Continuity Properties of Minimax and Sequential Optimization under Uncertainty

Eugene A. Feinberg

Stony Brook University

Workshop: Risk Management Approaches in Engineering Applications

University of Florida, Gainesville, FL
October 1, 2018

This talk is based on joint papers with Pavlo O. Kasyanov and Michael Z. Zgurovsky (National Technical University of Ukraine, Kiev).

The research was partially supported by NSF Grant CMMI-1636193.

Introduction

Decision making problems under uncertainty lead to Minimax equations. This talk discusses continuity properties of minimax equations and their solutions.

Let us consider a classic problem, when a decision maker minimizes the worst possible losses. The objective function is

$$v^\sharp(x) := \inf_{a \in \Phi_{\mathbb{A}}(x)} \sup_{b \in \Phi_{\mathbb{B}}(x,a)} f(x,a,b),$$

where
x is the state of the nature;
$\Phi_{\mathbb{A}}(x)$ is the decision set;
$\Phi_{\mathbb{B}}(x,a)$ is the set of possible values of parameters;
$f(x,a,b)$ is the payoff.

Applications:
Robust optimization, Optimization of risk measures (coherent risk measures, bPOE)

Introduction

Let $\mathbb{X}$, $\mathbb{A}$, and $\mathbb{B}$ be Borel subsets of Polish (complete, separable, metric) spaces, $\Phi_{\mathbb{A}} : \mathbb{X} \mapsto 2^{\mathbb{A}} \setminus \{\emptyset\}$ and $\Phi_{\mathbb{B}} : \mathbb{X} \times \mathbb{A} \mapsto 2^{\mathbb{B}} \setminus \{\emptyset\}$ be set-valued mappings defining feasible sets for values a and b, and $f : \mathbb{X} \times \mathbb{A} \times \mathbb{B} \mapsto \overline{\mathbb{R}}$ be the payoff function.

Consider the minimax function

$$v^{\sharp}(x) := \inf_{a \in \Phi_{\mathbb{A}}(x)} \sup_{b \in \Phi_{\mathbb{B}}(x,a)} f(x, a, b).$$

What can be said about continuity of the minimax functions $v^{\sharp}(x)$, the function $f^{\sharp}(x, a) := \sup_{b \in \Phi_{\mathbb{B}}(x,a)} f(x, a, b)$, and solution multifunctions $\Phi^*_{\mathbb{A}}(x) := \mathrm{Argmin}_{a \in \Phi_{\mathbb{A}}(x)} f^{\sharp}(x, a)$ and $\Phi^*_{\mathbb{B}}(x, a) := \mathrm{Argmax}_{b \in \Phi_{\mathbb{B}}(x,a)} f(x, a, b)$?

When are the sets $\Phi^*_{\mathbb{A}}(x)$ and $\Phi^*_{\mathbb{B}}(x, a)$ nonempty? When are these sets compact?

What can we say about solutions of sequential versions of such problems?

Introduction

The classic Berge theorem and Berge's maximum theorem give answers to continuity questions for optimization problems

$$v(x) = \inf_{a \in \Phi_{\mathbb{A}}(x)} f(x, a) \tag{1}$$

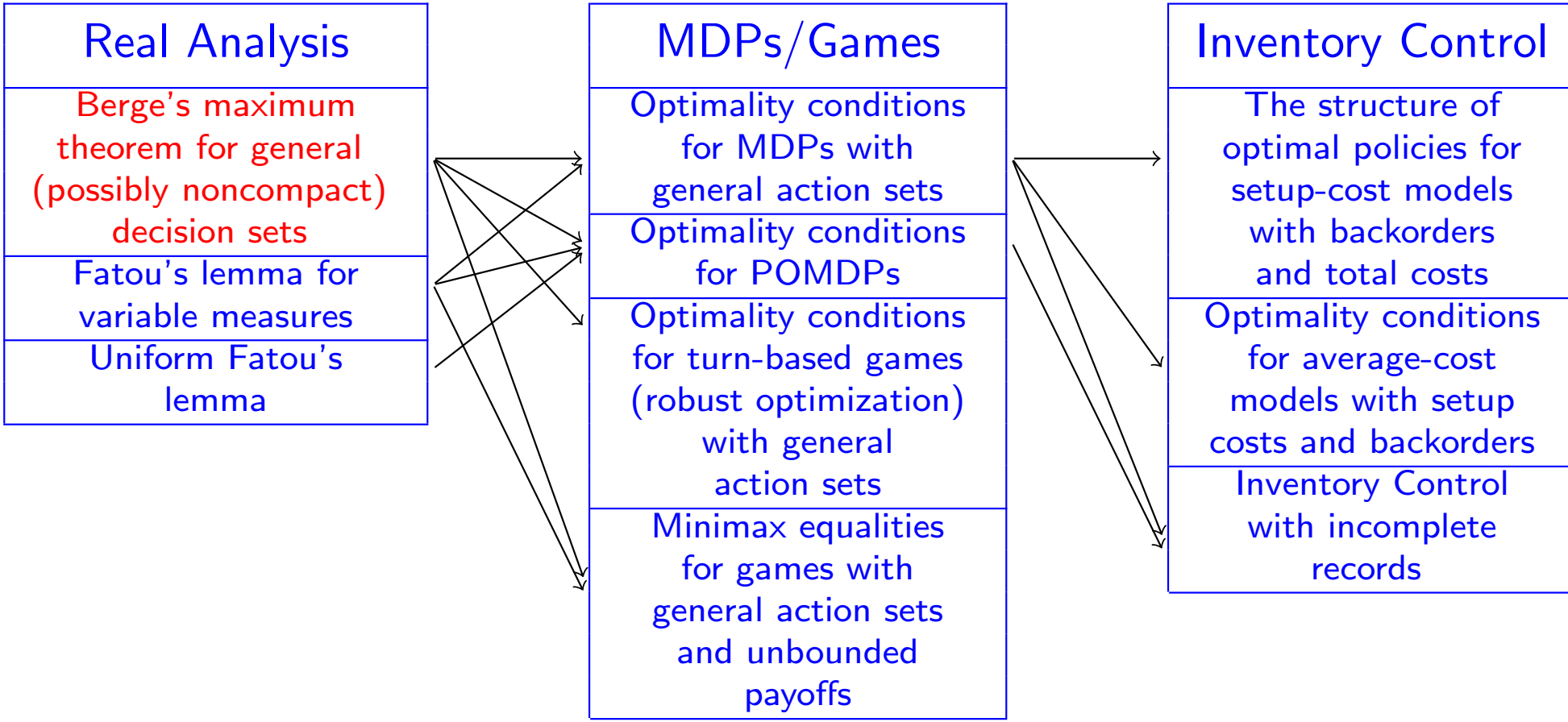
under the assumption that all the sets $\Phi_{\mathbb{A}}(x)$ are compact.

