Optimal Bundling under Dependent Heavy-Tailed Valuations

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Motivation: Extreme Valuations and Bundling

Extreme Valuations

- Dec 2007, two $250 tickets to a Led Zeppelin reunion concert, auctioned off in London for $168,000 to a 25 yo Kenneth Donnell
- Cable and satellite TV providers: "basic" + pay-per-view for **unusual** special events; high valuations of a small fraction of consumers, very heavy tailed
- Sporting and cultural events: season tickets; small fraction of consumers with high valuations
- Collectibles and antique items: sold in lots to customers with very diverse valuation

Question: Is bundling optimal?

Available results

mostly under thin-tailed distributions or similarly restrictive valuations:

Palfrey (83): bounded

Bakos & Brynjolfsson (99, 00), Fang and Norman (06), Zhou (17): log-concave

Geng & et al (05): correlated valuations

Ibragimov & Walden (10): independent valuations

Common result: optimal for monopolist to bundle when MC low, to sell separately when MC high; buyers never unanimously prefer separate auctioning

This paper: extremely heavy-tailed dependent valuations lead to reversal of standard conclusion

Setup

Single seller of $M = \{1, 2, \ldots, m\}$ goods

Bundles B: subsets of M

Bundling decisions $\mathcal{B}$: partitions of M, $\mathcal{B} = (B_1, \ldots, B_l)$

- $\underline{\mathcal{B}} = \{\{1\}, \{2\}, \ldots, \{m\}\}$: **separate provision**
- $\overline{\mathcal{B}} = \{1, 2, \ldots, m\}$: **single bundle**

Valuations for bundles

$$v(B) = \sum_{i \in B} X_i \text{ additive valuation}$$

X_i: representative consumer's valuation for good $i \in M$

How to allow for substitutes and complements?

Option 1: a copula function C for marginal distributions of X_i: negative dependence = substitutes, positive = complements

Option 2: $v(B) = \left(\sum_{i \in B} X_i\right)^r$: $r > 1$ complements, $0 < r < 1$ substitutes

Option 3: both

Probability of buying B:

$$\mathbb{P}\left(v(B) \geq kp_B\right) = \mathbb{P}_C\left(\left[\sum_{i \in B} X_i\right]^r \geq kp_B\right)$$

where $k = card(B)$; $p_B =$ per-good price of B

Seller's expected profit from selling B to n buyers:

$$n\left(kp_B - \sum_{i\in B} c_i\right)\mathbb{P}_C\left(\left[\sum_{i\in B} X_i\right]^r \geq kp_B\right)$$

where c_i = marginal cost of good i

Prices set by the monopolist

- prices per good $p_B \in [0, p_{\text{max}}]$
- p_{max}: regulatory price
- $\underline{p}$: price under **separate provision**, in $\underline{B}$
- $\overline{p}$: price per good **in single bundle**, $\overline{B}$

Key to results: tail and dependence properties of the joint distribution of valuations

$$H(x_1, \ldots, x_k) \Rightarrow \mathbb{P}_C \left(\left[\sum_{i \in B} X_i \right]^r \geq k p_B \right)$$

Roughly, bundle if bundling increases this probability

Main point: extremely heavy tailed valuations (with tail index $\alpha < 1$) and strongly dependent valuations (characterized by certain copulas and other dependence structures) interact to produce reversals of conventional results

Example: with extremely heavy tails, per-good valuations for bundles have more "spread" than valuations for component products; with a positive dependence between buyer's valuations, the seller may prefer to sell separately even with zero MC (eg, information goods) and bundle even when MC are large (eg, seasonal concert tickets)

Digression I: Heavy tailed univariate distributions

- **Power law family**

$$\mathbb{P}(|X| > x) \approx x^{-\alpha}, \text{ where } \alpha > 0 \text{ is } \textbf{tail index}$$

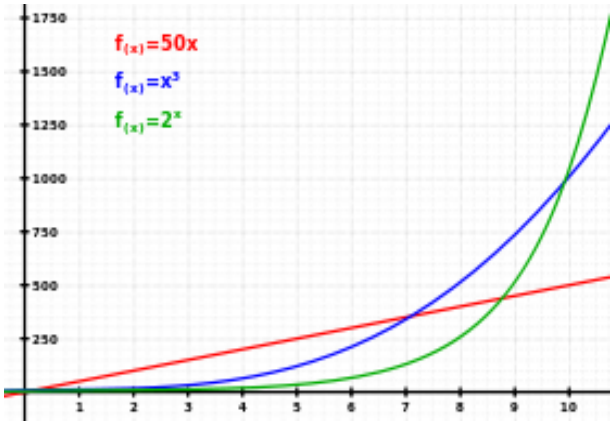
- **Key feature**

$$\mathbb{E}|X|^p < \infty \text{ iff } p < \alpha$$

Moments of order $p \geq \alpha$ are infinite

$\alpha \leq 4 \Rightarrow$ infinite fourth moments: $\mathbb{E}X^4 = \infty$ – **moderately heavy**
$\alpha \leq 2 \Rightarrow$ infinite variances: $\mathbb{E}X^2 = \infty$ – **moderately heavy**
$\alpha \leq 1 \Rightarrow$ infinite first moments: $\mathbb{E}|X| = \infty$ – **extremely heavy**

