability of Exceedance to Structural Reliability Problems

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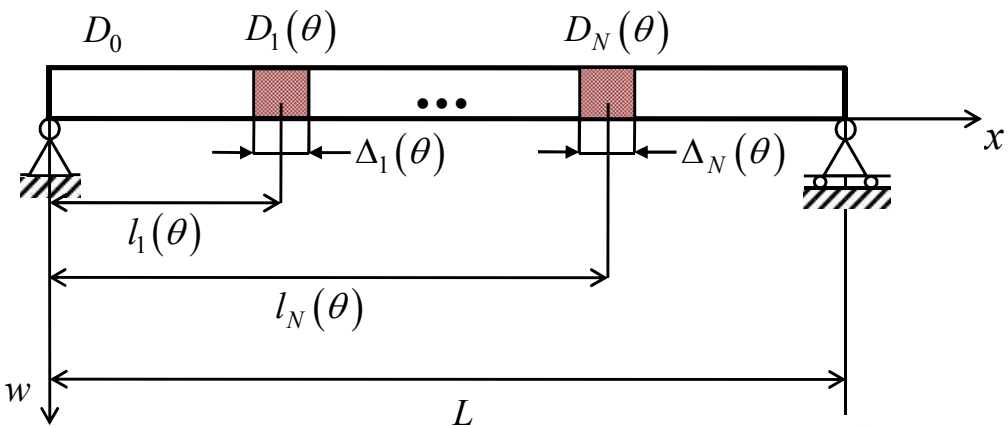
Objectives

- Apply the new Buffered Probability of Exceedance (bPOE) concept to structural reliability optimization problems.

- Optimize parameters of mechanical devices for excitation and the formation of wave motion taking into account the uncertainties in parameters (such as variability in material properties).

- Demonstrate how to apply the bPOE concept to optimization problems governed by partial differential equations with random and uncertain inputs.

Excitation of Beam Oscillation

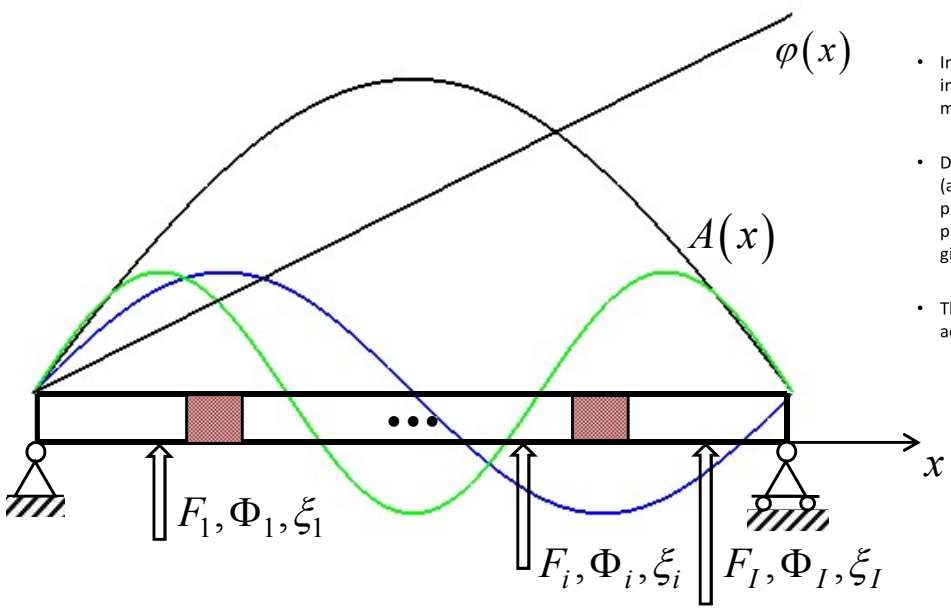


$$D_n = D_0\left(1+\theta_n^1\right)$$

$$l_n = \theta_n^2$$

$$\Delta_n = L\cdot\theta_n^3$$

$$n = \overline{1,N}$$



- Investigate vibrations of hinged beam with structural inhomogeneities (such as a crack). This is the simplest model of a mechanical device for excitation and formation of wave motion.
- Determine an optimal number of forces and their characteristics (application points, amplitude and phase of oscillation), which provide the best approximation of a given shape and point-wise phase vibrations of the beam in a given frequency range with a given accuracy.
- The mean square deviation is used to measure approximation accuracy.

Informal Problem Description

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Informal Problem Description (Cont'd)

- It is supposed that the system works inappropriately (i.e., "fails") if phase or amplitude deviate from the target more than some specified threshold.

- For minimization of probability of such deviation we use the new bPOE concept.

Hierarchy of Models

- **Model without Defects**
  Assumption: beam has no structural inhomogeneities (defects).

- **Deterministic Model with Defects.** Defects are characterized by a change in Young's modulus, their lengths and locations in the beam. Assumption: there is a complete information about parameters of defects.

- **Stochastic Model with Defects.** Assumption: only partial information about parameters of defects (lower and upper bounds of parameters) is available.

Model for Optimal Control Policy for Generating Vibration of Homogeneous Beam

- The problem of exciting a hinged elastic homogeneous beam by I forces with complex amplitudes $F_i, i = 1, \ldots, I,$ at a frequency ω is reduced to the following boundary value problem:

$$\frac{\partial^2}{\partial x^2}\left(ED\frac{\partial^2 w}{\partial x^2}\right) + \rho\frac{\partial^2 w}{\partial t^2} = \sum_{i=1}^{I} F_i e^{i\omega t}\delta(x - \xi_i), \quad x \in (0, L)$$

$$\begin{cases} w(0,t) = w(L,t) = 0 \\ \dfrac{\partial^2 w(0,t)}{\partial x^2} = \dfrac{\partial^2 w(L,t)}{\partial x^2} = 0 \end{cases} \qquad (1)$$

where E, D, ρ are Young's modulus, the moment of inertia of the cross section and density of the beam, respectively.

- We assume that $w(x,t) = w(x)e^{i\omega t}$, where $w(x)$ is complex amplitude of deflection.

Model for Optimal Control Policy for Generating Vibration of Homogeneous Beam (Cont'd)

We seek a solution of equation (2) in the following form:

$$w(x) = \sum_{i=1}^{I} F_i G(x, \xi_i), \tag{3}$$

where $G(x, \xi)$ is Green's function for (2):

$$\begin{aligned} &G_x^{(4)} - \pi^4 k^4 G = \delta(x - \xi), \ x \in (0,1) \\ &\begin{cases} G(0,\xi) = G(1,\xi) = 0, \ x \in (0,1) \\ G_x''(0,\xi) = G_x''(1,\xi) = 0 \end{cases} \end{aligned} . \tag{4}$$

Statement of the Control Problem

- Let the desired waveform is

$$W(x) = A(x)e^{i\Phi(x)}. \tag{5}$$

- Discrepancy $R(x) = w(x) - W(x) = A_R e^{i\phi_R}$, where $w(x)$ is a solution of (2) for a given parameters $I, F_i, \xi_i, i = 1,\ldots,I$.
- Let $I(\cdot)$ be a positive definite convex functional in L_2. It is used as an optimality criterion in the following system control problems.
- $I(R)$ has the form

$$I(R) = \int_0^1 |w|^2 dx - 2\mathrm{Re}\int_0^1 wW dx + \int_0^1 |W|^2 dx. \tag{7}$$

Statement of the Control Problem (Cont'd)

- **Problem 1.** For given I and $W(x)$ determine parameters $F_i, \xi_i, i = 1, \ldots, I$ minimizing $I(R)$.
- **Problem 2.** Determine the minimum value of I and the corresponding values of $F_i, \xi_i, i = 1, \ldots, I$, for which $I(R) \le \varepsilon^2$, where ε is the specified accuracy.
- Let $w = re^{i\phi}$ be a solution of (2). Then

$$I(R) = \int_0^1 R\bar{R}dx = \int_0^1 \left\{ (r - A)^2 + 4rA\sin^2\left(\frac{\phi - D}{2}\right) \right\} dx.$$

Necessary Conditions for the Minimum of the Functional

Let us introduce the following notations:

$$G_j = G(x, \xi_j),\ K_{ij} = \int_0^1 G_i G_j dx,\ \mathbf{K} = \{K_{ij}\}_{i,j=1}^{I},\ b_j = \int_0^1 G_j W dx,$$
$$\vec{b}^T = \{b_j\}_{j=1}^{I},\ \vec{F}^T = \{F_j\}_{j=1}^{I}. \tag{8}$$

Then,

$$\frac{\partial I(R)}{\partial \vec{F}} = 2\left(\mathbf{K}\vec{F} - \vec{b}\right), \tag{9}$$

where

$$\left(\frac{\partial I(R)}{\partial \vec{F}}\right)^T = \left\{\frac{\partial I(R)}{\partial u_j} + i\frac{\partial I(R)}{\partial v_j}\right\}_{j=1}^{I}.$$

Necessary Conditions for the Minimum of the Functional (Cont'd)

If the following equation fulfills

$$\frac{\partial I(R)}{\partial \vec{F}} = 0, \tag{10}$$

then the values of $\vec{F}$ minimizing $I(R)$ are determined by

$$\vec{F} = \mathbf{K}^{-1}\vec{b}. \tag{11}$$

With notations (8), formula (7) can be presented in the following form:

$$I(R) = \vec{F}^{T}\mathbf{K}\vec{F} - 2\mathrm{Re}\left(\vec{F}\vec{b}^{T}\right) + \int_0^1 |W|^2 dx. \tag{12}$$

Statement 3. $\boldsymbol{K}$ is a real, symmetric and positive definite matrix.