F., Kasyanov, Zadoianchuk (2013) and F., Kasyanov, Voorneveld (2014) generalized Berge's theorem and Berge's maximum theorem to possibly noncompact action sets. The generalizations were obtained by introducing the notion of $\mathbb{K}$-inf-compact functions.

This talk discusses continuity properties for minimax problems. Observe that (1) is a particular case of the minimax problem, when each set $\Phi_{\mathbb{B}}(x, a)$ is a singleton.

Introduction

Berge's theorems for noncompact decision sets and Fatou's lemma for variable probabilities provided new tools for solving problems in stochastic optimization, control, game theory, and their applications.



Plan of the Talk

- Preliminaries
- $\mathbb{K}$-inf-compact Functions
- Sufficient Conditions for $\mathbb{K}$-inf-compactness
- Continuity Properties of Minima
- Continuity Properties of Minimax
- Zero-Sum Stochastic Games with Noncompact Action Sets and Perfect Information
- References

Preliminaries

- Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, $\mathbb{S}$ be a metric space, $S \subset \mathbb{S}$ be a nonempty set.
- Let $\mathbb{K}(\mathbb{S})$ be the family of all nonempty compact subsets of $\mathbb{S}$.
- A function $f : S \subset \mathbb{S} \mapsto \overline{\mathbb{R}}$ is called lower semi-continuous at $s \in S$, if for each sequence $\{s^{(n)}\}_{n=1,2,\ldots} \subset S$, that converges to s in $\mathbb{S}$, the inequality $\liminf_{n\to\infty} f(s^{(n)}) \geq f(s)$ holds.
- A function $f : S \subset \mathbb{S} \mapsto \overline{\mathbb{R}}$ is called upper semi-continuous at $s \in S$, if $-f$ is lower semi-continuous at $s \in S$.
- A function $f : S \subset \mathbb{S} \mapsto \overline{\mathbb{R}}$ is called lower / upper semi-continuous, if f is lower / upper semi-continuous at each $s \in S$.
- Consider the level sets $\mathcal{D}_f(\lambda; S) := \{s \in S \,:\, f(s) \leq \lambda\}$, $\lambda \in \mathbb{R}$.
- A function $f : S \subset \mathbb{S} \mapsto \overline{\mathbb{R}}$ is called inf-compact on S, if all the level sets $\{\mathcal{D}_f(\lambda; S)\}_{\lambda\in\mathbb{R}}$ are compact in $\mathbb{S}$.
- A function $f : S \subset \mathbb{S} \mapsto \overline{\mathbb{R}}$ is called sup-compact on S, if $-f$ is inf-compact on S.

$\mathbb{K}$-inf-compactness

Let $\mathbb{X}$ and $\mathbb{Y}$ be metric spaces, $Z \subset \mathbb{X}$ be a nonempty set, $\Phi : \mathbb{X} \mapsto 2^{\mathbb{Y}}$ be a set-valued mapping with $\mathrm{Dom}\,\Phi := \{x \in \mathbb{X} : \Phi(x) \neq \emptyset\} \neq \emptyset$, and $\mathrm{Gr}_Z(\Phi) = \{(x, y) \in Z \times \mathbb{Y} : x \in \mathrm{Dom}\,\Phi,\ y \in \Phi(x)\}$. Note that $\mathrm{Gr}(\Phi) := \mathrm{Gr}_{\mathbb{X}}(\Phi)$.

Definition 1 ($\mathbb{K}$-inf-compact function; F., Kasyanov, and Zadoianchuk 2012, F. 2016)

A function $f : \mathrm{Gr}(\Phi) \subset \mathbb{X} \times \mathbb{Y} \mapsto \overline{\mathbb{R}}$ is called $\mathbb{K}$-inf-compact on $\mathrm{Gr}(\Phi)$, if for every $C \in \mathbb{K}(\mathrm{Dom}\,\Phi)$ this function is inf-compact on $\mathrm{Gr}_C(\Phi)$.

Necessary and Sufficient conditions for $\mathbb{K}$-inf-compactness

The function $f : \mathrm{Gr}(\Phi) \subset \mathbb{X} \times \mathbb{Y} \mapsto \overline{\mathbb{R}}$ is $\mathbb{K}$-inf-compact on $\mathrm{Gr}(\Phi)$ if and only if the following two assumptions hold:

(i) *$f : \mathrm{Gr}(\Phi) \subset \mathbb{X} \times \mathbb{Y} \mapsto \overline{\mathbb{R}}$ is lower semi-continuous;*

(ii) *if a sequence $\{x^{(n)}\}_{n=1,2,\ldots}$ with values in $\mathrm{Dom}\,\Phi$ converges in $\mathbb{X}$ and its limit x belongs to $\mathrm{Dom}\,\Phi$, then each sequence $\{y^{(n)}\}_{n=1,2,\ldots}$ with $y^{(n)} \in \Phi(x^{(n)})$, $n = 1, 2, \ldots,$ satisfying the condition that the sequence $\{f(x^{(n)}, y^{(n)})\}_{n=1,2,\ldots}$ is bounded above, has a limit point $y \in \Phi(x)$.*

Definition 2 ($\mathbb{K}$-sup-compact function)

A function $f : \mathrm{Gr}(\Phi) \subset \mathbb{X} \times \mathbb{Y} \mapsto \overline{\mathbb{R}}$ is called $\mathbb{K}$-sup-compact on $\mathrm{Gr}(\Phi)$, if the function $-f$ is $\mathbb{K}$-inf-compact on $\mathrm{Gr}(\Phi)$.

$\mathbb{K}$-inf-compactness: Examples

The real function f is $\mathbb{K}$-inf-compact and continuous on $\mathrm{Gr}(\Phi)$, but it is not inf-compact on $\mathrm{Gr}(\Phi)$:

- $\mathbb{X} := \mathbb{R}$,
- $\mathbb{Y} := \mathbb{R}$,
- $\Phi(x) := \mathbb{R}_+ = [0, +\infty)$,
- $f(x, y) := x + y$ for each $x \in \mathbb{R}$ and $y \in \mathbb{R}_+$;

or

- $f(x, y) := |x - y|$ for each $x \in \mathbb{R}$ and $y \in \mathbb{R}_+$.

These functions are not inf-compact because the sets $\{(x, y) \in \mathbb{R} \times \mathbb{R}_+ : x + y \leq 0\}$ and $\{(x, y) \in \mathbb{R} \times \mathbb{R}_+ : x = y\}$ are not compact.