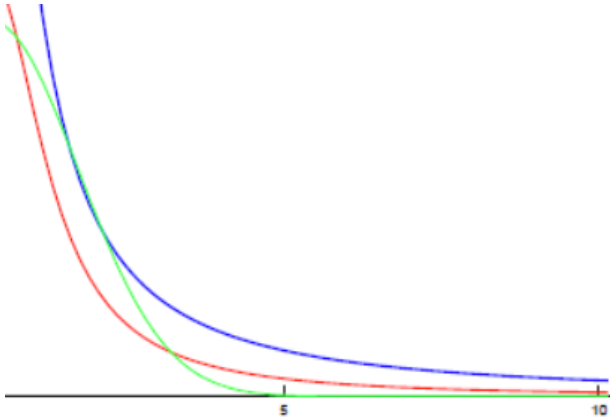
Polynomial vs exponential growth and decay



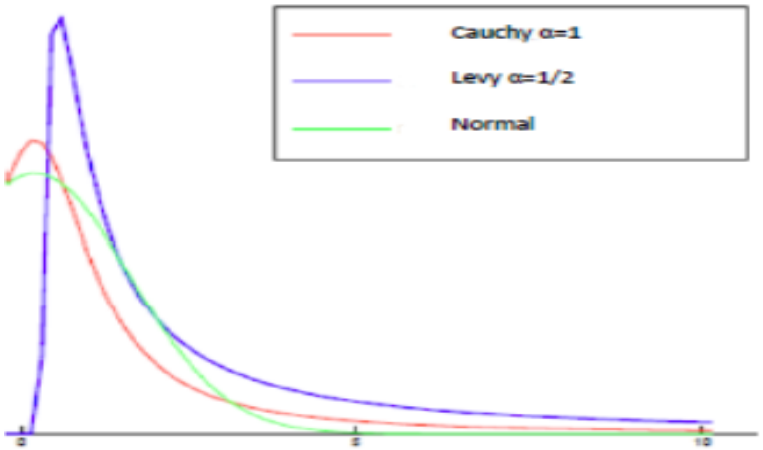
Any exponential growth will ultimately exceed any polynomial growth

Light vs heavy tails

Which is Gaussian?
Which is Cauchy $\alpha = 1$?
Which is Levy $\alpha = 0.5$?



A tale of two tails: 'Mild' and 'Wild' Chance

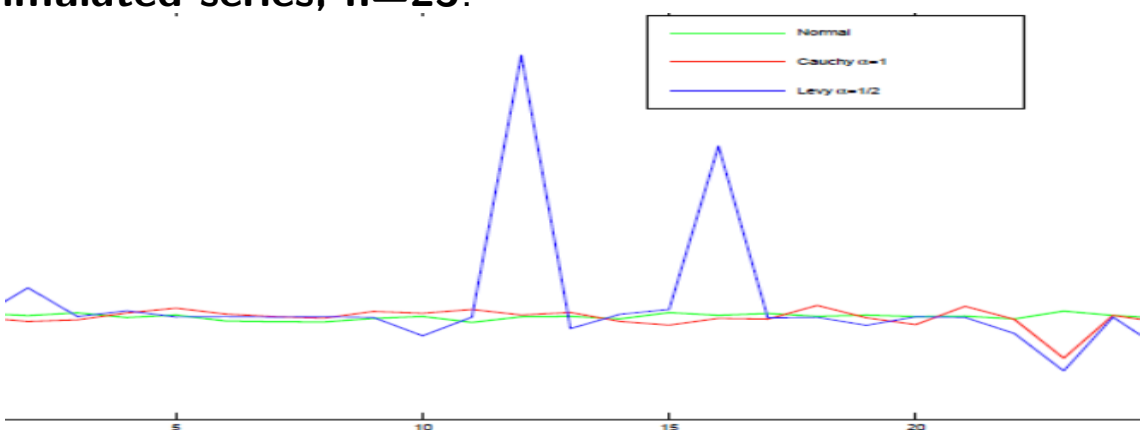


Tail of Levy is heavier than Cauchy which is heavier than Normal

$\mathbb{E}x \approx \int x \frac{C}{x^2} dx$ diverges for Cauchy

$\mathbb{E}x \approx \int x \frac{C}{e^{x^2}} dx$ converges for Normal

Simulated series, n=25:



More extreme observations

Special cases of power law:

- Student t distribution
- Cauchy and Levy distributions
- Pareto distribution
- Stable distributions with $\alpha < 2$

Empirical estimates of α:

- lower for markets of the type *nobody knows*, *winner takes it all*, *success breeds success*, i.e. uncertain individual demands, product success or failure, especially in information and creative goods markets
- returns from technological innovations: $\alpha <<< 1$
- loss distributions for operational risks: $\alpha < 1$
- profits in motion pictures industry: $1 < \alpha < 2$
- firm sizes, largest mutual fund sizes, city sizes: $\alpha \approx 1$ (Zipf's law)
- income: $\alpha \in (1.5, 3)$; wealth: $\alpha \approx 1.5$
- economic losses from earthquakes: $\alpha \in [0.6, 1.5]$
- citation indices, social network node degrees: $\alpha < 1$ (infinite expectation)

Stable distributions

$X \sim S_\alpha(\sigma)$: symmetric stable distribution, $\alpha \in (0, 2]$ if characteristic function

$\mathbb{E}e^{itX} = exp(-\sigma^\alpha |t|^\alpha)$

Key property: iid $X_1, ..., X_n \sim S_\alpha(\sigma)$

$\sum_i w_i X_i =_d \left(\sum_i w_i^\alpha\right)^{1/\alpha} X_1, \forall w_i$

- Normal $N(0, \sigma)$: $\alpha = 2 \Rightarrow \frac{1}{\sqrt{n}}(X_1 + \ldots + X_n) =_d X_1$
- Cauchy: $\alpha = 1$, $f(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$
- Levy: $\alpha = 1/2$, $f(x) = \frac{\sigma}{\sqrt{2\pi}} x^{-3/2} exp(-\frac{1}{2x})$

Note: these are power law distributions if $\alpha \in (0, 2)$:

$$\mathbb{P}(|X| > x) \approx \frac{C}{x^\alpha} \text{ so moments } \mathbb{E}|X|^p \text{ finite iff } p < \alpha$$

Digression II: Copulas and dependence

Main idea: to separate effects of dependence from effects of marginals in multivariate distributions

Copulas: Functions that join marginals to form multidimensional cdf

$$\underbrace{H(x_1, \ldots, x_T)}_{\text{multivariate c.d.f.}} = \underbrace{C(\underbrace{F_1(x_1), \ldots, F_T(x_T)}_{\text{univariate marginal c.d.f.'s}})}_{\text{Copula function}}$$

$$\Updownarrow$$

$$\underbrace{h(x_1, \ldots, x_T)}_{\text{multivariate p.d.f.}} = \underbrace{c(F_1(x_1), \ldots, F_T(x_T))}_{\text{copula p.d.f.}} \prod_{t=1}^{T} \underbrace{f_t(x_t)}_{\text{marginal p.d.f.}}$$

- **Recall**

  If X has cdf F, then $F(X) \sim$ uniform (on $[0, 1]$)

  If $U \sim$ uniform, then $F^{-1}(U) \equiv X$ has cdf F

- **Definition**

  A copula is a multivariate distribution whose marginals are uniform:

copula cdf	$C(u_1, \ldots, u_T)$
copula density	$c(u_1, \ldots, u_T) = \frac{d^T C(u_1, \ldots, u_T)}{du_1 \ldots du_T}$

- **Mechanism**

  Suppose you want X_1 to have cdf F_1, X_2 to have cdf F_2:

  pick a copula, let $U_1 = F_1(X_1), U_2 = F_2(X_2)$

  now you have a joint distribution for (X_1, X_2): $C(F_1(x_1), F_2(x_2))$

  So given marginals, the only "choice" is which copula.

- **Sklar's (1959) Theorem ("True" copula)**

  Suppose we want X_1, X_2 to have cdf's F_1, F_2 and joint cdf H. There is always a copula (if H is continuous it is unique) such that

$$H(x_1, x_2) = C(F_1(x_1), F_2(x_2)).$$

- **Examples**
  - Independence or product $\quad C(u,v) = \Pi(u,v) = uv \qquad c(u,v) = 1$
  - Logistic $\quad C(u,v) = \frac{uv}{u+v-uv}$
  - Gaussian $\quad C(u,v;\rho) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v); \rho)$
  - Gumbel $\quad C(u,v,\rho) = \exp\left[-((-\ln u)^\rho + (-\ln v)^\rho)^{1/\rho}\right], \qquad \rho \in [1, \infty)$
  - Frank $\quad C(u,v,\rho) = \begin{cases} uv, & \rho = 0 \\ -\frac{1}{\rho}\ln\left[1 + \frac{(e^{-\rho u}-1)(e^{-\rho v}-1)}{e^{-\rho}-1}\right], & \rho \in (-\infty, \infty)\backslash\{0\} \end{cases}$