Necessary Conditions for the Minimum of the Functional (Cont'd)

Statement 4. When (10) is satisfied, expression (12) takes the form:

$$I(R) = -\vec{b}^T K^{-1} \bar{\vec{b}} + \int_0^1 |W|^2 dx. \tag{13}$$

Statement 5. For $I(R)$ the following equality takes place:

$$\frac{\partial I(R)}{\partial \xi_j} = 2\sum_{i \neq j} \frac{\partial K_{ij}}{\partial \xi_j}\left(u_i u_j + v_i v_j\right) + \frac{\partial K_{jj}}{\partial \xi_j}\left(u_j^2 + v_j^2\right) - 2\left(u_j \operatorname{Re}\left(\frac{\partial b_j}{\partial \xi_j}\right) + v_j \operatorname{Im}\left(\frac{\partial b_j}{\partial \xi_j}\right)\right) \tag{14}$$

Statement 6. If (10) takes place, then

$$\frac{\partial I(R)}{\partial \xi_j} = -\frac{\partial \vec{b}^T}{\partial \xi_j} K^{-1} \vec{b} - \vec{b}^T K^{-1} \frac{\partial \bar{\vec{b}}}{\partial \xi_j} + \vec{b}^T K^{-1} \frac{\partial K}{\partial \xi_j} K^{-1} \bar{\vec{b}}. \tag{15}$$

Green's Function

The solution of (4) is

$$G(x,\xi)=\begin{cases} g(x,\xi),\ x<\xi, \\ g(\xi,x),\ x>\xi, \end{cases}, \tag{16}$$

where

$$g(x,\xi)=\frac{\sin k\pi x \sin k\pi(1-\xi)\ \mathrm{sh}\ k\pi - \mathrm{sh}\ k\pi x\ \mathrm{sh}\ k\pi(1-\xi)\sin k\pi}{2k^3\pi^3 \sin k\pi\ \mathrm{sh}\ k\pi}. \tag{17}$$

Optimal Control Model for Generation of Vibration of Beam with Defect (Cont'd)

- We assume that:

  (1) the beam is excited by a single force of intensity F, applied at point $x > \xi$; (2) the defect of length ℓ located at point $x = \chi \neq \xi$ is characterized by a change in Young's modulus ΔE.

- The control of oscillations is described by the following equation:

$$\frac{d^2}{dx^2}\left(E(x)I\frac{d^2w}{dx^2}\right) - \omega^2\rho w = F\delta(x-\xi), \tag{18}$$

where

$$E(x) = E_0(1 - \Delta E f(x)), \tag{19}$$

$$f(x) = H(x - (\chi - \ell)) - H(x - (\chi + \ell)). \tag{20}$$

Optimal Control Model for Generation of Vibration of Beam with Defect (Cont'd)

- After some transformations and the introduction of dimensionless quantities, we obtain:

$$(1-\Delta E f(x))w_x^{(4)}(x,\xi)-\pi^4 k^4 w(x,\xi)=F\delta(x-\xi)+2\Delta E w_x''(x,\xi)\sum_{k=0}^{\infty}\frac{\delta^{(2k+1)}(x-\chi)}{(2k+1)!}e^{2k+1}+$$

$$+4\Delta E w_x^{(3)}(x,\xi)\sum_{k=0}^{\infty}\frac{\delta^{(2k+2)}(x-\chi)}{(2k+1)!}e^{2k+1},$$

where x, χ, ξ, F are dimensionless quantities.

- We want to find a solution in the form $w=w_0+\hat{w}$, where $\hat{w}=\bar{\bar{O}}(w_0)$ in the sense of $\ell, \Delta E \sim 0$.

Optimal Control Model for Generation of Vibration of Beam with Defect (Cont'd)

- We obtain:

$$\begin{cases} w_{0x}^{(4)}(x,\xi) - \pi^4 k^4 w_0(x,\xi) = F\delta(x-\xi), \\ \hat{w}_x^{(4)}(x,\xi) - \pi^4 k^4 \hat{w}(x,\xi) = \Delta E\left\{ f\left(w_0+\hat{w}\right)_x^{(4)} + 2\left(w_0+\hat{w}\right)_x^{(2)} \sum_{k=0}^{\infty} \frac{\delta^{(2k+1)}(x-\chi)}{(2k+1)!} e^{2k+1} + \right. \\ \left. +4\left(w_0+\hat{w}\right)_x^{(3)} \sum_{k=0}^{\infty} \frac{\delta^{(2k+2)}(x-\chi)}{(2k+1)!} e^{2k+1} \right\}. \end{cases} \quad (24)$$

- We want to find $\hat{w}$ in the form:

$$\hat{w} = \varepsilon_1 w_1 + \varepsilon_2 w_2 + \cdots, \quad (25)$$

where $\varepsilon_{i+1} = \bar{\bar{O}}(\varepsilon_i)$ in the sense $\ell, \Delta E \sim 0$.

Optimal Control Model for Generation of Vibration of Beam with Defect (Cont'd)

- It makes sense to consider $\tilde{G}(x,\xi) = G_0(x,\xi) + \varepsilon_1 G_1(x,\xi) + \cdots + \varepsilon_k G_k(x,\xi)$ as an asymptotic approximation. Further we assume that

$$G(x,\xi) = G_0(x,\xi) + \varepsilon_1 G_1(x,\xi,\chi), \tag{31}$$

where G_0 is defined in (16)-(17), $\varepsilon_1 = 2\ell\Delta E$, and G_1 is defined as:

$$\begin{aligned} &G_{1x}^{(4)} - \pi^4 k^4 G_1 = G_{0x}^{(2)}(\chi,\xi)\delta''(x-\chi) \\ &\begin{cases} G_1(0,\xi,x) = G_1(1,\xi,x) = 0 \\ G_{1x}^{(2)}(0,\xi,x) = G_{1x}^{(2)}(1,\xi,x) = 0 \end{cases} \end{aligned} . \tag{32}$$

- Hence, we have:

$$G_1(x,\xi) = \frac{\partial^2 G_0(\chi,\xi)}{\partial x^2}\frac{\partial^2 G_0(x,\chi)}{\partial \xi^2}. \tag{33}$$

Investigation of the Functional for the 1st Approximation

- Let $G(x,\xi,\chi)=G_0(x,\xi)+\varepsilon_1 G_1(x,\xi,\chi)$, where G_0 is defined in (16)-(17), G_1 is defined in (33), where

$$K_{ij}=\int_0^1 G_0(x,\xi_i)G_0(x,\xi_j)dx+\varepsilon_1\int_0^1\left(G_0(x,\xi_i)G_1(x,\xi_j,\chi)+G_0(x,\xi_j)G_1(x,\xi_i)\right)dx+ \\ +\varepsilon_1^2\int_0^1\left(G_1(x,\xi_i)G_1(x,\xi_j)\right)dx=K_{ij}^{\ 0}+\varepsilon_1 K_{ij}^{\ 1}+\varepsilon_1^2 K_{ij}^{\ 2}. \tag{34}$$

- The matrix $\mathbf{K}=\{K_{ij}\}_{i,j=1}^{I}$, where K_{ij} are computed by formula (34), is a generalization of the matrix $\mathbf{K}$ defined in (8), for the case of beam with defects.
- **Statement 7.** $\mathbf{K}$ is a real, symmetric and positive definite matrix.