$\mathbb{K}$-inf-compactness

- Inventory control cost function:

  Let $\mathbb{X} := \mathbb{R}$, $\mathbb{Y} := \mathbb{R}_+$,

  $$f(x, y) = K\mathbf{I}\{y > 0\} + cy + \mathbb{E}h(x + y - D),$$

  where $K > 0$, $c > 0$, $h : \mathbb{R} \mapsto \mathbb{R}_+$ is continuous with $h(x) \to +\infty$ as $|x| \to +\infty$, and D is a nonnegative random variable (demand) with $\mathbb{E}D < +\infty$.

- $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \overline{\mathbb{R}}$ is proper, lower semi-continuous, and $f(x, y)$ is level-bounded in x locally uniformly in y (Rockafellar and Wets).

Continuity of Multifunctions: Classic Definitions

- A set-valued mapping $F : \mathbb{X} \mapsto 2^{\mathbb{Y}}$ is upper semi-continuous at $x \in \operatorname{Dom} F$ if, for each neighborhood $\mathcal{G}$ of the set $F(x)$, there is a neighborhood of x, say $U(x)$, such that $F(x^*) \subset \mathcal{G}$ for all $x^* \in U(x) \cap \operatorname{Dom} F$.
- A set-valued mapping $F : \mathbb{X} \mapsto 2^{\mathbb{Y}}$ is lower semi-continuous at $x \in \operatorname{Dom} F$ if, for each open set $\mathcal{G}$ with $F(x) \cap \mathcal{G} \neq \emptyset$, there is a neighborhood of x, say $U(x)$, such that if $x^* \in U(x) \cap \operatorname{Dom} F$, then $F(x^*) \cap \mathcal{G} \neq \emptyset$.
- A set-valued mapping $F : \mathbb{X} \mapsto 2^{\mathbb{Y}}$ is called upper / lower semi-continuous, if it is upper / lower semi-continuous at all $x \in \operatorname{Dom} F$.
- A set-valued mapping $F : \mathbb{X} \mapsto 2^{\mathbb{Y}}$ is called continuous (at $x \in \operatorname{Dom} F$), if it is upper semi-continuous and lower semi-continuous (at x).

A set-valued mapping $F : \mathbb{X} \mapsto 2^{\mathbb{Y}} \setminus \{\varnothing\}$ is lower semicontinuous at $x \in \operatorname{Dom} F$ iff for each sequence $x_n \to x$ with $x_n \in \operatorname{Dom} F$, each $a \in F(x)$ is a limit point of a sequence $\{a_n\}$ with $a_n \in F(x_n)$.

Sufficient Conditions for $\mathbb{K}$-inf-compactness

Sufficient Conditions for $\mathbb{K}$-inf-compactness

Let $\Phi : \mathbb{X} \mapsto 2^{\mathbb{Y}}$ be a set-valued mapping and $f : \mathrm{Gr}(\Phi) \subset \mathbb{X} \times \mathbb{Y} \mapsto \overline{\mathbb{R}}$ be a function. Then the following statements hold:

(a) *if $f : \mathrm{Gr}(\Phi) \subset \mathbb{X} \times \mathbb{Y} \mapsto \overline{\mathbb{R}}$ is inf-compact on $\mathrm{Gr}(\Phi)$, then the function f is $\mathbb{K}$-inf-compact on $\mathrm{Gr}(\Phi)$;*

(b) *if $f : \mathrm{Gr}(\Phi) \subset \mathbb{X} \times \mathbb{Y} \mapsto \overline{\mathbb{R}}$ is lower semi-continuous and $\Phi : \mathbb{X} \mapsto 2^{\mathbb{Y}}$ is upper semi-continuous and compact-valued at each $x \in \mathrm{Dom}\,\Phi$, then the function f is $\mathbb{K}$-inf-compact on $\mathrm{Gr}(\Phi)$.*

Sufficient Condition (b) can be rephrased in the following form: If conditions of Berge's maximum theorem hold, then the function f is $\mathbb{K}$-inf-compact.

Continuity Properties of Minima: Berge's Theorem

(1) Berge's theorem for possibly noncompact action sets (F., Kasyanov, and Zadoianchuk 2013)

If the function $f : \mathbb{X} \times \mathbb{Y} \mapsto \overline{\mathbb{R}}$ *is* $\mathbb{K}$*-inf-compact on* $\mathrm{Gr}(\Phi)$*, then the function*

$$f^*(x) = \min_{y \in \Phi(x)} f(x, y)$$

is lower semi-continuous, and the set $\Phi^*(x) = \mathrm{Argmin}_{y \in \Phi(x)} f(x, y)$ *is compact, if* $f^*(x) < +\infty$*, and* $\Phi^*(x) = \Phi(x)$ *if* $f^*(x) = +\infty$*.*

If $\mathbb{X} = \mathbb{R}^n$, $\mathbb{Y} = \mathbb{R}^m$, and $\Phi(x) = \mathbb{R}^m$, this theorem becomes the Parametric Optimization Theorem in Rockafellar and Wets; the speaker thanks Johannes Royset for pointing this out.

(2) Berge's theorem

If the function $f : \mathbb{X} \times \mathbb{Y} \mapsto \mathbb{R}$ *and the set-valued mapping* $\Phi(x)$ *satisfy the following properties:*

(i) f *is lower semi-continuous;*

(ii) *the sets* $\Phi(x)$ *are compact for all* $x \in \mathbb{X}$*;*

(iii) *the set-valued mapping* $x \mapsto \Phi(x)$ *is upper semi-continuous;*

then the function f^* *is lower semi-continuous.*

Advantages of (1) compared to (2): (1) is more general and it is much easier to verify the $\mathbb{K}$-inf-compactness of f than the upper semi-continuity of the set-valued mapping Φ.