- Eyraud-Farlie-Gumbel-Morganstern (EFGM)

$$C(u, v, \rho) = uv(1 + \rho(1 - u)(1 - v)), \qquad \rho \in [-1, 1]$$

- General copula by inversion
  - start with c.d.f.'s $K(x_1, x_2)$, $u = F_1(x_1)$ and $v = F_2(x_2)$
  - obtain $x_1 = F_1^{-1}(u)$, $x_2 = F_2^{-1}(v)$ and

$$C(u, v) \quad = \quad K(F_1^{-1}(u), F_2^{-1}(v))$$

- Archimedean copulas
  - start with a generator function $\varphi : (0, 1) \to [0, \infty]$, $\varphi' < 0$ and $\varphi'' > 0$
  - obtain

$$C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v))$$

  - e.g., Gumbel copula is Archimedean with $\varphi(t) = (-\log t)^\rho$

- **Note**
  - copula dependence parameter ρ parameterizes dependence between U and V; it has nothing to do with correlation except for normal copula
  - copulas have bounds

$$\underbrace{\underbrace{\max(u+v-1,0)}_{\text{Fréchet-Hoeffding lower bound copula}}}_{\text{countermonotonicity}} \leq C(u,v) \leq \underbrace{\underbrace{\min(u,v)}_{\text{Fréchet-Hoeffding upper bound copula}}}_{\text{comonotonicity}}$$

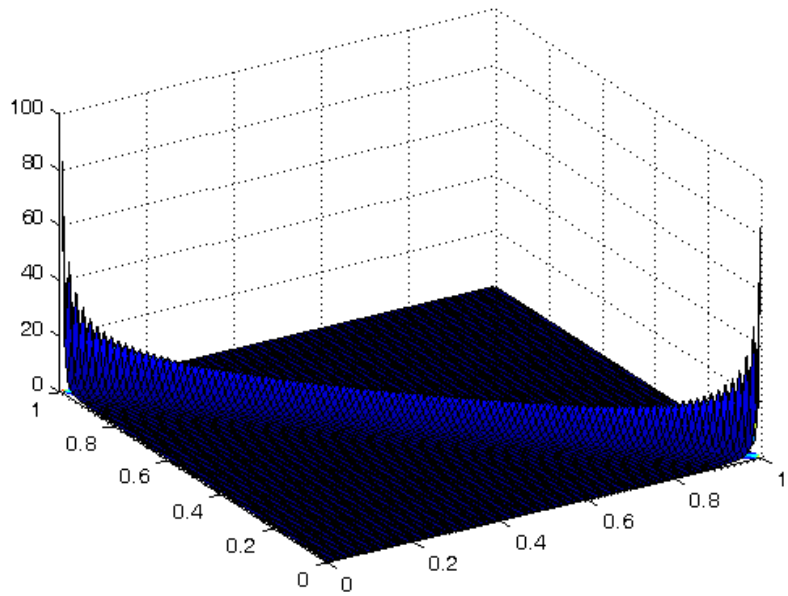
  - copulas have unequal ranges of dependence as measured by any common measure

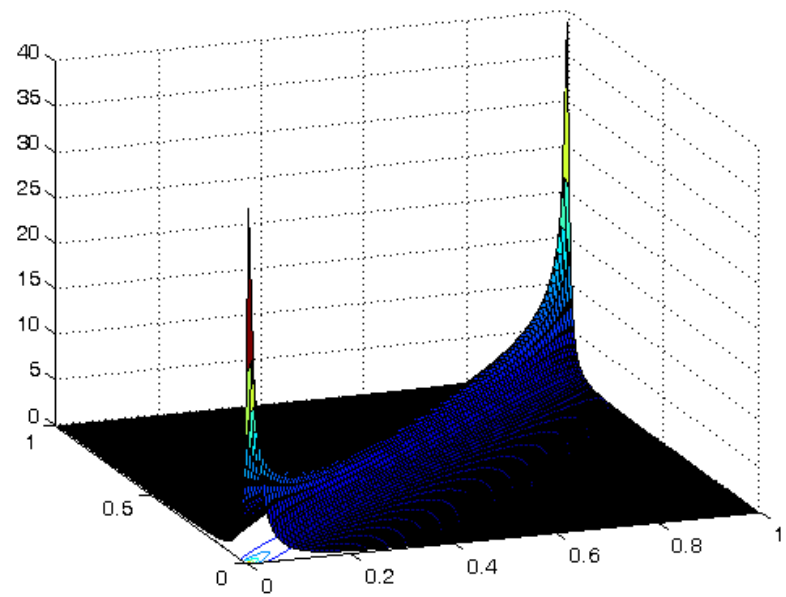
$$\text{e.g. Kendall's } \tau \quad \equiv \quad \mathbb{P}[\underbrace{(U_1-U_2)(V_1-V_2)>0}_{\text{concordance}}] - \mathbb{P}[\underbrace{(U_1-U_2)(V_1-V_2)<0}_{\text{discordance}}]$$

$$\tau_{\text{Frank}} \quad \in \quad [-1,1] \quad \text{while} \quad \tau_{\text{FGM}} \quad \in \quad [-2/9, 2/9]$$

  - to separate dependence from effects of marginals: F's capture heavy-tailedness, skewness, existence of moments, etc; C captures long memory vs short memory, volatility clustering, tail dependence, etc

Parametric copula density example: Plackett, $\rho = 0.002$ and $\rho = 100$





$$C(u, v; \gamma) = \frac{1 + (\rho - 1)(u + v) - \sqrt{(1 + (\rho - 1)(u + v))^2 - 4\rho(\rho - 1)uv}}{2(\rho - 1)}$$

End of digressions

Back to bundling problem

Seller's problem:

$$\max_B \left(kp_B - \sum_{i \in B} c_i \right) \mathbb{P}_C \left(\left[\sum_{i \in B} X_i \right]^r \geq kp_B \right)$$

where $k = card(B)$; $p_B =$ per-good price of B; $c_i =$ marginal cost of good i

We study properties of this problem under X_i from possibly extremely heavy tailed power law distribution under certain dependence structures (ie, a generic power-type copula family and multiplicative common shock structure).

