Risk Management with
POE, VaR, CVaR, and bPOE

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Lower Bound vs. Average of Tail

Presentation compares two concepts for measuring and optimization of tails of probabilistic distributions

- Optimistic concept based on lower bound of outcomes in the tail
- Conservative concept based on average value of the tail
Concept 1: Lower Bound of the Tail

Two equivalent variants:

- Fix threshold $x$, which is the lower bound of outcomes in the tail, and constrain Probability of Exceedance (POE): $p(x) \leq 1 - \alpha$

- Fix probability of the tail, $1 - \alpha$, and constrain quantile, called Value-at-Risk (VaR) in finance: $q(\alpha) \leq x$

$$q(\alpha) \leq x \iff p(x) \leq 1 - \alpha$$
Concept 2: Average Value of the Tail

Two equivalent variants:

- Fix mean value of the tail, \( x \), and constrain tail probability, called Buffered Probability of Exceedance (bPOE): \( \bar{p}(x) \leq 1 - \alpha \)

- Fix probability of the tail, \( 1 - \alpha \), and constrain mean value of the tail, called Superquantile, Average VaR, Tail VaR, CVaR, and Expected Shortfall in finance: \( \bar{q}(\alpha) \leq x \)

\[
\bar{q}(\alpha) \leq x \iff \bar{p}(x) \leq 1 - \alpha
\]
CDF and POE

- $X =$ random “loss”
- Cumulative Distribution Function (CDF) = $F(x) = \mathbb{P}\{X \leq x\}$
- Probability of Exceedance (POE) = $p(x) = \mathbb{P}\{X > x\} = 1 - F(x)$, also known as Survival, Survivor, or Reliability function.
Risk Management with POE and CDF

Requirement: **probability that loss exceeds threshold** \( x \) **is small**

\[
p(x) \leq 1 - \alpha \quad \text{e.g.,} \quad 1 - \alpha = 1 - 0.95 = 0.05
\]

- **Nuclear:** probability that release of radiation exceeds some level
- **Finance:** default probability of a company \((\text{Assets-Liability} < 0)\)

Equivalently: **probability that loss is below threshold** \( x \) **is large**

\[
p(x) = 1 - F(x) \leq 1 - \alpha \quad \Longrightarrow
\]

\[
F(x) \geq \alpha \quad \text{e.g.,} \quad \alpha = 0.95
\]

- **Material Science:**
  material should withstand the load \( x \) with high probability
Quantile \( q(\alpha) \) is inverse of CDF.

Quantile is a solution of equation \( F(x) = \alpha \), i.e. \( F(q(\alpha)) = \alpha \).

Quantile is a solution of equation \( p(x) = 1 - \alpha \), i.e., \( p(q(\alpha)) = 1 - \alpha \).
Risk Management with Quantiles (VaR)

Requirement: Quantile with confidence $\alpha$ is less than some threshold $q(\alpha) \leq x$

- Finance: e.g., VaR for daily loss is below $1$ billion
Equivalence of POE and Quantile Constraints

Some engineering areas use POE other areas use Quantiles.

**Constraints on POE and quantiles are equivalent.** It is a matter of convenience.

Finance uses quantiles (Value-at-Risk or VaR) specified in USD.

Nuclear engineering uses POE, maybe because probabilities are more understandable to people than radiation dosages.
POE(\(x\)) \leq 1 - \alpha \implies \text{quantile}(\alpha) \leq x

Continuous and strictly increasing CDF

\[ p(x) \leq 1 - \alpha \]

\[ \implies F(x) \geq \alpha \]

\[ \implies F^{-1}(F(x)) \geq F^{-1}(\alpha) = q(\alpha) \]

\[ \implies x \geq q(\alpha) \]

\[ \implies q(\alpha) \leq x \]
\text{quantile}(\alpha) \leq x \quad \implies \quad \text{POE}(x) \leq 1 - \alpha

Continuous and strictly increasing CDF

\[ q(\alpha) = F^{-1}(\alpha) \leq x \]

\[ \implies F(F^{-1}(\alpha)) \leq F(x) \]

\[ \implies \alpha \leq F(x) \]

\[ \implies \alpha \leq 1 - p(x) \]

\[ \implies p(x) \leq 1 - \alpha \]
POE and Quantile have poor mathematical properties:

- nonconvex in random variable
- discontinuous for discrete distributions w.r.t. parameters
- difficult to manage (optimize)
- are not conservative: do not take into account the values of outcomes in the tail of the distribution
Superquantile (CVaR) vs Quantile (VaR)

Superquantile $\bar{q}(\alpha) = \text{average of the tail in excess of quantile (VaR)}$

$\bar{q}(\alpha) = \text{inverse of } \bar{F}(x) \text{ which is CDF of Superdistribution (red curve)}$
Formal Superquantile (CVaR) definition:
continuous distributions:
\[ \bar{q}(\alpha) = \mathbb{E}\{ X \mid X > q(\alpha) \} \]
general (including discrete) distributions:
\[ \bar{q}(\alpha) = \frac{1}{1 - \alpha} \int_{\alpha}^{1} q(\alpha) \, d\alpha = \min\{ C + \frac{1}{1 - \alpha} \mathbb{E}[X - C]^+ \}, \]
where \( [X - C]^+ = \max\{0, X - C\} \)

- takes into account values of outcomes in the tail of the distribution
- coherent risk measure (the best from theoretical perspective)
- convex in random variable
- continuous w.r.t. parameters
- easy to manage and optimize with convex and linear programming, (Rockafellar & Uryasev (2000))
bPOE vs POE

Buffered Probability of Exceedance (bPOE) = $1 - \bar{F}(x) = 1 - \alpha$, where $\alpha$ satisfies equation $\bar{q}(\alpha) = x$.

Superdistribution $\bar{F}(x)$ (Rockafellar & Royset (2013)). Special case of bPOE with $x = 0$ (Rockafellar & Royset (2010)). General bPOE case and optimization representation (Norton & Uryasev (2014), Mafusalov & Uryasev (2014)).
bPOE properties

bPOE: will be a new hit in risk management, similar to CVaR

- optimization representation: \( \bar{p}(x) = \min_{a \geq 0} \mathbb{E}[a(X - x) + 1]^+ \)
- takes into account values of outcomes in the tail of the distribution
- quasi-convex in random variable \( X \)
- lowest quasi-convex (in \( X \)) upper bound of POE
- bPOE is about twice bigger than POE
- continuous w.r.t. parameters
- easy to manage (optimize with convex and linear programming)

\[ \bar{q}(\alpha) \leq x \quad \iff \quad \bar{p}(x) \leq 1 - \alpha \]
Low Bound vs. Average of Tail

\[ q(\alpha) \leq x \iff p(x) \leq 1 - \alpha \]

\[ \bar{q}(\alpha) \leq x \iff \bar{p}(x) \leq 1 - \alpha \]
Risk Management in Different Fields

\[ p(x) \leq 1 - \alpha \quad \text{nuclear, material, finance} \]

\[ q(\alpha) \leq x \quad \text{finance} \]

\[ \bar{q}(\alpha) \leq x \quad \text{finance} \]

\[ \bar{p}(x) \leq \alpha \quad \text{optimization of large physical systems} \]
Example: bPOE Minimization

- \( L(z) = c_0 + \sum_{i=1}^{n} c_i z_i \) is a linear function w.r.t. \( z = (z_1, \ldots, z_n) \) with random coefficients \((c_0, c_1, \ldots, c_n)\)

- minimize bPOE of \( L(z) \) w.r.t. \( z \)

\[
\min_z \bar{p}(x, L(z)) = \min_z \min_{a \geq 0} \mathbb{E} \left[ a(L(z) - x) + 1 \right]^+
\]

\[
= \min_{z, a \geq 0} \mathbb{E} \left[ a(c_0 + \sum_{i=1}^{n} c_i z_i - x) + 1 \right]^+
\]

\[
= \min_{z, a \geq 0} \mathbb{E} \left[ (c_0 - x) a + \sum_{i=1}^{n} c_i a z_i + 1 \right]^+
\]

\[
= \min_{y, a \geq 0} \mathbb{E} \left[ (c_0 - x) a + \sum_{i=1}^{n} c_i y_i + 1 \right]^+
\]

- change of variables \( az \rightarrow y \) reduces the problem to convex and linear programming w.r.t. variables \( y, a \)