Continuity Properties of Minima: Berge's Maximum Theorem

Berge's maximum theorem for possibly noncompact action sets (F., Kasyanov, and Voorneveld 2014)

If

(i) $\Phi : \mathbb{X} \mapsto 2^{\mathbb{Y}} \setminus \{\varnothing\}$ *is a lower semi-continuous set-valued mapping;*

(ii) $f : \mathbb{X} \times \mathbb{Y} \mapsto \mathbb{R}$ *is a* $\mathbb{K}$*-inf-compact and upper semi-continuous function on* $\mathrm{Gr}(\Phi)$,

then the function

$$f^*(x) = \inf_{y \in \Phi(x)} f(x, y)$$

is continuous and the set-valued mapping $\Phi^*(x) = \mathrm{Argmin}_{y \in \Phi(x)} f(x, y)$ *is upper semi-continuous and compact-valued.*

Berge's maximum theorem

If

(i) $\Phi : \mathbb{X} \mapsto 2^{\mathbb{Y}} \setminus \{\varnothing\}$ *is a continuous compact-valued mapping;*

(ii) $f : \mathbb{X} \times \mathbb{Y} \mapsto \mathbb{R}$ *is a continuous function;*

then the function f^* *is continuous and the set-valued mapping* Φ^* *is upper semi-continuous and compact-valued.*

Continuity Properties of Minima: Useful Facts

Upper semi-continuity of the minimum (Hu and Papageorgiou 1997)

If a set-valued mapping $\Phi : \mathbb{X} \mapsto 2^{\mathbb{Y}} \setminus \{\emptyset\}$ *is lower semi-continuous and a function* $f : \mathrm{Gr}(\Phi) \subset \mathbb{X} \times \mathbb{Y} \mapsto \overline{\mathbb{R}}$ *is upper semi-continuous, then the value function* $f^* : \mathbb{X} \mapsto \overline{\mathbb{R}}$,

$$f^*(x) = \inf_{y \in \Phi(x)} f(x, y), \tag{2}$$

is upper semi-continuous.

Consider a function $f : \mathbb{X} \times \mathbb{Y} \mapsto \overline{\mathbb{R}}$. If $f(x, y)$ is upper semi-continuous in x for each y, then the value function $f^*(x)$ is upper semi-continuous.

Continuity Properties of Minima: Useful Facts

Properties of continuous minima (F., Kasyanov and Voorneveld 2014, F. and Kasyanov 2015)

Let $\Phi : \mathbb{X} \mapsto 2^{\mathbb{Y}} \setminus \{\emptyset\}$, *a function* $f : \mathrm{Gr}(\Phi) \subset \mathbb{X} \times \mathbb{Y} \mapsto \overline{\mathbb{R}}$ *be* $\mathbb{K}$*-inf-compact on* $\mathrm{Gr}(\Phi)$, *and the value function* $f^* : \mathbb{X} \mapsto \mathbb{R} \cup \{-\infty\}$ *be continuous. Then the infimum in*

$$f^*(x) = \inf_{y \in \Phi(x)} f(x, y)$$

can be replaced with the minimum and the solution multifunction $\Phi^* : \mathbb{X} \mapsto 2^{\mathbb{Y}} \setminus \{\emptyset\}$,

$$\Phi^*(x) = \{y \in \Phi(x) : \ f^*(x) = f(x, y)\},$$

is compact-valued and upper semi-continuous.

Continuity of Minima

Let $\Phi(x) = \mathbb{Y}$ *for each* $x \in \mathbb{X}$ *and the function* $f : \mathbb{X} \times \mathbb{Y} \mapsto \mathbb{R}$ *is* $\mathbb{K}$*-inf-compact on* $\mathrm{Gr}(\Phi)$. *Then the function* $f^*(x)$ *is continuous and the set-valued mapping* $\Phi^*(x)$ *is upper semi-continuous and compact-valued.*

The last theorem implies that the value function for the inventory control problem is continuous.

Continuity Properties of Minimax: Notations

- Let $\mathtt{X}, \mathtt{A}$ and $\mathtt{B}$ be metric spaces;
- $\Phi_{\mathtt{A}} : \mathtt{X} \mapsto 2^{\mathtt{A}} \setminus \{\varnothing\}$ and $\Phi_{\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{A}}) \subset \mathtt{X} \times \mathtt{A} \mapsto 2^{\mathtt{B}} \setminus \{\varnothing\}$ be set-valued mappings;
- $\mathtt{f} : \mathrm{Gr}(\Phi_{\mathtt{B}}) \subset \mathtt{X} \times \mathtt{A} \times \mathtt{B} \mapsto \overline{\mathbb{R}}$ be a function;
- the worst-loss function at $(x, a) \in \mathrm{Gr}(\Phi_{\mathtt{A}})$:

$$\mathtt{f}^{\sharp}(x, a) := \sup_{b \in \Phi_{\mathtt{B}}(x,a)} \mathtt{f}(x, a, b); \tag{3}$$

- the minimax function at $x \in \mathtt{X}$:

$$\mathrm{v}^{\sharp}(x) := \inf_{a \in \Phi_{\mathtt{A}}(x)} \sup_{b \in \Phi_{\mathtt{B}}(x,a)} \mathtt{f}(x, a, b); \tag{4}$$

- the solution multifunctions at $x \in \mathtt{X}$ and $a \in \Phi_{\mathtt{A}}(x)$:

$$\Phi^*_{\mathtt{A}}(x) := \left\{a^* \in \Phi_{\mathtt{A}}(x) \,:\, \mathrm{v}^{\sharp}(x) = \mathtt{f}^{\sharp}(x, a^*)\right\}; \tag{5}$$

$$\Phi^*_{\mathtt{B}}(x, a) := \left\{b \in \Phi_{\mathtt{B}}(x, a) \,:\, \mathtt{f}^{\sharp}(x, a) = \mathtt{f}(x, a, b)\right\}. \tag{6}$$

Continuity Properties of Minimax: Preliminaries

- The set-valued mapping $\Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}} : \mathtt{X} \times \mathtt{B} \mapsto 2^{\mathtt{A}}$ is uniquely defined by

$$\mathrm{Gr}(\Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}}) := \{(x, b, a) \in \mathtt{X} \times \mathtt{B} \times \mathtt{A} \,:\, (x, a, b) \in \mathrm{Gr}(\Phi_{\mathtt{B}})\}. \tag{7}$$

- The function $\mathtt{f}^{\mathtt{A}\leftrightarrow\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}}) \subset (\mathtt{X} \times \mathtt{B}) \times \mathtt{A} \mapsto \overline{\mathbb{R}}$ is defined as

$$\mathtt{f}^{\mathtt{A}\leftrightarrow\mathtt{B}}(x, b, a) := \mathtt{f}(x, a, b), \quad (x, a, b) \in \mathrm{Gr}(\Phi_{\mathtt{B}}). \tag{8}$$

- The following equality holds:

$$\mathrm{Dom}\, \Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}} = \mathrm{proj}_{\mathtt{X}\times\mathtt{B}}\mathrm{Gr}(\Phi_{\mathtt{B}}), \tag{9}$$

  where $\mathrm{proj}_{\mathtt{X}\times\mathtt{B}}\mathrm{Gr}(\Phi_{\mathtt{B}})$ is a projection of $\mathrm{Gr}(\Phi_{\mathtt{B}})$ on $\mathtt{X} \times \mathtt{B}$.