Main results

Theorem 1: For valuation function $v(B) = \left(\sum_{i \in B} X_i\right)^r$, $0 < r < 1$, and valuations $X_i \sim S_\alpha$, $r < \alpha < 1$, with certain dependence structures, risk-neutral seller strictly prefers $\underline{\mathcal{B}}$ to any other bundling decision (ie, goods are sold separately)

Note: high degree of substitutability $r < \alpha$, plus $\alpha < 1$.

Theorem 2: For valuation function $v(B) = \left(\sum_{i \in B} X_i\right)^r$, $\forall r > 0$ and valuations $X_i \sim S_\alpha$, $0 < \alpha < \min\{1, r\}$, with certain dependence structures, risk-neutral seller strictly prefers $\overline{\mathcal{B}}$ to any other bundling decision (ie, goods are sold as a single bundle)

Note: complements $r > 1$ or weak substitutes $\alpha < r < 1$, plus $\alpha < 1$.

Theorem 3: For valuation function $v(B) = \sum_{i \in B} X_i$ and valuations $X_i \sim S_\alpha$, $0 < \alpha < 1$, with certain dependence structures, risk-neutral seller strictly prefers $\overline{\mathcal{B}}$ to any other bundling decision (ie, goods are sold as a single bundle)

Note: independently priced goods $r = 1$, plus $\alpha < 1$.

Intuition and examples: bundle if it increases probability of sale.

For extremely heavy tails, per-good valuations of bundles do not concentrate as size of bundle grows unlike with thin or moderately heavy tails.

Seller bundles even with large c_i in order to increase the chance of capturing the high valuations. Except when goods are strong substitutes and the effect of high valuations is offset by the strong dependence between components of the bundle.

Optimal bundling for goods with extreme valuations is opposite to thin-tailed valuations

Bundled when valuations are extreme, no matter the c_i, or when complements: observed in seasonal tickets for sporting and cultural events and performances, Wagner's *Der Ring des Nibelungen* bundle.

Separate provision for strong substitutes with extreme valuations: observed in cable and satellite TV providers with "basic" bundle plus pay-per-view for things like boxing matches, other information goods

Note: results hold independent of $c_i \geq 0$ but for $p_{\max} < \infty$.

Tool for proofs – majorization of vectors: v is majorized by w ($v \prec w$)

$$\begin{aligned} \sum_{i=1}^{l} v_{[i]} &\leq \sum_{i=1}^{l} w_{[i]}, l = 1, \ldots, k-1 \\ \sum_{i=1}^{k} v_{[i]} &= \sum_{i=1}^{k} w_{[i]} \\ v_{[1]} \geq \ldots \geq v_{[k]}, & \qquad w_{[1]} \geq \ldots \geq w_{[k]} \end{aligned}$$

Natural notation for bundling analysis: $v \prec w \Leftrightarrow$ weights v are less diverse

$$\underbrace{\overline{w} \equiv \left(\frac{1}{k}, \ldots, \frac{1}{k}\right)}_{\text{equal weights; pure bundling}} \prec (w_1, \ldots, w_k) \prec \underbrace{(1, 0, \ldots, 0) \equiv \underline{w}}_{\text{one risk; separate provision } Z=X}$$

$$Z_{\overline{w}} = \frac{1}{k} \sum_i X_i \qquad\qquad Z_{\underline{w}} = X_1$$

Examples

$$a = (a_1, \ldots, a_k) \prec b = (b_1, \ldots, b_k)$$

Diversification: portfolio a is more diversified than portfolio b

Income inequality: income distribution a has less inequality than income distribution b

Bundling: a represents a "more mixed" bundle than b

Extremal examples

$\overline{w} = \left(\frac{1}{k}, \ldots, \frac{1}{k}\right)$: most diversified portfolio, least income inequality, pure bundle

$\underline{w} = (1, 0, \ldots, 0)$: least diversified portfolio, most income inequality, separate provision

Foundations of majorization – Marshall & Olkin (11)

Schur-convexity – Proschan (65)

A function $\phi : \mathbf{R}_+^k \to \mathbf{R}$ is called *Schur-convex*

if $a \prec b \Rightarrow \phi(a) \leq \phi(b)$ for all $a, b \in \mathbf{R}_+^k$.