Derived Formulas

To get the elements of the $\mathbf{K} = \{K_{ij}\}_{i,j=1}^{I}$ and $\vec{b}$ we use formulas:

$$K_{ij}^{0} = \int_0^1 G_0(x,\xi_i) G_0(x,\xi_j)\,dx,$$

$$b_j^{0} = \int_0^1 G_0(x,\xi_j)\, w(x)\, dx,$$

$$K_{ij}^{1} = \frac{\partial^2 G_0(\chi,\xi_j)}{\partial x^2} \int_0^1 G_0(x,\xi_i) \frac{\partial^2 G_0(x,\chi)}{\partial \xi^2}\,dx + \frac{\partial^2 G_0(\chi,\xi_i)}{\partial x^2} \int_0^1 G_0(x,\xi_j) \frac{\partial^2 G_0(x,\chi)}{\partial \xi^2}\,dx \tag{36}$$

$$K_{ij}^{2} = \frac{\partial^2 G_0(\chi,\xi_j)}{\partial x^2} \frac{\partial^2 G_0(\chi,\xi_j)}{\partial x^2} \int_0^1 \left(\frac{\partial^2 G_0(x,\chi)}{\partial \xi^2} \right)^2 dx,$$

$$b_j^{1} = \frac{\partial^2 G_0(\chi,\xi_j)}{\partial x^2} \int_0^1 \left(\frac{\partial^2 G_0(x,\chi)}{\partial \xi^2} \right)^2 w(x)\,dx.$$

Gradients are calculated directly from (36).

Multiple Defects

- Consider the case when there are multiple defects in the beam.
- The k-th defect is at χ_k and is characterized by $\varepsilon_{1(k)} = 2\ell_k \Delta E_k$.
- Formulas (10), (11), (12) do not change because of the linearity of the problem, where

$$\begin{aligned} &\mathbf{K} = \sum_{k=1}^{K} \mathbf{K}_{(k)}, \quad \vec{b} = \sum_{k=1}^{K} \vec{b}_{(k)}, \\ &\mathbf{K}_{(k)} = \mathbf{K}^0_{(k)} + \varepsilon_{1(k)} \mathbf{K}^1_{(k)} + \varepsilon^2_{1(k)} \mathbf{K}^2_{(k)}, \\ &\vec{b}_{(k)} = \vec{b}^0_{(k)} + \varepsilon_{1(k)} \vec{b}^1_{(k)}. \end{aligned} \tag{37}$$

Risk Measures (CDF, VaR, POE)

- Cumulative distributions function (CDF) of a random variable x:

$$F_X(x) = P\{X \leq x\},\ x \in (-\infty, \infty).$$

- Quantile function $q_\alpha(X)$ (in financial applications, it is called Value-at-Risk (VaR)):

$$F_X^{-1}(\alpha) = q_\alpha(X) = \inf\{x: \alpha \leq F_X(x)\} \text{ for } \alpha \in (0,1).$$

- Probability of Exceedance (POE):

$$p_x(X) = P(X > x) = 1 - F_X(x),\ x \in (-\infty, \infty).$$

- Both POE and VaR are "optimistic" risk functions which are based on the lower bound of outcomes in the tail.
- Both characteristics do not provide information about large losses, which may occur with low probability.
- POE and VaR are difficult to optimize for discrete distributions, because they are non-convex, non-smooth, and have multiple local extrema.

Risk Measures (CVaR)

- CVaR (superquantile) for a distribution, which does not jump at the VAR, $q_\alpha(X)$, equals the expected loss exceeding quantile

$$\bar{q}_\alpha(X) = E[X : X \geq q_\alpha(X)].$$

- In general case, when F_X has a jump at VaR, $q_\alpha(X)$, the definition of CVaR is more complicated:

$$\bar{q}_\alpha(X) = \min_u \left(u + \frac{E[X-u]^+}{1-\alpha} \right),$$

where $[\cdot]^+ = \max\{\cdot, 0\}$.

- CVaR has a much better mathematical properties compared to VaR. It is convex in random variables. Moreover, it can be optimized with linear and convex programming algorithms.

Risk Measures (bPOE)

- Buffered Probability of Exceedance (bPOE):

$$\bar{p}_x(X) = 1 - \bar{q}_x^{-1}(X) \text{ for } EX < x < \sup X.$$

- CVaR) and bPOE are conservative counterparts of VaR and POE.
- bPOE is an upper bound for POE. It provides more information about the magnitude of outcomes in the tail of the distribution, compared to POE.
- bPOE can be calculated with the following formula:

$$\bar{p}_x(X) = \min_{a \geq 0} E[a(X - x) + 1]^+ .$$

- This one-dimensional optimization problem is convex, and in the case of finite probability space it can be reduced to LP.
- Minimizing of bPOE is intended to reduce the probability of an undesirable tail events. Compared to bPOE minimization, the CVaR minimization fixes the probability of the tail and minimizes the average of the tail outcomes

Optimization Problem Statement

- Let the beam contains an undefined number of defects N_{def} with undefined characteristics ℓ_{def}, dE_{def}, in unknown locations χ_{def}.
- The beam is excited by I forces $F_j = u_j + iv_j$ applied at the points ξ_j, $j = 1, \ldots, I$.
- For the given form of vibration $W(x) = \mathrm{Re}\,W(x) + i\,\mathrm{Im}\,W(x)$, we want to determine an optimal allocation of forces and their characteristics.
- It is assumed the following prior information for the random defects: the minimum and maximum number of defects, the boundaries of their possible location, the boundaries of linear dimensions, and changes in Young's modulus of defects.

Optimization Problem Statement (Cont'd)

Assuming that the beam defects are described by the random vector $\vec{\theta}^T = \{N_{def}, \ell_{def}, dE_{def}, \chi_{def}\}$, we define the risk functional J that includes the mean square deviation of the deflection of the beam (and the deflection phase) from the given oscillation form:

$$J(\vec{F},\vec{\xi}) = \frac{1}{2}\rho\left(\int_0^1 \left|w(\vec{\theta},\vec{F},\vec{\xi},x - W(x)\right|^2 dx\right) + \frac{\lambda}{2} D(\vec{F},\vec{\xi}), \tag{63}$$

where ρ is the measure of risk, $D(\cdot,\cdot)$ is the regularizing functional defined on the control space, and λ is the penalty coefficient. Denote the mean square deviation:

$$L(\vec{\theta},\vec{F},\vec{\xi}) = \int_0^1 \left|w(\vec{\theta},\vec{F},\vec{\xi},x) - W(x)\right|^2 dx\,. \tag{64}$$

We use bPOE measure of risk ρ. The control problem is an unconditional minimization of the functional J on a given domain of control space $(\vec{F},\vec{\xi}) \in U$:

$$\min_{\vec{F},\vec{\xi}\in U} J(\vec{F},\vec{\xi}). \tag{65}$$

Optimization Problem Statement (Cont'd)

- To solve problem (65), we fixed the controls $(\vec{F}, \vec{\xi})$ and generated N_{Sample} random realizations of defects $\vec{\theta}_1, \ldots, \vec{\theta}_{N_{Sample}}$, using an a priori predetermined distribution law of $\vec{\theta}$.
- For fixed controls $(\vec{F}, \vec{\xi})$, for each generated random realization of defects, $\vec{\theta}_i, i = 1, \ldots, N_{Sample}$, we solved corresponding boundary value problem (28), using the results presented in previous slides.
- We obtained N_{Sample} realizations, $L_1(\vec{\theta}_1, \vec{F}, \vec{\xi}), \ldots, L_{N_{Sample}}(\vec{\theta}_{N_{Sample}}, \vec{F}, \vec{\xi})$, of random variable $L(\vec{\theta}, \vec{F}, \vec{\xi})$.

Optimization Problem Statement (Cont'd)

- Let us consider $L(\vec{\theta},\vec{F},\vec{\xi})$ as a discrete random variable with values $L_1(\vec{\theta}_1,\vec{F},\vec{\xi}),\ldots,L_{N_{Sample}}(\vec{\theta}_{N_{Sample}},\vec{F},\vec{\xi})$ and equal probabilities.
- Then

$$bPOE_z\left(L(\vec{\theta},\vec{F},\vec{\xi})\right)=\begin{cases}0, & \text{if } z\geq \max_j L_j(\vec{\theta}_j,\vec{F},\vec{\xi})\\ 1-\alpha, & \text{if } average\left(L(\vec{\theta},\vec{F},\vec{\xi})\right)<z<\max_j L_j(\vec{\theta}_j,\vec{F},\vec{\xi}),\\ 1, & \text{if } z\leq average\left(L(\vec{\theta},\vec{F},\vec{\xi})\right)\end{cases} \qquad (66)$$

where z is a threshold; α is a confidence level such that $CVaR_\alpha\left(L(\vec{\theta},\vec{F},\vec{\xi})\right)=z$.