Sufficient Conditions for $\mathbb{K}$-inf-compactness of $\mathtt{f}^{\mathtt{A}\leftrightarrow\mathtt{B}}$

The function $\mathtt{f}^{\mathtt{A}\leftrightarrow\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}}) \subset (\mathtt{X} \times \mathtt{B}) \times \mathtt{A} \mapsto \overline{\mathbb{R}}$ is $\mathbb{K}$-inf-compact on $\mathrm{Gr}(\Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}})$ if and only if the following two conditions hold:

(i) *the function $\mathtt{f} : \mathrm{Gr}(\Phi_{\mathtt{B}}) \subset \mathtt{X} \times \mathtt{A} \times \mathtt{B} \mapsto \overline{\mathbb{R}}$ is lower semi-continuous;*

(ii) *if a sequence $\{x^{(n)}, b^{(n)}\}_{n=1,2,\ldots}$ with values in $\mathrm{Dom}\, \Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}}$ converges and its limit (x, b) belongs to $\mathrm{Dom}\, \Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}}$, then each sequence $\{a^{(n)}\}_{n=1,2,\ldots}$ with $(x^{(n)}, a^{(n)}, b^{(n)}) \in \mathrm{Gr}(\Phi_{\mathtt{B}})$, $n = 1, 2, \ldots,$ satisfying the condition that the sequence $\{\mathtt{f}(x^{(n)}, a^{(n)}, b^{(n)})\}_{n=1,2,\ldots}$ is bounded above, has a limit point $a \in \Phi_{\mathtt{A}}(x)$.*

Continuity Properties of Minimax: Preliminaries

Recall that a set-valued mapping $\Phi_{\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{A}}) \subset \mathtt{X} \times \mathtt{A} \mapsto 2^{\mathtt{B}} \setminus \{\varnothing\}$ is lower semicontinuous iff for each $x \in \mathtt{X}$ and $a \in \Phi_{\mathtt{A}}(x)$, for every sequence $(x_n, a_n) \to (x, a)$ with $x_n \in \mathtt{X}$, $a_n \in \Phi_{\mathtt{A}}(x_n)$, every $b \in \Phi_{\mathtt{B}}(x, a)$ is a limit point of a sequence $\{b_n\}$ with $b_n \in \Phi_{\mathtt{B}}(x_n, a_n)$.

Definition 3 ($\mathtt{A}$-lower semi-continuity)

A set-valued mapping $\Phi_{\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{A}}) \subset \mathtt{X} \times \mathtt{A} \mapsto 2^{\mathtt{B}} \setminus \{\varnothing\}$ is called $\mathtt{A}$-lower semi-continuous, if the following condition holds:

> if a sequence $x_n \to x$ with $x_n, x \in \mathtt{X}$, and $a \in \Phi_{\mathtt{A}}(x)$, $a_n \in \Phi_{\mathtt{A}}(x_n)$, then each $b \in \Phi_{\mathtt{B}}(x, a)$ is a limit point of a sequence $\{b_n\}_{n=1,2,\ldots}$ with $b_n \in \Phi_{\mathtt{B}}(x_n, a_n)$.

Sufficient conditions for $\mathtt{A}$-lower semi-continuity

Let $\Phi_{\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{A}}) \subset \mathtt{X} \times \mathtt{A} \mapsto 2^{\mathtt{B}} \setminus \{\varnothing\}$ be a lower semi-continuous set-valued mapping. Then the following statements hold:

(a) *if $\Phi_{\mathtt{A}} : \mathtt{X} \mapsto 2^{\mathtt{A}} \setminus \{\varnothing\}$ is upper semi-continuous and compact-valued at each $x \in \mathtt{X}$, then $\Phi_{\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{A}}) \subset \mathtt{X} \times \mathtt{A} \mapsto 2^{\mathtt{B}} \setminus \{\varnothing\}$ is $\mathtt{A}$-lower semi-continuous;*

(b) *if $\Phi_{\mathtt{B}}(x, a)$ does not depend on $a \in \Phi_{\mathtt{A}}(x)$ for each $x \in \mathtt{X}$, that is, $\Phi_{\mathtt{B}}(x, a_*) = \Phi_{\mathtt{B}}(x, a^*)$ for each $(x, a_*), (x, a^*) \in \mathrm{Gr}(\Phi_{\mathtt{A}})$, then $\Phi_{\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{A}}) \subset \mathtt{X} \times \mathtt{A} \mapsto 2^{\mathtt{B}} \setminus \{\varnothing\}$ is $\mathtt{A}$-lower semi-continuous.*

$\mathbb{K}$-inf-compactness of the worst-loss function

Let $\Phi_{\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{A}}) \subset \mathtt{X}\times\mathtt{A} \mapsto 2^{\mathtt{B}}\setminus\{\varnothing\}$ be an $\mathtt{A}$-lower semi-continuous set-valued mapping and the function $\mathtt{f}^{\mathtt{A}\leftrightarrow\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}}) \subset (\mathtt{X}\times\mathtt{B})\times\mathtt{A} \mapsto \overline{\mathbb{R}}$ be $\mathbb{K}$-inf-compact on $\mathrm{Gr}(\Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}})$. Then the worst-loss function $\mathtt{f}^\sharp : \mathrm{Gr}(\Phi_{\mathtt{A}}) \subset \mathtt{X}\times\mathtt{A} \mapsto \overline{\mathbb{R}}$ is $\mathbb{K}$-inf-compact on $\mathrm{Gr}(\Phi_{\mathtt{A}})$.

Upper semi-continuity of the worst-loss function

If a function $\mathtt{f} : \mathrm{Gr}(\Phi_{\mathtt{B}}) \subset (\mathtt{X}\times\mathtt{A})\times\mathtt{B} \mapsto \overline{\mathbb{R}}$ is $\mathbb{K}$-sup-compact on $\mathrm{Gr}(\Phi_{\mathtt{B}})$, then the worst-loss function $\mathtt{f}^\sharp : \mathrm{Gr}(\Phi_{\mathtt{A}}) \subset \mathtt{X}\times\mathtt{A} \mapsto \overline{\mathbb{R}}$ is upper semi-continuous. Moreover, the supremum in the definition of $\mathtt{f}^\sharp(x,a)$ can be replaced with the maximum and the solution multifunction $\{\Phi_{\mathtt{B}}^*(x,a)\}_{(x,a)\in\mathrm{Gr}(\Phi_{\mathtt{A}})}$ satisfies the following properties:

(a) *the graph $\mathrm{Gr}(\Phi_{\mathtt{B}}^*)$ is a Borel subset of $\mathtt{X}\times\mathtt{A}\times\mathtt{B}$;*

(b) *if $\mathtt{f}^\sharp(x,a) = -\infty$, then $\Phi_{\mathtt{B}}^*(x,a) = \Phi_{\mathtt{B}}(x,a)$, and, if $\mathtt{f}^\sharp(x,a) > -\infty$, then $\Phi_{\mathtt{B}}^*(x,a)$ is compact.*

Continuity of the worst-loss function

Let $\Phi_{\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{A}}) \subset \mathtt{X}\times\mathtt{A} \mapsto 2^{\mathtt{B}}\setminus\{\varnothing\}$ be a lower semi-continuous set-valued mapping, $\mathtt{f} : \mathrm{Gr}(\Phi_{\mathtt{B}}) \subset (\mathtt{X}\times\mathtt{A})\times\mathtt{B} \mapsto \overline{\mathbb{R}}$ be a $\mathbb{K}$-sup-compact function on $\mathrm{Gr}(\Phi_{\mathtt{B}})$, and the function $\mathtt{f}^{\mathtt{A}\leftrightarrow\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}}) \subset (\mathtt{X}\times\mathtt{B})\times\mathtt{A} \mapsto \overline{\mathbb{R}}$ be $\mathbb{K}$-inf-compact on $\mathrm{Gr}(\Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}})$. Then the worst-loss function $\mathtt{f}^\sharp : \mathrm{Gr}(\Phi_{\mathtt{A}}) \subset \mathtt{X}\times\mathtt{A} \mapsto \overline{\mathbb{R}}$ is continuous.

Continuity Properties of Minimax

Lower Semi-continuity of Minimax

Let $\Phi_B : \mathrm{Gr}(\Phi_A) \subset X \times A \mapsto 2^B \setminus \{\varnothing\}$ *be an* A*-lower semi-continuous set-valued mapping and the function* $f^{A\leftrightarrow B} : \mathrm{Gr}(\Phi_B^{A\leftrightarrow B}) \subset (X \times B) \times A \mapsto \overline{\mathbb{R}}$ *be* $\mathbb{K}$*-inf-compact on* $\mathrm{Gr}(\Phi_B^{A\leftrightarrow B})$*. Then the minimax function*

$$v^\sharp(x) := \inf_{a\in\Phi_A(x)} \sup_{b\in\Phi_B(x,a)} f(x,a,b)$$

is lower semi-continuous and the set $\Phi_A^*(x) = \mathrm{Argmin}_{a\in\Phi_A(x)} \sup_{b\in\Phi_B(x,a)} f(x,a,b)$ *is compact if* $v^\sharp(x) < +\infty$*, and* $\Phi_A^*(x) = \Phi_A(x)$ *if* $v^\sharp(x) = +\infty$*. Moreover, the graph* $\mathrm{Gr}(\Phi_A^*)$ *is a Borel subset of* $X \times A$*.*

Upper Semi-continuity of Minimax

Let $\Phi_A : X \mapsto 2^A \setminus \{\varnothing\})$ *be a lower semi-continuous set-valued mapping and* $f : \mathrm{Gr}(\Phi_B) \subset (X \times A) \times B \mapsto \overline{\mathbb{R}}$ *be a* $\mathbb{K}$*-sup-compact function on* $\mathrm{Gr}(\Phi_B)$*. Then the minimax function* $v^\sharp : X \mapsto \overline{\mathbb{R}}$ *is upper semi-continuous and the set* $\Phi_B^*(x,a) = \mathrm{Argmax}_{b\in\Phi_B(x,a)} f(x,a,b)$ *is compact, if* $\sup_{b\in\Phi_B(x,a)} f(x,a,b) > -\infty$*, and* $\Phi_B^*(x,a) = \Phi_B(x,a)$ *if* $\sup_{b\in\Phi_B(x,a)} f(x,a,b) = -\infty$*.*
Moreover, the graph $\mathrm{Gr}(\Phi_B^*)$ *is a Borel subset of* $X \times A \times B$*.*

Continuity Properties of Minimax

Continuity of Minimax and Solution Multifunctions

Let $\Phi_{\mathtt{A}} : \mathtt{X} \mapsto 2^{\mathtt{A}} \setminus \{\varnothing\}$ *be a lower semi-continuous set-valued mapping,* $\Phi_{\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{A}}) \subset \mathtt{X} \times \mathtt{A} \mapsto 2^{\mathtt{B}} \setminus \{\varnothing\}$ *be an* $\mathtt{A}$*-lower semi-continuous set-valued mapping,* $\mathtt{f} : \mathrm{Gr}(\Phi_{\mathtt{B}}) \subset (\mathtt{X} \times \mathtt{A}) \times \mathtt{B} \mapsto \overline{\mathbb{R}}$ *be a* $\mathbb{K}$*-sup-compact function on* $\mathrm{Gr}(\Phi_{\mathtt{B}})$*, and the function* $\mathtt{f}^{\mathtt{A}\leftrightarrow\mathtt{B}} : \mathrm{Gr}(\Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}}) \subset (\mathtt{X} \times \mathtt{B}) \times \mathtt{A} \mapsto \overline{\mathbb{R}}$ *be* $\mathbb{K}$*-inf-compact on* $\mathrm{Gr}(\Phi_{\mathtt{B}}^{\mathtt{A}\leftrightarrow\mathtt{B}})$*. Then the worst-loss function*

$$\mathtt{f}^{\sharp}(x, a) := \sup_{b \in \Phi_{\mathtt{B}}(x,a)} \mathtt{f}(x, a, b)$$

is continuous, the minimax function

$$\mathtt{v}^{\sharp}(x) := \inf_{a \in \Phi_{\mathtt{A}}(x)} \sup_{b \in \Phi_{\mathtt{B}}(x,a)} \mathtt{f}(x, a, b)$$

is continuous, the solution multifunction $\Phi_{\mathtt{A}}^{*}(x) = \mathrm{Argmin}_{a \in \Phi_{\mathtt{A}}(x)} \sup_{b \in \Phi_{\mathtt{B}}(x,a)} \mathtt{f}(x, a, b)$ *is upper semi-continuous and compact-valued, and the solution multifunction* $\Phi_{\mathtt{B}}^{*}(x, a) = \mathrm{Argmax}_{b \in \Phi_{\mathtt{B}}(x,a)} \mathtt{f}(x, a, b)$ *is upper semi-continuous and compact-valued.*

Zero-Sum Games with Perfect Information: Introduction

In a two-player zero-sum stochastic games with perfect information, Player II knows the action chosen by Player I. This model is equivalent to the model of turn-based games. This model is also called the robust Markov Decision Process. For such games, it sufficient for players to use only pure strategies.