A function $\phi : \mathbf{R}_+^n \to \mathbf{R}$ is called *Schur-concave*

if $a \prec b \Rightarrow \phi(a) \geq \phi(b)$ for all $a, b \in \mathbf{R}_+^k$.

Key example: $\phi_\alpha(w_1, ..., w_n) = \sum_{i=1}^k w_i^\alpha$

Strictly Schur-convex for $\alpha > 1$

Strictly Schur-concave for $\alpha < 1$.

Examples from diversification of risk

Suppose $X_1, \ldots, X_k$ iid $S_2(\sigma)$, $\alpha = 2$, Normal distribution

- $Z_{\underline{w}} = \frac{1}{k}\sum_i X_i =_d \frac{1}{\sqrt{k}} X_1 = \frac{1}{\sqrt{k}} Z_{\overline{w}}$
- $\mathsf{VaR}_q(Z_{\underline{w}}) = \frac{1}{\sqrt{k}}\mathsf{VaR}_q(Z_{\overline{w}}) < \mathsf{VaR}_q(Z_{\overline{w}})$
- diversification reduces riskiness: VaR reduced by increasing k in $Z_{\underline{w}}$

Now suppose $X_1, \ldots, X_k$ iid $S_{1/2}(\sigma)$, $\alpha = 1/2$, Levy distribution

- $Z_{\underline{w}} = \frac{1}{k}\sum_i X_i =_d \left[\sum_{i=1}^k \left(\frac{1}{k}\right)^{1/2}\right]^2 X_1 = kX_1 = kZ_{\overline{w}}$
- $\mathsf{VaR}_q(Z_{\underline{w}}) = k\mathsf{VaR}_q(Z_{\overline{w}}) > \mathsf{VaR}_q(Z_{\overline{w}})$
- diversification increases riskiness: VaR increased by increasing k in $Z_{\underline{w}}$

Note: Tails of marginals matter – opposite conclusions on benefits of diversification
Note: iid risks for now

General results (not only for stables but for all power laws):

- moderately heavy tails $\alpha > 1$ (finite first moments and above)

$$v \prec w \Rightarrow \mathbb{P}\left(\sum_{i=1}^{k} v_i X_i > x\right) < \mathbb{P}\left(\sum_{i=1}^{k} w_i X_i > x\right)$$

- extremely heavy tails $\alpha < 1$ (infinite first moments)

$$v \prec w \Rightarrow \mathbb{P}\left(\sum_{i=1}^{k} v_i X_i > x\right) > \mathbb{P}\left(\sum_{i=1}^{k} w_i X_i > x\right)$$

Note: the bundle sale probability is a Schur-concave function of bundle component weights if $\alpha < 1$: a majorized vector of weights gives a more spread-out, less peaked, distribution

Mostly for independent risks

What happens for $\alpha = 1$?

E.g., $X_1, \ldots, X_k$ iid $S_1(\sigma)$, $\alpha = 1$, Cauchy distribution

Density

$$f(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$$

Heavy power law tails

$$\mathbb{P}(|X| > x) \approx \frac{C}{x}$$

Infinite first moment

Bundling has **no effect** at all

$$Z_w = \sum_{i=1}^{k} w_i X_i =_d X_1, \text{ for any } w_i \geq 0, \sum_i w_i = 1$$

What are the "certain dependence structures"?

Dependence structure 1: Model with common shocks

$$(X_1, \ldots, X_k) = (ZY_1, \ldots, ZY_k)$$

where $Y = (Y_1, \ldots, Y_k)$ iid from $S_\alpha(\sigma)$, $\alpha \in (0, 2]$, and $Z \geq 0$: arbitrary positive r.v., independent of Y; **a common shock** to all valuations

- exhibit both dependence and heavy tails in marginals
  - $\alpha < 1$: infinite first moments of marginals
  - $\alpha > 1$: $\mathbb{E}|X_i|^p < \infty$ if $\mathbb{E}Z^p < \infty$ and $\mathbb{E}|Y_i|^p < \infty$
    so if $\mathbb{E}Z = \infty$ then infinite first moment too

Note: If tail index of $Z > \alpha$ then $v(B) = \left(\sum_{i \in B} X_i\right)^r$ have tail index α/r, while X has tail index α, so $\mathbb{E}v(B) < \infty$ if $r < \alpha < 1$ and $\mathbb{E}v(B) = \infty$ if $\alpha < \min\{1, r\}$. If tail index of $Z < \alpha$ then the order-p moment of valuations is finite iff the tail index of Z is in $[pr, 1]$.