- ***Optimization problem***

Minimizing risk

$$\min_{\vec{F},\vec{\xi}}\left[\frac{1}{2}bPOE_z\left(L(\vec{\theta},\vec{F},\vec{\xi})\right)+\frac{\lambda}{2}D(\vec{F},\vec{\xi})\right]. \qquad (67)$$

Results of Simulations:
Deterministic Case without Defects

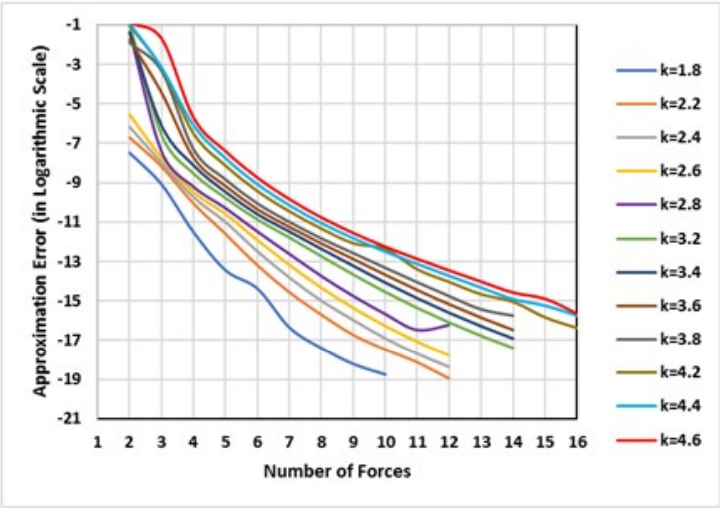


Figure 1. Dependence of Approximation Error (Objective Function) on a Number of Applied Forces and Wave Number.

- For any frequency (wave number, k) the increase in the number of applied forces, results in the reduction of the approximation error.
- For any fixed number of applied forces, the approximation error increases with the increase of the wave number, k.

Results of Simulations
Deterministic Case without Defects

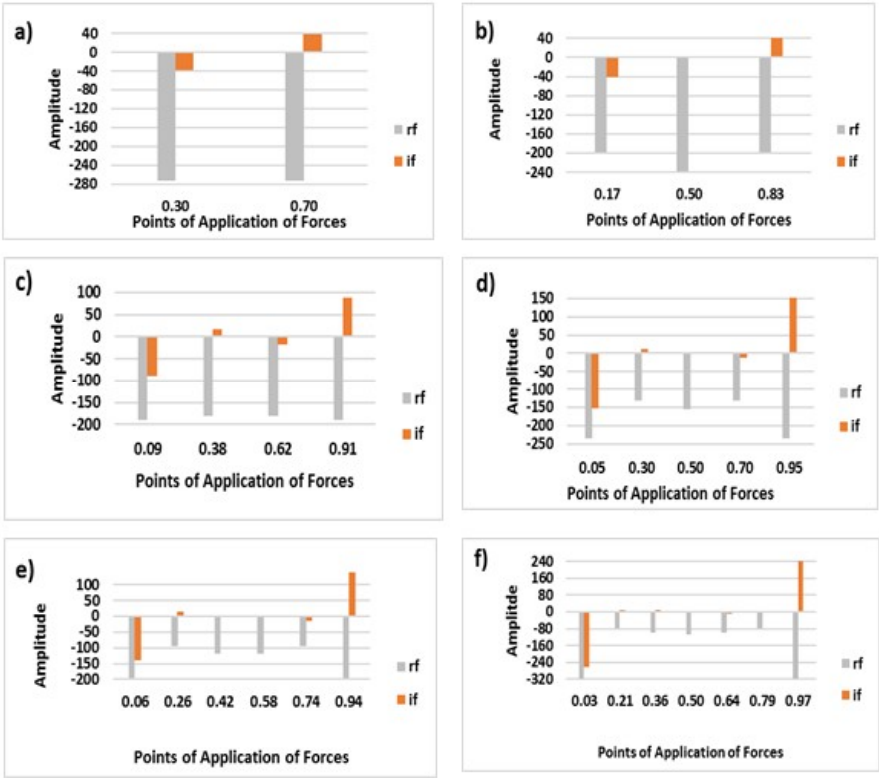


Figure 2. Optimal characteristics of applied forces (application places, real, rf, and imaginary, if, parts of their amplitudes), for wave number k =1.8 in cases: 2 forces (a), 3 forces (b), 4 forces (c), 5 forces (d), 6 forces (e), 7 forces (f).

Results of Simulations
Deterministic Case without Defects

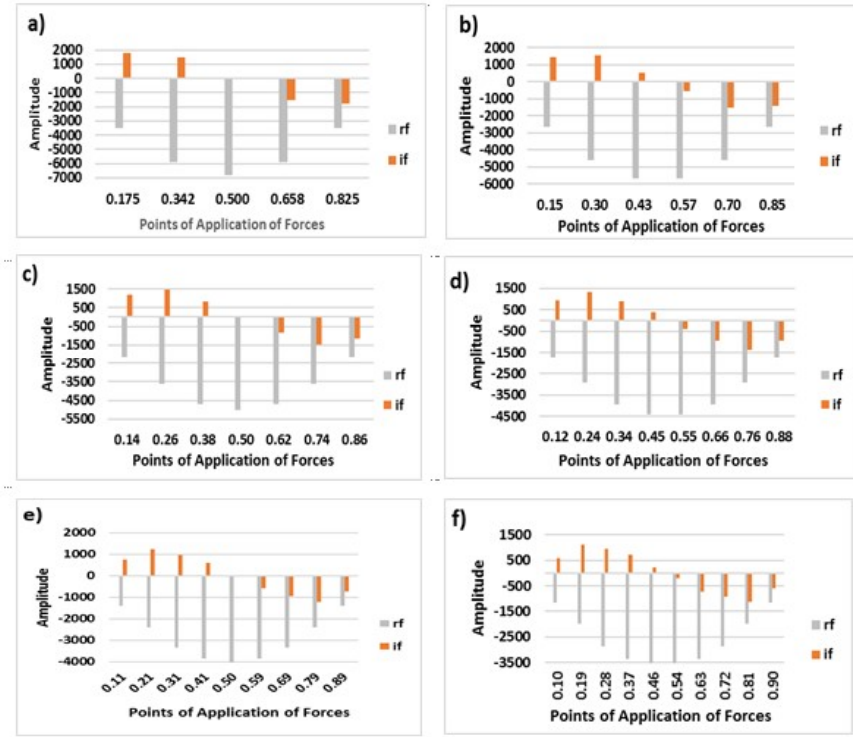


Figure 3. Optimal characteristics of applied forces (application places, real, rf, and imaginary, if, parts of their amplitudes), for wave number k =4.6 in cases: 5 forces (a), 6 forces (b), 7 forces (c), 8 forces (d), 9 forces (e), 10 forces (f).

Results of Simulations
Deterministic Case with one Defect

- Any defect is characterized by:
  1. its position in the beam xx;
  2. geometric size $\Delta L \in (0,0.1)$;
  3. relative change in Young's modulus ΔE.
- The defect may be located anywhere in the beam: $xx \in (0,1)$.
- The relative change in the Young's modulus of ΔE can vary within interval (-0.5, 0.5). Negative ΔE value corresponds to the consolidation; positive, to weakening; $\Delta E = 0$ means that there are no defects.
- We use parameter $Delta = \Delta E \times \Delta L$ in the model as a defect characteristic.

Results of Simulations
Deterministic Case with one Defect

- First, we suppose that there is full information about these defect parameters.
- We investigated how value of defect ($Delta$), and defect location, xx, influence the solution of the problem of optimal control of elastic beam oscillation and corresponding approximation error (approximation accuracy).
- We fixed the wave number =1.8, number of forces = 4, and randomly generated single defects with different pair of parameters (xx, $Delta$) 90000 times.
- For each fixed value of (xx, $Delta$) we solved the problem of optimal control of elastic beam oscillation, and determined the optimal characteristics of forces, and minimal value of objective function, which is considered as approximation error (accuracy).

Results of Simulations
Deterministic Case with one Defect

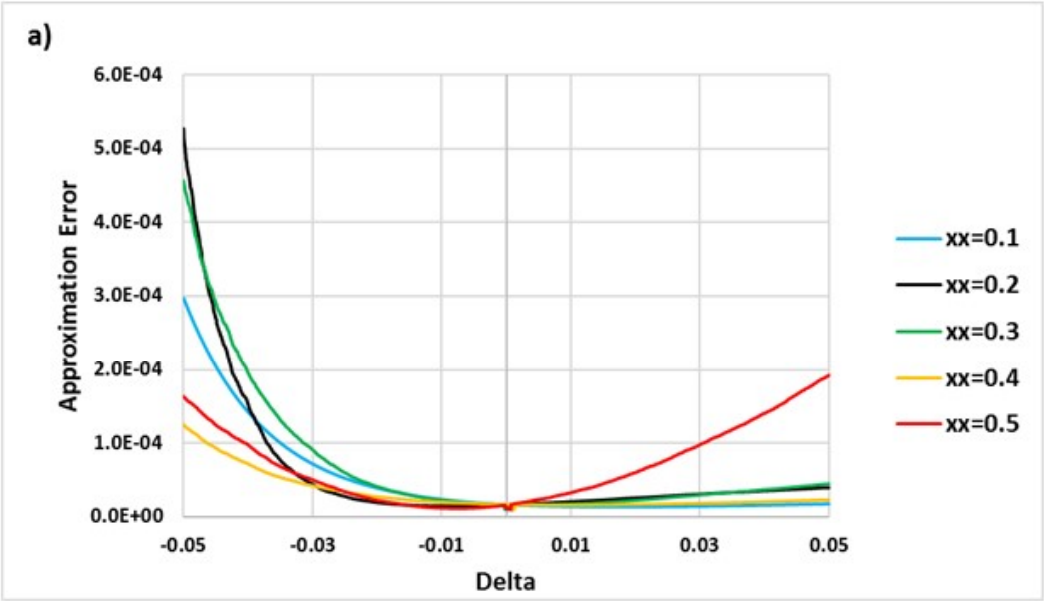


Figure 4a. Dependence of approximation error on defect value (Delta), for defect locations xx = 0.1, 0.2, 0.3, 0.4, 0.5. Wave number =1.8; number of forces = 4.