We consider the case of Borel state and action sets and the discounted criterion. The problem is nontrivial when payoffs are not bounded. Such games have been studied by several authors over long time including Küenle (1986), Petersen, James, and Dupuis (2000) González-Trejo, Hernández-Lerma, and Hoyos-Reyes (2003), Iyengar (2005), Jaśkiewicz and Nowak (2011, 2014) and summarized in the survey by Jaśkiewicz and Nowak(2017).

We describe more general results. The higher level of generality is achieved by applying the generalization of Berge's maximum theorem to noncompact action sets (Feinberg, Kasyanov, and Zadojanchuk 2013, Feinberg, Kasyanov, and Voorneveld 2014, Feinberg and Kasyanov 2015) and their applications to minimax equalities (Feinberg, Kasyanov, Zgurovsky 2017, 2018).

Model Formulation

A *discrete-time two-person zero-sum stochastic game with perfect information* is defined by a tuple $\{\mathbb{X}, \mathbb{A}, \mathbb{B}, A, B, c, q\}$,

- $\mathbb{X}$ is the state space, which is a Borel subset of a Polish (complete separable metric) space;
- $\mathbb{A}$ is the *action space* of the *Player I*, which is a Borel subset of a Polish space;
- $\mathbb{B}$ is the *action space* of the *Player II*, which is a Borel subset of a Polish space;
- $A(x)$ is the set of actions available to Player I at a state $x \in \mathbb{X}$, where $A : \mathbb{X} \mapsto 2^{\mathbb{A}} \setminus \{\emptyset\}$. It is assumed that $\mathrm{Gr}(A) \in \mathcal{B}(\mathbb{X} \times \mathbb{A})$ and there exist a measurable mapping $\phi_{\mathbb{A}} : \mathbb{X} \mapsto \mathbb{A}$ such that $\phi_{\mathbb{A}}(x) \in A(x)$ for each $x \in \mathbb{X}$;
- $B(x, a)$ is the set of actions available to Player II at a state $x \in \mathbb{X}$ after Player I chose the action $a \in A(x)$, where $B : \mathrm{Gr}(A) \mapsto 2^{\mathbb{B}} \setminus \{\emptyset\}$. It is assumed that $\mathrm{Gr}(B) \in \mathcal{B}(\mathbb{X} \times \mathbb{A} \times \mathbb{B})$ and there exist a measurable mapping $\phi_{\mathbb{B}} : \mathbb{X} \times \mathbb{A} \mapsto \mathbb{B}$ such that $\phi_{\mathbb{B}}(x, a) \in B(x, a)$ for each $(x, a) \in \mathrm{Gr}(A)$;
- $c(x, a, b) \in [-\infty, +\infty]$ is the *payoff* of Player I to Player II if actions $a \in A(x)$ and $b \in B(x, a)$ are chosen a state $x \in \mathbb{X}$; c is a *Borel* function on $\mathrm{Gr}(B)$;
- $q(X|x, a, b)$ be the Borel-measurable transition probability representing that the next state is in $X \in \mathcal{B}(\mathbb{X})$, given that the actions $a \in A(x)$ and $b \in B(x, a)$ are chosen at the state $x \in \mathbb{X}$.

Expected Total Discounted Payoff

For a discount factor $\alpha \in [0, 1)$, strategies $\pi^{\mathbb{A}}$ and $\pi^{\mathbb{B}}$ of Players I and II respectively, and an initial state $x \in \mathbb{X}$, let

$$v_\alpha^{\pi^{\mathbb{A}},\pi^{\mathbb{B}},\oplus}(x) := \mathbb{E}_x^{\pi^{\mathbb{A}},\pi^{\mathbb{B}}} \sum_{t=0}^{\infty} \alpha^t c^+(x_t, a_t, b_t),$$

$$v_\alpha^{\pi^{\mathbb{A}},\pi^{\mathbb{B}},\ominus}(x) := \mathbb{E}_x^{\pi^{\mathbb{A}},\pi^{\mathbb{B}}} \sum_{t=0}^{\infty} \alpha^t c^-(x_t, a_t, b_t),$$

where $z^+ := \max\{z, 0\}$ and $z^- := \min\{z, 0\}$ for each $z \in \mathbb{R}$.

The expected total discounted payoff of Player I to Player II over the infinite horizon is

$$v_\alpha^{\pi^{\mathbb{A}},\pi^{\mathbb{B}}}(x) := \mathbb{E}_x^{\pi^{\mathbb{A}},\pi^{\mathbb{B}}} \sum_{t=0}^{\infty} \alpha^t c(x_t, a_t, b_t) = v_\alpha^{\pi^{\mathbb{A}},\pi^{\mathbb{B}},\oplus}(x) + v_\alpha^{\pi^{\mathbb{A}},\pi^{\mathbb{B}},\ominus}(x) \quad (10)$$

Additional conditions ensure that at least one of the summands in (10) is finite for each pair of policies $\pi^{\mathbb{A}}$ and $\pi^{\mathbb{B}}$.

Definition of Optimal Strategies for Player I

Let

$$v_\alpha^\sharp(x) := \inf_{\pi^{\mathbb{A}} \in \Pi^{\mathbb{A}}} \sup_{\pi^{\mathbb{B}} \in \Pi^{\mathbb{B}}} v_\alpha^{\pi^{\mathbb{A}}, \pi^{\mathbb{B}}}(x),$$

where $\Pi^{\mathbb{A}}$ and $\Pi^{\mathbb{B}}$ are the set of all strategies for players I and II respectively. This is the minimax of the game starting at the state $x \in \mathbb{X}$.

A strategy $\pi_*^{\mathbb{A}} \in \Pi^{\mathbb{A}}$ is called optimal for player I if

$$\sup_{\pi^{\mathbb{B}} \in \Pi^{\mathbb{B}}} v_\alpha^{\pi_*^{\mathbb{A}}, \pi^{\mathbb{B}}}(x) = v_\alpha^\sharp(x)$$

for all $x \in \mathbb{X}$.