Bundling and dependence

Dependence matters: extreme examples

Minimize $\mathbb{P}_C(w_1 X_1 + w_2 X_2 > x)$ s.t. $w_1, w_2 \geq 0, w_1 + w_2 = 1$

Recall: Optimal bundle under **independence**:

$\alpha > 1$	$\alpha < 1$
$\Downarrow$	$\Downarrow$
$\left(\frac{1}{2}, \frac{1}{2}\right)$	$(1, 0)$
single bundle	separate provision

Now look at extreme dependence

Bundling and dependence

Extreme positive dependence

$X_1 = X_2$ (a.s.), ie comonotonic risks

$$\mathbb{P}_C(w_1 X_1 + w_2 X_2 > x) = \mathbb{P}_C(X_1 > x) \forall w$$

no effect on exceedance probability (similar to Cauchy) regardless of heavy-tailedness

Extreme negative dependence

$X_1 = -X_2$ (a.s.), ie countermonotonic risks

$$\mathbb{P}_C(w_1 X_1 + w_2 X_2) = \mathbb{P}_C[(w_1 - w_2) X_1] \forall w$$

optimal to **diversify fully** $w_j = \frac{1}{2}$ (similar to Normal) regardless of heavy-tailedness
Point: Optimal bundling affected by both dependence and properties of marginals

Bundling under copulas and power law marginals

Let $f(\cdot;\alpha)$ denote power law marginal pdf, and $c(\cdot,\cdot;\rho)$ denote a copula

$$\mathbb{P}\left(\underbrace{w_1X_1 + w_2X_2}_{w'X} > x\right) = \int_{w'z>x} f(z_1;\alpha)f(z_2;\alpha)c\left(F(z_1;\alpha), F(z_2;\alpha);\rho\right)dz,$$

Note: No general solution in terms of $\mathbb{P}(X_1 > x)$

Whether bundling increases the probability of sale depends on interplay of α and ρ.

But for some families we can say more

Bundling: EFGM Copula and Heavy Tails

Recall EFGM copula

$$\begin{aligned} C(u,v,\rho) &= uv(1+\rho(1-u)(1-v)) \\ c(u,v,\rho) &= 1+\rho(2u-1)(2v-1) \end{aligned}$$

Add power law marginals

$$\mathbb{P}(X_1 > x) = \mathbb{P}(X_2 > x) = x^{-\alpha},\ x \geq 1$$

Then

$$\mathbb{P}_C\left(w_1X_1 + w_2X_2 > x\right) = \int_{w'z>x} \underbrace{f(z_1;\alpha)}_{\alpha z_1^{-\alpha-1}} \underbrace{f(z_2;\alpha)}_{\alpha z_2^{-\alpha-1}} \underbrace{c\left(F(z_1;\alpha), F(z_2;\alpha);\rho\right)}_{1+\rho(1-2z_1^{\alpha})(1-2z_2^{\alpha})} dz,$$

Let $(\xi_1(\alpha_1), \xi_2(\alpha_2))$ be independent power law rv; let $w_1 = w_2 = 1/2$

Key component $\int_{\frac{s+t}{2}>z} f_1(s) f_2(t)(1-2F_1(s))(1-2F_2(t))dsdt$

$$
\begin{aligned}
&= \int_{\frac{s+t}{2}>z} \alpha^2 s^{-\alpha-1} t^{-\alpha-1}(2s^{-\alpha}-1)(2t^{-\alpha}-1)dsdt \\
&= 4\alpha^2 \underbrace{\int_{\frac{s+t}{2}>z} s^{-2\alpha-1} t^{-2\alpha-1} dsdt}_{\mathbb{P}\left(\frac{\xi_1(2\alpha)+\xi_2(2\alpha)}{2}>z\right)} - 2\alpha^2 \underbrace{\int_{\frac{s+t}{2}>z} s^{-2\alpha-1} t^{-\alpha-1} dsdt}_{\mathbb{P}\left(\frac{\xi_1(2\alpha)+\xi_2(\alpha)}{2}>z\right)} \\
&\quad -2\alpha^2 \underbrace{\int_{\frac{s+t}{2}>z} s^{-\alpha-1} t^{-2\alpha-1} dsdt}_{\mathbb{P}\left(\frac{\xi_1(\alpha)+\xi_2(2\alpha)}{2}>z\right)} + \alpha^2 \underbrace{\int_{\frac{s+t}{2}>z} s^{-\alpha-1} t^{-\alpha-1} dsdt}_{\mathbb{P}\left(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2}>z\right)}
\end{aligned}
$$

All terms are probabilities under independence

For independent power laws, there are asymptotic results as $z \to \infty$:

for all $\beta > 0$,

$$\mathbb{P}\Big(\frac{\xi_1(\beta) + \xi_2(\beta)}{2} > z\Big) \sim 2\mathbb{P}(\xi_1(\beta) > 2z) \sim 2^{1-\beta} z^{-\beta}$$

for $\beta_1 < \beta_2$,

$$\mathbb{P}\Big(\frac{\xi_1(\beta_1) + \xi_2(\beta_2)}{2} > z\Big) \sim \mathbb{P}(\xi_1(\beta_1) > 2z) \sim 2^{-\beta_1} z^{-\beta_1}$$

Point: we can express $\mathbb{P}_C\Big(\frac{X_1+X_2}{2} > z\Big)$ in terms of $\mathbb{P}\Big(\frac{\xi_1(\alpha)+\xi_2(\alpha)}{2} > z\Big)$