Results of Simulations
Deterministic Case with one Defect

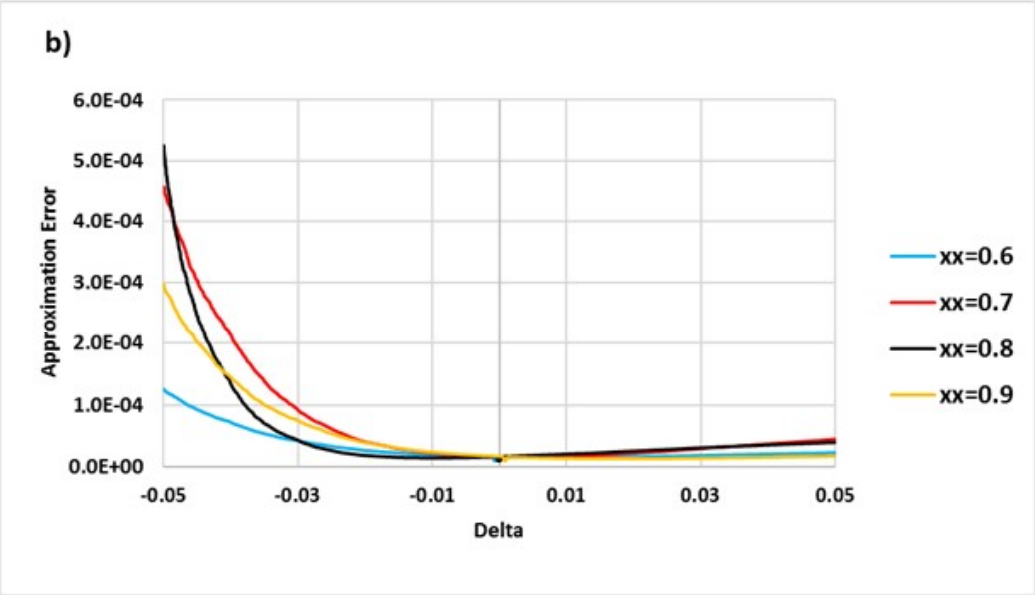


Figure 4b. Dependence of approximation error on defect value (Delta), for defect locations xx = 0.6, 0.7, 0.8, 0.9. Wave number =1.8; number of forces = 4.

Results of Simulations
Deterministic Case with one Defect

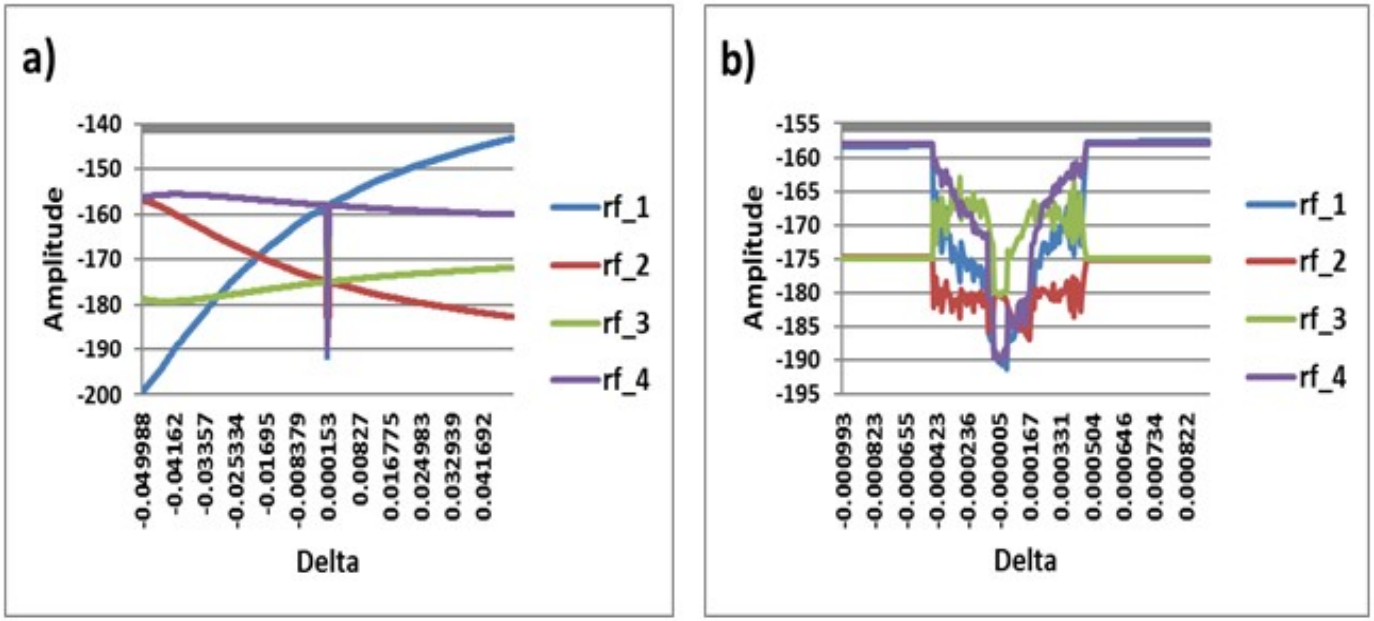


Figure 5. Dependence of real parts of forces on defect size (Delta) for fixed defect location (xx = 0.4). Wave number =1.8; number of forces = 4; ranges of defect sizes are: (a) Delta ∈ (-0.05,0.05), (b) Delta ∈ (-0.001,0.001).

Results of Simulations
Deterministic Case with one Defect

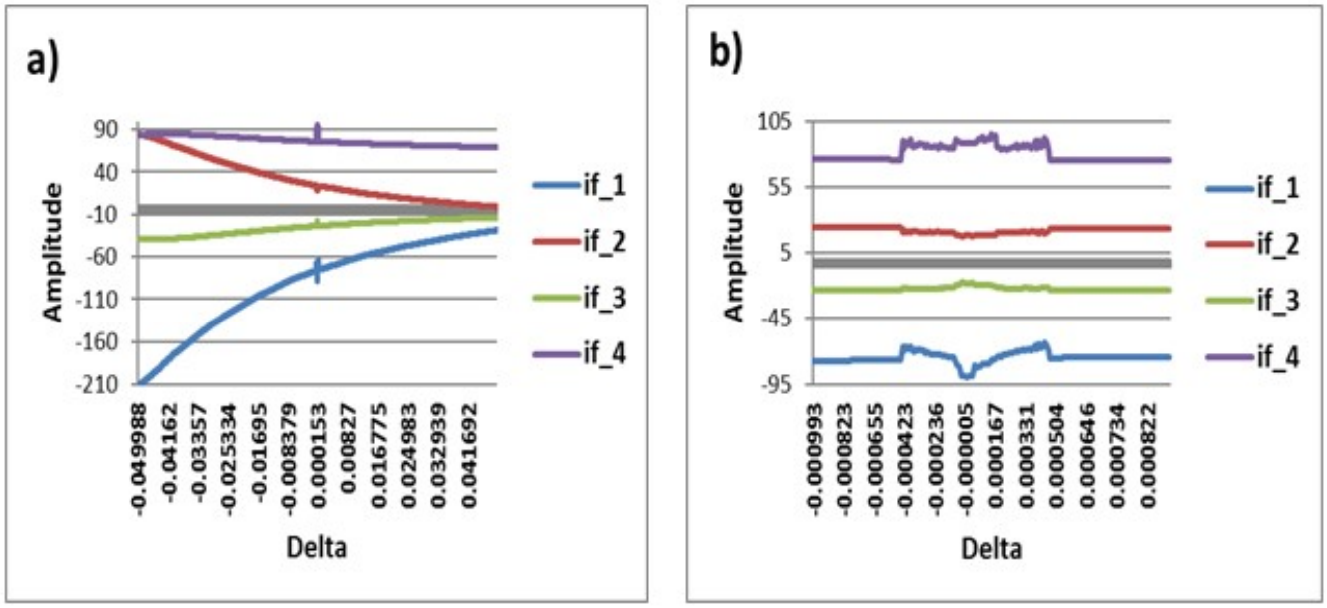


Figure 6. Dependence of imaginary parts of forceson defect size (Delta) for fixed defect location (xx = 0.4). Wave number =1.8; number of forces = 4; ranges of defect sizes are: (a) Delta $\in$ (-0.05,0.05), (b) Delta $\in$ (-0.001,0.001).

