Soft Margin Support Vector Classification as Buffered Probability Minimization

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Outline

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VaR and POE

Some Notation...

- Y:= a real valued random variable
- $z \in \mathbb{R}$:= a threshold level
- $\alpha \in [0, 1]$:=probability level

Quantile (VaR) & Probability of Exceedance (POE)

- $q_\alpha(Y)$:= $\min\{z : P(Y \leq z) \geq \alpha\}$ =quantile of Y at $\alpha \in [0, 1]$

- *POE at z*: probability level where $q_\alpha(Y) = z$
  or $\{1 - \alpha : q_\alpha(Y) = z\} = P(Y > z) = p_z(Y)$

CVaR and bPOE

Some Notation...

- Y:= a real valued random variable
- $z \in \mathbb{R}$:= a threshold level
- $\alpha \in [0, 1]$:=probability level

Superquantile (CVaR) & Buffered Probability of Exceedance (bPOE)

- $\bar{q}_\alpha(Y) := E[Y|Y > q_\alpha(Y)]$ = *superquantile* of Y at $\alpha \in [0, 1]$

- *bPOE at z*: probability level where $\bar{q}_\alpha(Y) = z$
  or $\{1 - \alpha : \bar{q}_\alpha(Y) = z\} = \bar{p}_z(Y)$

POE vs bPOE?

Probability of Exeedance (POE)

- POE is concerned with the proportion of events exceeding a threshold $z \in \mathbb{R}$.
- DOES NOT consider the magnitude of these events.
- Considers only a count of events exceeding a threshold $z \in \mathbb{R}$.

Buffered Probability of Exceedance (bPOE)

- bPOE is concerned with the proportion of events, that when considered together, have average magnitude equal to some threshold $z \in \mathbb{R}$.
- bPOE is a probability measurement that accounts for the magnitude of tail events.

Why is bPOE important?

POE can hide critical information

- When the distribution of Y is heavy tailed, the magnitude of tail events is important.

Example: Hurricane Damage Data

Threshold (Damage in $ billions)	POE (%)	bPOE (%)
100	1	3
50	4	10
10	15	69
1	48	100
0.1	79	100

- Damage Data = Heavy-tailed distribution
- Notice: bPOE reflects heavy-tail (e.g. at threshold $10 billion)

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Calculation of bPOE & Superquantile

Calculation of superquantile

Let $E[\cdot]^+ = E[\max(\cdot, 0)]$. The superquantile of Y at probability level $\alpha \in [0, 1]$ equals:

$$\bar{q}_\alpha(Y) = \min_\gamma \gamma + \frac{E[Y - \gamma]^+}{1 - \alpha}$$

Calculation of bPOE

bPOE of Y at threshold $z \in \mathbb{R}$ equals:

$$\bar{p}_z(Y) = \inf_{\gamma < z} \frac{E[Y - \gamma]^+}{z - \gamma} \tag{1}$$

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Easy Probability Minimization

Consider a linear loss function $L(w, Y) = w^T Y$ with parameters $w \in \mathbb{R}^n$ and real valued random vector $Y \in \mathbb{R}^m$.

Minimize POE of loss at threshold $z = 0$...Hard Problem!

$$\min_w P(L(w, Y) > 0)$$

Minimize bPOE of loss at threshold $z = 0$...Easy Problem!

$$\min_w \bar{p}_0(L(w, Y)) = \min_w E[L(w, Y) + 1]^+$$

The Binary Classification Problem

THE DATA:

We have samples $\{(X_1, y_1), \ldots, (X_N, y_N)\}$ where

- $X_i \in \mathbb{R}^n$: vector of features for sample i
- $y_i \in \{-1, +1\}$: class label of sample i

THE TASK:

Using the labeled samples, construct a linear function $f : \mathbb{R}^n \to \mathbb{R}$ that predicts labels.

$$X' \to \boxed{f(X') \leq 0} \to y' = -1$$

$$X' \to \boxed{f(X') > 0} \to y' = +1$$

Application Examples:

- Credit scoring: $X =$ financial indicators; $y =$ creditworthy or not?
- Medical: $X =$ patient health indicators; $y =$ has disease or not?
- E-mail filtering: $X =$ e-mail text; $y =$ spam or not?
- Recommendation: $X =$ past purchases; $y =$ beer or wine?

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Margin Maximization: Support Vector Machines

Let's Think Geometrically

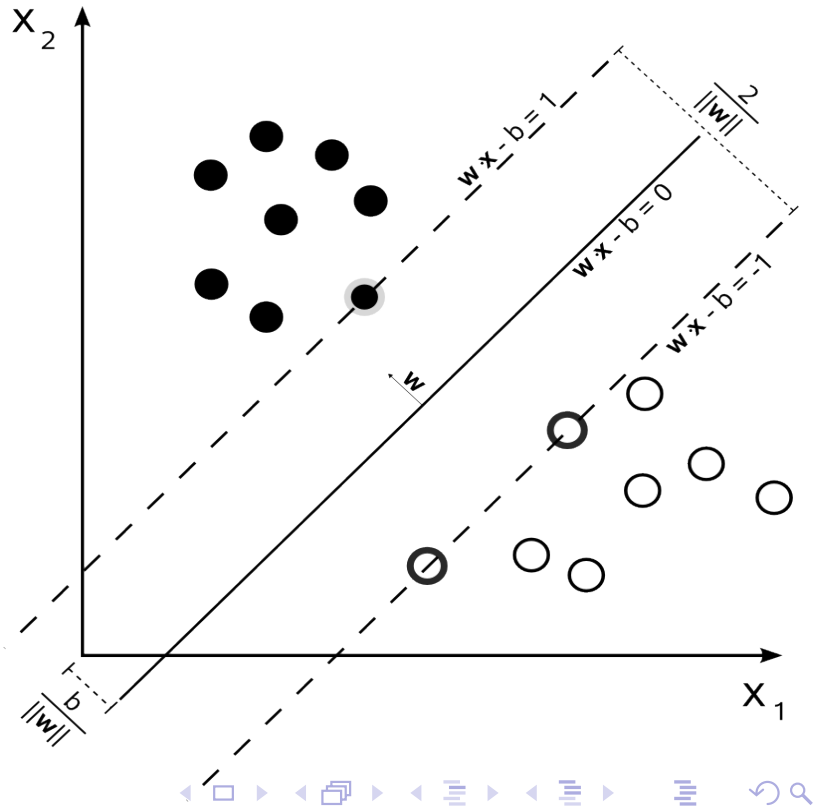
Find a 'separating' hyperplane

What is our function (i.e. hyperplane)?

$$f(X) = w^T X - b$$

What makes a good hyperplane? One with *maximum margin*

$$\max_{w,b} \frac{1}{\|w\|_2} \equiv \min_{w,b} \|w\|_2$$



Finding the Maximum Margin Hyperplane: SVM formulation

Step 1: Define our random loss function:

- $L(w, b, X_i, y_i) = -y_i(w^T X_i - b)$
- If prediction is wrong, $-y_i(w^T X_i - b) > 0$
- If prediction is right, $-y_i(w^T X_i - b) < 0$

Step 2: Define our formulation to solve for max margin hyperplane:

Hard Margin Support Vector Classifier

$$\begin{aligned} \min_{w,b} \quad & \|w\|_2 \\ s.t. \quad & 0 \geq -y_i(w^T X_i - b) + 1 \quad , \forall\, i \in \{1, ..., N\} \end{aligned} \tag{2}$$

Q: What if data not linearly separable?
A : Introduce tradeoff between in-sample accuracy and margin size

Soft Margin Support Vector Classifier: The C-SVM

$$\begin{aligned} \min_{w,b,\xi} \quad & C\|w\|_2 + \sum_{i=1}^{N} \xi_i \\ s.t. \quad & \xi_i \geq -y_i(w^T X_i - b) + 1 \quad , \forall\, i \in \{1, ..., N\} \\ & \xi \geq 0 \end{aligned} \tag{3}$$

Hard Margin Support Vector Classifier

$$\begin{aligned} \min_{w,b} \quad & \|w\|_2 \\ s.t. \quad & 0 \geq -y_i(w^T X_i - b) + 1 \quad , \forall\, i \in \{1, ..., N\} \end{aligned} \tag{4}$$

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Let's Think Probabilistically

Find hyperplane that minimizes probability of losses

Step 1: Define our random loss function:

- $L(w, b, X_i, y_i) = -y_i(w^T X_i - b)$
- If prediction is wrong, $-y_i(w^T X_i - b) > 0$
- If prediction is right, $-y_i(w^T X_i - b) < 0$

Step 2: Define formulation: Minimize probability of losses exceeding some threshold $-C \leq 0$:

$$\min_{w,b} \quad p_{-C}\left(\frac{-y(w^T X - b)}{\|w\|}\right) \equiv \min_{w,b} \quad P\left(\frac{-y(w^T X - b)}{\|w\|} \geq -C\right)$$

Problem! With empirical observations, probability minimization is non-convex, discontinuous.
Idea: Minimize buffered probability of exceedance instead!

$$\min_{w,b} \quad \bar{p}_{-C}\left(\frac{-y(w^T X - b)}{\|w\|}\right) \tag{5}$$

$$\downarrow$$

Using bPOE formula, (5) becomes the EC-SVM

$$\begin{aligned} \min_{w,b,\xi} \quad & \sum_{i=1}^{N} \xi_i \\ s.t. \quad & \xi_i \geq C\|w\| - y_i(w^T X_i - b) + 1 \\ & \xi \geq 0 \end{aligned} \tag{6}$$

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bPOE minimization and C-SVM: Equivalent problems!

Soft Margin Maximization: The C-SVM

$$\begin{aligned} \min_{w,b,\xi} \quad & C\|w\|_2 + \sum_{i=1}^{N} \xi_i \\ s.t. \quad & \xi_i \geq -y_i(w^T X_i - b) + 1 \\ & \xi \geq 0 \end{aligned} \tag{7}$$

bPOE minimization: The EC-SVM

$$\begin{aligned} \min_{w,b,\xi} \quad & \sum_{i=1}^{N} \xi_i \\ s.t. \quad & \xi_i \geq C\|w\| - y_i(w^T X_i - b) + 1 \\ & \xi \geq 0 \end{aligned} \tag{8}$$

What did we prove?

- C-SVM and EC-SVM are equivalent for $C \geq 0$.
- Equivalence holds for ANY general norm.

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Why is this interesting? What else have we proved?

C-SVM

- Derived with geometric intuition
- Previously, no interpretation for C-parameter
- Previously, no interpretation for optimal objective value

EC-SVM

- Derived from probabilistic intuition
- Free parameter, C, has statistical interpretation (bPOE threshold)
- Optimal objective value is a probability level

"..the parameter C has no intuitive meaning."
Shawe-Taylor, Cristianini (2004)

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A Larger Framework

A Larger Framework: Superquantiles and Soft Margins

$\hat{C} \in [0, +\infty)$		$C \in (-\infty, +\infty)$	$=$	$-z$
$\uparrow$		$\uparrow$		$\uparrow$
C-SVM	$\subset$	EC-SVM	$\equiv$	$\min_{w,b} \bar{p}_z\left(-y(w^T X - b)\right)$
$\Updownarrow$		$\Updownarrow$		$\Updownarrow$
ν-SVM	$\subset$	Eν-SVM	$\equiv$	$\min_{w,b} \bar{q}_\alpha\left(-y(w^T X - b)\right)$
$\downarrow$		$\downarrow$		$\downarrow$
$\hat{\nu} \in (\nu_{min}, \nu_{max}]$		$\nu \in [0, 1]$	$=$	$1 - \alpha$

Key:

- $\Updownarrow$ — Formulations generate same set of optimal hyperplanes.
- $\subset$ — The right hand side formulation is an "extension" of the left hand formulation.
- $\equiv$ — Formulations are objectively equivalent.
- $\uparrow$ or $\downarrow$ — Arrow points to parameter values for the formulation

Summary

- SVM's are an extremely popular tool for classification
- We showed that SVM's can also be viewed as a simple bPOE minimization problem
- We derived an equivalent SVM that is more interpretable than the classical formulation
- We showed that it fits into a larger framework, fully connecting soft margin classification with superquantile concepts

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$$\begin{aligned} \max_{\beta} \quad & \sum_{i=1}^{N} \beta_i \\ s.t. \quad & \| \sum_{i=1}^{N} \beta_i y_i x_i \|^* \leq C \\ & \sum_{i=1}^{N} \beta_i y_i = 0 \\ & 0 \leq \beta_i \leq 1 \ , \ \forall \ i \in \{1, \ldots, N\} \end{aligned}$$

$$\begin{aligned} \max_{\beta} \quad & \sum_{i=1}^{N} \beta_i \\ s.t. \quad & \| \sum_{i=1}^{N} \beta_i y_i x_i \|^* \leq C \sum_{i=1}^{N} \beta_i \\ & \sum_{i=1}^{N} \beta_i y_i = 0 \\ & 0 \leq \beta_i \leq 1 \ , \ \forall \ i \in \{1, \ldots, N\} \end{aligned}$$

$$\begin{aligned} \max_{\beta} \quad & \sum_{i=1}^{N} \beta_i \\ s.t. \quad & \sum_{i=1}^{N} \sum_{i=1}^{N} \beta_i \beta_j (y_i y_j \phi(x_i)^T \phi(x_j) - C^2) \leq 0 \\ & \sum_{i=1}^{N} \beta_i y_i = 0 \\ & 0 \leq \beta_i \leq 1 \ , \ \forall \ i \in \{1, \ldots, N\} \end{aligned}$$