Weak Continuity of Transition Probabilities

The transition probability $q(\cdot|x,a,b)$ is assumed to be weakly continuous in $(x,a,b) \in \mathrm{Gr}(B)$:

if $(x^{(n)}, a^{(n)}, b^{(n)}) \to (x,a,b)$, then $q(\cdot|x^{(n)}, a^{(n)}, b^{(n)}) \xrightarrow{weakly} q(\cdot|x,a,b)$, that is,

$$\lim_{n\to\infty} \int_{\mathbb{X}} f(y) q(dy|x^{(n)}, a^{(n)}, b^{(n)}) = \int_{\mathbb{X}} f(y) q(dy|x,a,b),$$

if $(x,a,b) = \lim_{n\to\infty}(x^{(n)}, a^{(n)}, b^{(n)})$, for all bounded and continuous functions $f : \mathrm{Gr}(B) \mapsto \mathbb{R}$.

Preliminaries

Recall that:

- The set-valued mapping $B^{\mathbb{A}\leftrightarrow\mathbb{B}} : \mathbb{X} \times \mathbb{B} \mapsto 2^{\mathbb{A}} \setminus \{\emptyset\}$ is uniquely defined by

$$\mathrm{Gr}(B^{\mathbb{A}\leftrightarrow\mathbb{B}}) := \{(x, b, a) \in \mathbb{X} \times \mathbb{B} \times \mathbb{A} \,:\, (x, a, b) \in \mathrm{Gr}(B),$$

  that is

$$B^{\mathbb{A}\leftrightarrow\mathbb{B}}(x, b) = \{a \in A(x) : b \in B(x, a)\}.$$

- The function $c^{\mathbb{A}\leftrightarrow\mathbb{B}} : \mathrm{Gr}(B^{\mathbb{A}\leftrightarrow\mathbb{B}}) \subset (\mathbb{X} \times \mathbb{B}) \times \mathbb{A} \mapsto \overline{\mathbb{R}}$ is defined as

$$c^{\mathbb{A}\leftrightarrow\mathbb{B}}(x, b, a) := c(x, a, b), \quad (x, a, b) \in \mathrm{Gr}(B).$$

Sufficient Conditions for the Existence of Optimal Strategies for Player I

Jaśkiewicz and Nowak (2011)		F., Kasyanov, Zgurovsky (2018)
(i) $A(x)$ is compact for each $x \in \mathbb{X}$ and set-valued mapping $x \mapsto A(x)$ is upper semi-continuous. (ii) c is lower semi-continuous.	$\xRightarrow{\text{(i)+(ii)}}$	$c^{\mathbb{A}\leftrightarrow\mathbb{B}}$ is $\mathbb{K}$-inf-compact.
(iii) Set-valued mapping $(x,a) \to B(x,a)$ is lower semi-continuous.	$\xRightarrow{\text{(i)+(iii)}}$	The set-valued mapping $(x,a) \to B(x,a)$ is $\mathbb{A}$-lower semi-continuous.

There exists a continuous function $\omega : X \mapsto [1,\infty)$ and a constant $\beta > 0$ such that $\alpha\beta < 1$,

$$\int_{\mathbb{X}} \omega(y) q(dy|x,a,b) \le \beta\omega(x) \quad \text{for each } (x,a,b) \in \mathrm{Gr}(B),$$

the function $(x,a,b) \mapsto \int_{\mathbb{X}} \omega(y) q(dy|x,a,b)$ is continuous, and

$$-\omega(x) \le \inf_{a\in A(x)} \inf_{b\in B(x,a)} c(x,a,b) \quad \text{for each } x \in \mathbb{X}.$$

References

MDPs/POMDPs

E. A. Feinberg, P. O. Kasyanov, and N. V. Zadoianchuk. Average cost Markov decision processes with weakly continuous transition probabilities. *Mathematics of Operations Research*, 37(4), 591–607, 2012.

E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky. Convergence of probability measures and Markov decision models with incomplete information. *Proceedings of the Steklov Institute of Mathematics*, 287(1), 96–117, 2014.

E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky. Partially observable total-cost Markov decision processes with weakly continuous transition probabilities. *Mathematics of Operations Research*, 41(2), 656–681, 2016.

E. A. Feinberg and M. E. Lewis. On the convergence of optimal actions for Markov decision processes and the optimality of (s, S) inventory policies. *Naval Research Logistic*, DOI:10.1002/nav.21750, 2017.

E. A. Feinberg and Y. Liang. On the optimality equation for average cost Markov decision processes and its validity for inventory control. *Annals of Operations Research*, DOI: 10.1007/s10479-017-2561-9, 2017.

Inventory Control

E. A. Feinberg. Optimality conditions for inventory control. *Tutorials in Operations Research*, 14–44, INFORMS, 2016.

E.A. Feinberg and Y. Liang. Structure of optimal solutions to periodic-review total-cost stochastic inventory control problems. *ACM SIGMETRICS Performance Evaluation Review*, 44(2), 21–23, 2016.

E. A. Feinberg and Y. Liang. Structure of optimal solutions to periodic-review total-cost inventory control problems. *Annals of Operations Research*, DOI: 10.1007/s10479-017-2548-6, 2017.

E. A. Feinberg and Y. Liang. Stochastic setup-cost inventory model with backorders and quasiconvex cost functions. Preprint arXiv:1705.06814, http://arxiv.org/pdf/1705.06814.pdf, 2017.

References (the most relevant ones are typed in blue)

Berge's Maximum Theorem

E. A. Feinberg, P. O. Kasyanov, and N. V. Zadoianchuk. Berge's theorem for noncompact image sets. *Journal of Mathematical Analysis and Applications*, 37(1), 255–259, 2013.

E. A. Feinberg, P. O. Kasyanov, and M. Voorneveld. Berge's maximum theorem for noncompact image sets. *Journal of Mathematical Analysis and Applications*, 413(2), 1040–1046, 2014.

E. A. Feinberg and P. O. Kasyanov. Continuity of minima: Local results. *Set-Valued and Variational Analysis*, 23(3), 485–499, 2015.

Fatou's Lemma

E. A. Feinberg, P. O. Kasyanov, and N. V. Zadoianchuk. Fatou's lemma for weakly converging probabilities. *Theory of Probability and Its Applications*, 58(4), 683–689, 2014.

E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky. Uniform Fatou's lemma. *Journal of Mathematical Analysis and Applications*, 444(1), 550–567, 2016.

E. A. Feinberg, P. O. Kasyanov, and Y. Liang. Fatou's lemma and Lebesgue's convergence theorem for varying measures and some of their applications.

Games

E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky. Continuity of equilibria for two-person zero-sum games with noncompact action sets and unbounded payoffs. *Annals of Operations Research*, 2017, published online: DOI 10.1007/s10479-017-2677-y.

E. A. Feinberg, P. O. Kasyanov, and M. Z. Zgurovsky. An example showing that A-lower semi-continuity is essential for minimax continuity theorems. *Operations Research Letters* 46(4):385–388, 2018.