Results of Simulations
Deterministic Case with Multiple Defects

- We fixed optimal solution of deterministic problem without defects with wave number k =1.8 and number of forces = 4.

- Then we generated three series of scenarios. Each series contains one thousand scenarios. Each scenario from the first series contains one generated defect; scenario from the second series contains no more than five generated defects; scenario from the third series contains no more than ten generated defects.

- For each scenario we recalculated approximation error, obtained by solving optimization problem without defects.

- As a result, we obtained three series of approximation errors.

- Then we built cumulative distribution functions using these serios of scenarios.

Results of Simulations
Deterministic Case with Multiple Defects

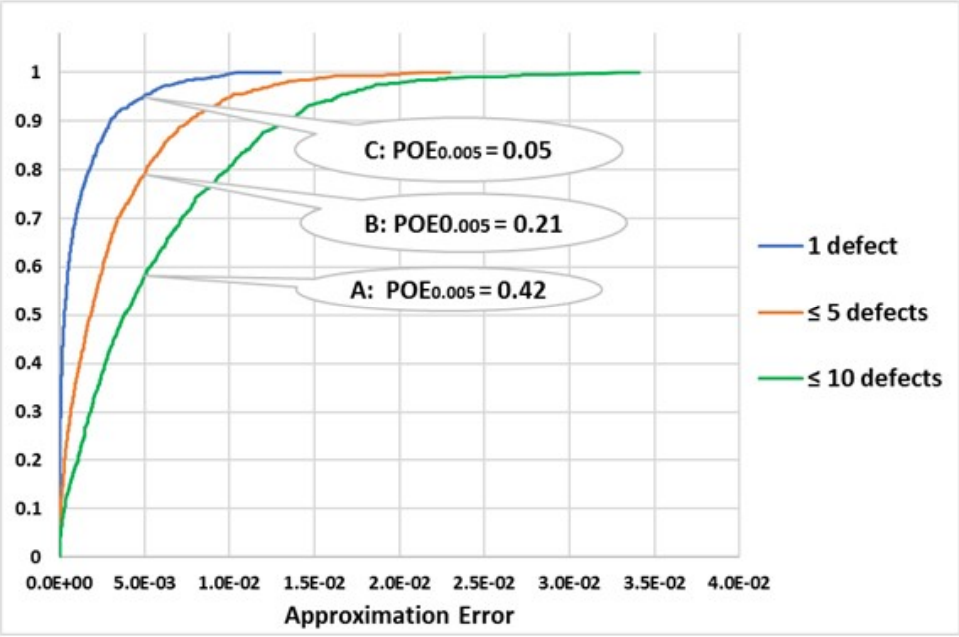


Figure 7. Probability Distribution Functions for Approximation Errors, and Values of Probability of Exceedance (POE) Corresponding to Multiple Defects.

Results of Simulations
Stochastic Case with Multiple Defects (Approach)

- Numerical simulations were conducted for given form of vibration $W(x) = \operatorname{Re} W(x) + i \operatorname{Im} W(x)$, different numbers of forces I, and wave numbers k=1.8 and k=4.6.
- For fixed I and k we solved **deterministic problem** of optimal control of beam oscillation without defects and obtained optimal characteristics of forces (real and imaginary parts of their amplitudes $F_j = u_j + iv_j$, application places ξ_j, $j = 1,\ldots,I$).
- For fixed optimal deterministic solution $(\vec{F},\vec{\xi})$, $\vec{F} = (F_1,\ldots,F_I)^T$, $\vec{\xi} = (\xi_1,\ldots,\xi_I)^T$ we performed M=10 times the following procedure:
  1. Generate $N_{Sample} = 1000$ samples $\vec{\theta}^T = \{N_{def}, \ell_{def}, dE_{def}, \chi_{def}\}$ of random defects.
  2. Determine scenarios $L_1(\vec{\theta}_1,\vec{F},\vec{\xi}),\ldots,L_{N_{Sample}}(\vec{\theta}_{N_{Sample}},\vec{F},\vec{\xi})$ of random mean square deviation $L(\vec{\theta},\vec{F},\vec{\xi}) = \int_0^1 \left| w(\vec{\theta},\vec{F},\vec{\xi},x) - W(x) \right|^2 dx$, using fixed optimal deterministic solution $(\vec{F},\vec{\xi})$, and generated defects $\vec{\theta}$.
  3. Calculate initial (original) value of $bPOE_z(L(\vec{\theta},\vec{F},\vec{\xi}))$, using a threshold z and generated scenarios.
  4. Solve optimization problem, using regularizing functional $D(\vec{F},\vec{\xi})$:

$$\min_{\vec{F},\vec{\xi}\in U} bPOE_z(L(\vec{\theta},\vec{F},\vec{\xi})) + \lambda D(\vec{F},\vec{\xi}).$$

Results of Simulations
Generation of Random Defects

Random defects were generated, using the following prior information:

1. Number of defects in one sample $1 \le N_{def} \le 5$;

2. Vector of defects sizes $\ell_{def} = (\Delta L_1, \ldots, \Delta L_{N_{def}})^T$, $0 < \Delta L_m < 0.1$, $m = 1, \ldots, N_{def}$;

3. Vector of relative change in Yung's modulus $dE_{def} = (\Delta E_1, \ldots, \Delta E_{N_{def}})^T$, $-0.5 \le \Delta E_m \le 0.5$, $m = 1, \ldots, N_{def}$;

4. Vector of defects positions $\chi_{def} = (xx_1, \ldots, xx_{N_{def}})^T$, $0.1 \le xx_m \le 0.9$, $m = 1, \ldots, N_{def}$.

Results of Simulations
Stochastic Case #1

Parameters: Number of Samples =1000; Wave Number = 1.8;

Number of Forces = 5; Threshold= 0.011; Number of Runs $\boldsymbol{M}$ = 10, Number of Defects $\leq$ 5.

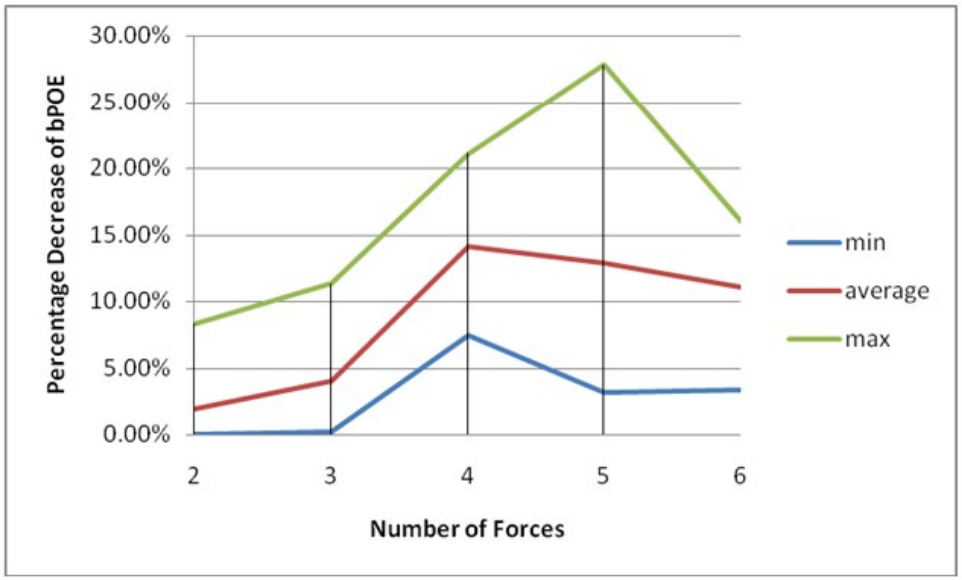
Results of Modeling:

Deterministic optimal mean square deviation = 4.96E-06

	# of Run									
	1	2	3	4	5	6	7	8	9	10
bPOE before Optimization	7.96E-2	8.33E-2	8.51E-2	8.25E-2	9.10E-2	9.80E-2	7.54E-2	1.11E-1	7.89E-2	7.06E-2
bPOE after Optimization	6.49E-2	6.89E-2	6.15E-2	7.28E-2	8.46E-2	9.49E-2	6.83E-2	9.85E-2	6.58E-2	6.71E-2
Percentage Decrease of ***bPOE***	18.4%	17.3%	27.8%	11.8%	7.1%	3.2%	9.4%	11.5%	16.6%	5.0%

Results of Simulations
Stochastic Case #1

- For fixed number of forces we determined minimum, maximum, and average values of **Percentage Decrease of bPOE** over 10 runs.
- The following chart demonstrates dependence of these characteristics on number of forces for Wave Number = 1.8, and Threshold= 0.011.

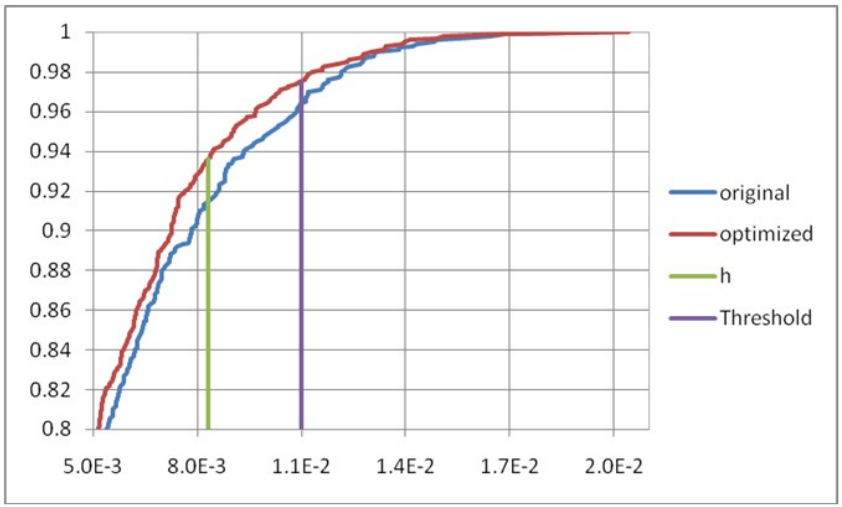


Results of Simulations
Stochastic Case #1, Run #3, Tails of CDF

Let h be a value such that $Treshold = E[L(\vec{\theta},\vec{F},\vec{\xi}) | L(\vec{\theta},\vec{F},\vec{\xi}) > h]$.

The following chart compares tails of CDF of the random value $L(\vec{\theta},\vec{F},\vec{\xi})$ before optimization (original) with that after optimization (optimized).

Wave Number = 1.8; Number of Forces = 5; Threshold= 0.011; Run#3.



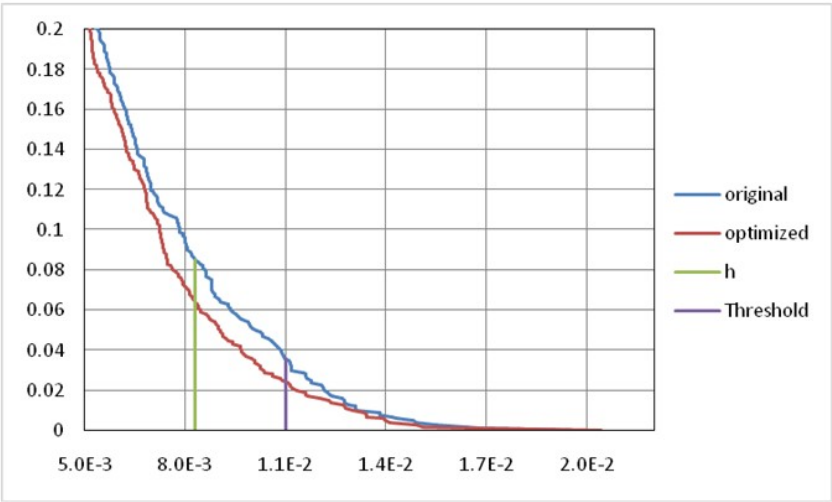
By minimizing $bPOE_{Threshold}(L(\vec{\theta},\vec{F},\vec{\xi}))$ we maximize $\Pr(L(\vec{\theta},\vec{F},\vec{\xi}) \le h)$.

Results of Simulations
Stochastic Case #1, Run #3, Tails of POE

Let h be a value such that $Treshold = E[L(\vec{\theta},\vec{F},\vec{\xi}) | L(\vec{\theta},\vec{F},\vec{\xi}) > h]$.

The following chart compares tails of POE of the random value $L(\vec{\theta},\vec{F},\vec{\xi})$ before optimization (original) with that after optimization (optimized).

Wave Number = 1.8; Number of Forces = 5; Threshold= 0.011; Run#3.

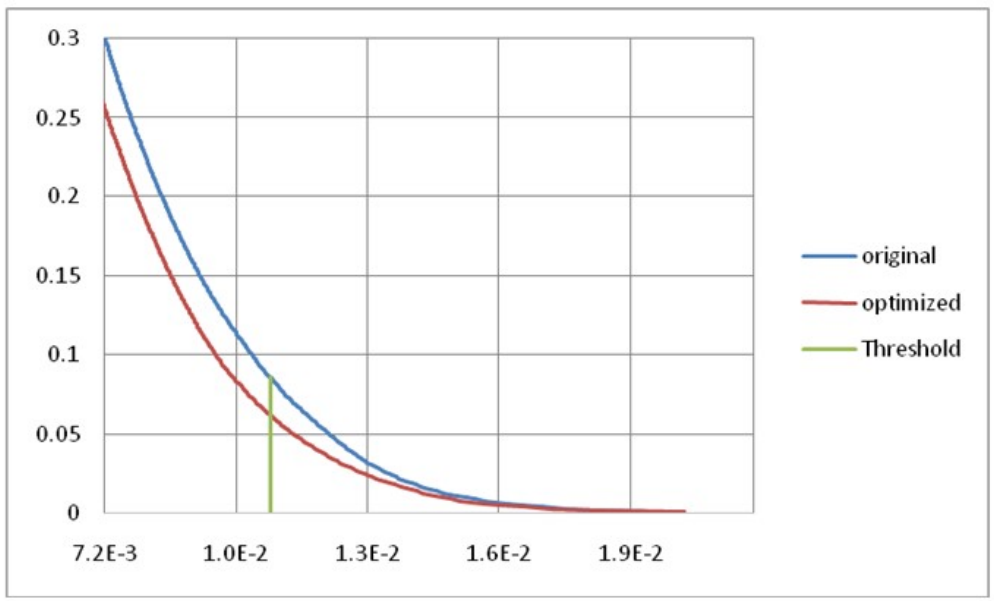


By minimizing $bPOE_{Threshold}(L(\vec{\theta},\vec{F},\vec{\xi}))$ we minimize $\Pr(L(\vec{\theta},\vec{F},\vec{\xi}) > h)$.

Results of Simulations
Stochastic Case #1, Run #3, Tails of bPOE

The following chart compares tails of bPOE of the random value $L(\vec{\theta}, \vec{F}, \vec{\xi})$ before optimization (original) with that after optimization (optimized).

Wave Number = 1.8; Number of Forces = 5; Threshold= 0.011; Run#3.



Results of Simulations
Stochastic Case #2

Parameters: Number of Samples =1000; Wave Number = 4.6;

Number of Forces = 8; Threshold= 0.0046; Number of Runs $\boldsymbol{M}$ = 10, Number of Defects $\leq$ 5.

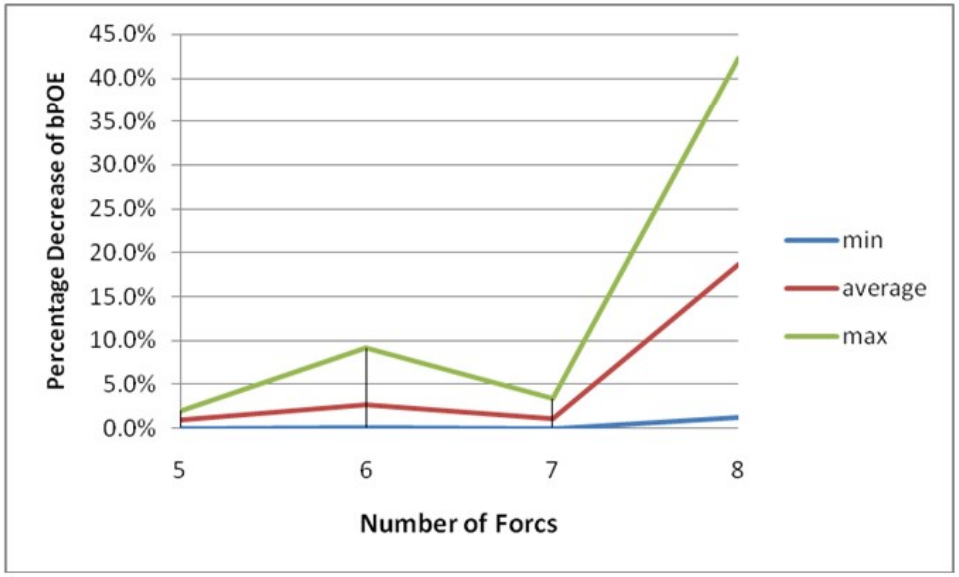
Results of Modeling:

Deterministic optimal mean square deviation = 5.61E-05

	# of Run									
	1	2	3	4	5	6	7	8	9	10
bPOE before Optimization	6.62E-03	1.64E-02	1.74E-02	1.40E-02	1.64E-02	1.28E-02	1.15E-02	1.78E-02	7.64E-03	6.04E-03
bPOE after Optimization	3.82E-03	1.04E-02	1.31E-02	1.12E-02	1.42E-02	1.12E-02	1.07E-02	1.75E-02	7.55E-03	4.36E-03
Percentage Decrease of ***bPOE***	42.31%	36.52%	24.78%	19.81%	13.33%	13.01%	6.77%	1.46%	1.20%	27.82%

Results of Simulations
Stochastic Case #2

- For fixed number of forces we determined minimum, maximum, and average values of **Percentage Decrease of bPOE** over 10 runs.
- The following chart demonstrates dependence of these characteristics on number of forces for Wave Number = 4.6, and Threshold= 0.0046.

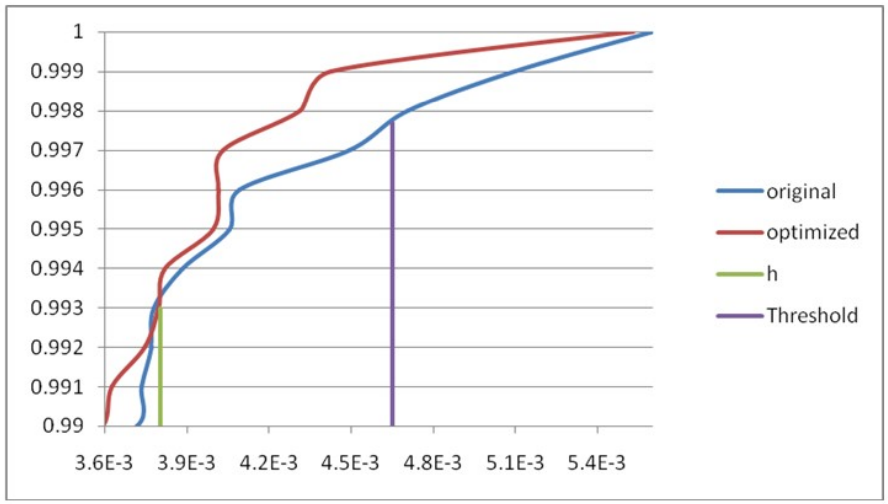


Results of Simulations
Stochastic Case #2, Run #5, Tails of CDF

Let h be a value such that $Treshold = E[L(\vec{\theta},\vec{F},\vec{\xi}) | L(\vec{\theta},\vec{F},\vec{\xi}) > h]$.

The following chart compares tails of CDF of the random value $L(\vec{\theta},\vec{F},\vec{\xi})$ before optimization (original) with that after optimization (optimized).

Wave Number = 4.6; Number of Forces = 8; Threshold= 0.0046; Run#5.



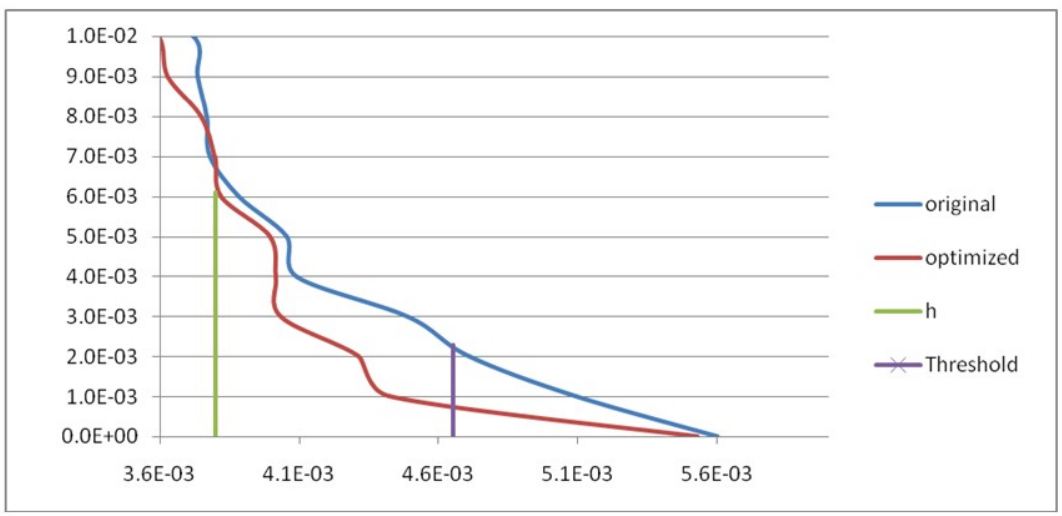
By minimizing $bPOE_{Threshold}(L(\vec{\theta},\vec{F},\vec{\xi}))$ we maximize $\Pr(L(\vec{\theta},\vec{F},\vec{\xi}) \le h)$.

Results of Simulations
Stochastic Case #2 , Run #5, Tails of POE

Let h be a value such that $Treshold = E[L(\vec{\theta},\vec{F},\vec{\xi}) | L(\vec{\theta},\vec{F},\vec{\xi}) > h]$.

The following chart compares tails of POE of the random value $L(\vec{\theta},\vec{F},\vec{\xi})$ before optimization (original) with that after optimization (optimized).

Wave Number = 4.6; Number of Forces = 8; Threshold= 0.0046; Run#5.

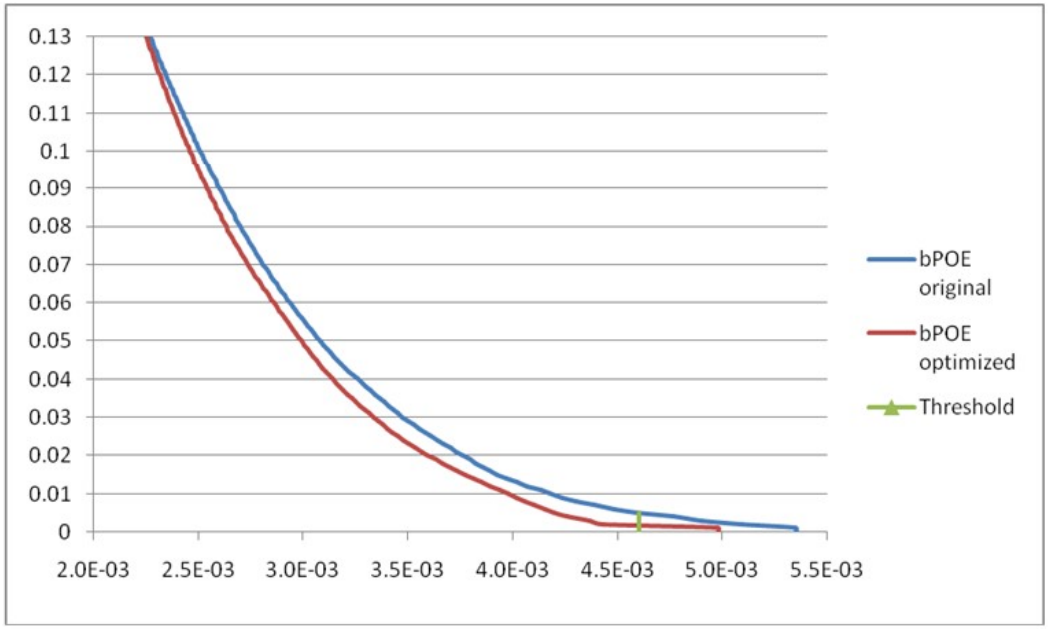


By minimizing $bPOE_{Threshold}(L(\vec{\theta},\vec{F},\vec{\xi}))$ we minimize $\Pr(L(\vec{\theta},\vec{F},\vec{\xi}) > h)$.

Results of Simulations
Stochastic Case #2, Run #5, Tails of bPOE

The following chart compares tails of bPOE of the random value $L(\vec{\theta}, \vec{F}, \vec{\xi})$ before optimization (original) with that after optimization (optimized).

Wave Number = 4.6; Number of Forces = 8; Threshold= 0.0046; Run#5.



THANK YOU FOR YOUR ATTENTION!