Theorem: Schur-concavity under EFGM and heavy tails

- Moderately heavy tails $\alpha > 1$: finite first moments

$$\mathbb{P}_C\left(\frac{X_1 + X_2}{2} > x\right) < \mathbb{P}(X_1 > x), \quad \text{for sufficiently large valuations}$$

- Extremely heavy tails $\alpha < 1$: infinite first moments

$$\mathbb{P}_C\left(\frac{X_1 + X_2}{2} > x\right) > \mathbb{P}(X_1 > x), \quad \text{for sufficiently large valuations}$$

Key restriction: Asymptotics w.r.t. $x \to \infty$

Key conclusion: Same threshold value for α as for iid case

Extensions: Power-Type Copulas

k-dimensional EFGM

$$C(u_1, \ldots, u_k; \rho) = \prod_{t=1}^{k} u_t(1 + \rho(1 - u_1) \ldots (1 - u_k))$$

where $\rho \in [-1, 1]$

Power copula

$$C(u_1, \ldots, u_k; \rho) = \prod_{t=1}^{k} u_t(1 + \rho(u_1^l - u_1^{l+1}) \ldots (u_k^l - u_k^{l+1}))$$

where $l \geq 0$ (EFGM: $l = 0$)

Extensions: Power-Type Copulas

Generalized EFGM

$$C(u_1, \ldots, u_k; \rho) = \prod_{t=1}^{k} u_t \left(1 + \sum_{t=2}^{k} \sum_{1 \le j_1 < \ldots < j_k \le k} \rho_{j_1 \ldots j_k} (1 - u_{j_1}) \ldots (1 - u_{j_k}) \right)$$

where $\rho_j \in [-1, 1]$

Polynomial copulas

$$C(u_1, u_2; \rho) = u_1 u_2 \left(1 + \sum_{k \ge 1, q \ge 1}^{k+q \le m-2} \rho_{kq} (u_1^k - 1)(u_2^q - 1) \right)$$

Copula with cubic section

$$C(u_1, u_2; \rho) = u_1 u_2 + 2\rho u_1 u_2 (1 - u_1)(1 - u_2)(1 + u_1 + u_2 - 2u_1 u_2)$$

Iterated EFGM

$$C(u.v) = uv[1 + \alpha(1-u)(1-v) + \beta uv(1-u)(1-v)]$$
$$C(u.v) = uv[1 + \alpha(1-u)(1-v) + \beta(1-u^2)(1-v^2)]$$
$$C(u,v) = uv + \sum_j \alpha_j (uv)^{1/2}[(1-u)(1-v-)]^{(j+1)/2}$$

Sarmanov copula

$$C(u.v) = uv[1 + 3\alpha(1-u)(1-v) + 5\alpha^2(1-u)(1-2u)(1-v)(1-2v)]$$

Bairamov-Kotz copula

$$C(u.v) = uv[1 + \alpha(1-u^\alpha)^\beta(1-v\alpha)^\beta]$$

Bernstein-Kantorovich copula

$$c_J(\mathbf{u}) = \sum_{v_1=0}^{J-1} \cdots \sum_{v_k=0}^{J-1} \gamma_v \prod_{l=1}^{k} \underbrace{\frac{J!}{v_l!(J-v_l-1)!} u_l^{v_l} (1-u_l)^{J-v_l-1}}_{\beta\text{-density}}$$

Yet, this does not cover the entire class of copula functions!

Extensions: First Order Approximations

Ali-Mikhail-Haq

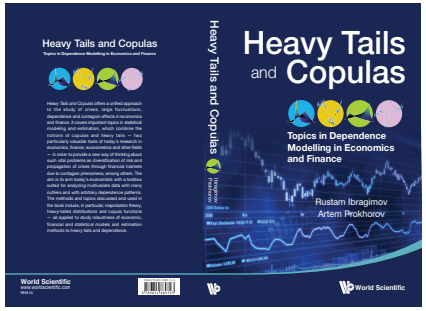
$$C(u_1, \ldots, u_k; \rho) = (1-\rho)\left[\prod_{i=1}^{k}\left(\frac{1-\rho}{u_i}+\rho\right)-\rho\right]^{-1}$$

Frank: Archimedian

$$C(u_1, u_2; \rho) = -\frac{1}{\rho}\ln\left[1+\frac{(e^{-\rho u_1}-1)(e^{-\rho u_2}-1)}{e^{-\rho}-1}\right]$$

Plackett: Tail dependent

$$C(u_1, u_2; \rho) = \frac{1+(\rho-1)(u_1+u_2)-\sqrt{(1+(\rho-1)(u_1+u_2))^2-4\rho(\rho-1)u_1u_2}}{2(\rho-1)}$$

Concluding remarks

- We have looked at optimal bundling for heavy tailed valuations with arbitrary dependence
- We have shown that when marginals are extremely heavy tailed ($\alpha < 1$) and/or have a certain large but not comprehensive class of dependence structures the conventional predictions of optimal bundling behavior of multiproduct monopolist are reversed
- More on heavy tails and copulas in the book: