On the attainable distributions of diffusion processes pertaining to a chain of distributed systems

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Introduction

Let $x(t)$ be an $\mathbb{R}^d$-valued Markov process with a transition kernel $q(s, y; t, x)$, $0 \leq s \leq t \leq T$.

Start with an initial density $p_0(x)$, then the density at time T is given by

$$p_T(x) = \int q(0, y; T, x) p_0(y) dy.$$

Suppose that we observe a different density $\pi_T(x)$ at time T. Then, our assumption about the transition kernel of $x(t)$ seems wrong.

Then, the following question arises: What is the transition kernel $\tilde{q}(s, y; t, x)$ that is close, *in some sense*, to $q(s, y; t, x)$ and for which

$$\pi_T(x) = \int \tilde{q}(0, y; T, x) p_0(y) dy.$$

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Remark

The above question is closely related to the problem of assigning initial and final conditions to the Markov process $x(t)$; and such a problem has been well studied in the context of reciprocal processes (e.g., Jamison (1974/75); Beurling (1960)).

Introduction . . .

Definition (1)

Let $x(t)$, $0 \leq t \leq T$, be a stochastic process defined on a measure space $(\Omega, \mathscr{F}, P)$. For $0 \leq s < t \leq T$, define the following σ-algebras:

$$\mathscr{A}_{s,t} = \sigma\big\{x(\tau)|s \leq \tau \leq t\big\},$$
$$\mathscr{B}_{s,t} = \sigma\big\{x(\tau)|\tau \leq s \text{ or } \tau \geq t\big\}.$$

Then, we say that $x(t)$ is a **reciprocal process** if

$$P\big\{A \cap B|x(s), x(t)\big\} = P\big\{A|x(s), x(t)\big\}P\big\{B|x(s), x(t)\big\}$$

for any $0 \leq s < t \leq T$ and $A \in \mathscr{A}_{s,t}$, $B \in \mathscr{B}_{s,t}$.

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Under some technical assumptions, a reciprocal process admits an **intermediate density** $p(s, x; t, y; u, z)$, i.e.,

$$p(s, x; t, y; u, z) = \frac{q(s, x; t, y)q(t, y; u, z)}{q(s, x; u, z)}, \; 0 \leq s < t < u \leq T,$$

which expresses the conditional density of $x(t)$ given $x(s) = x$ and $x(u) = z$.

Remark

Suppose that we are given two probability measures μ_0 and μ_T and a transition kernel $q(s, y; t, x)$ for $0 \leq s < t \leq T$. Then, we can construct a reciprocal process (with an intermediate density $p(s, x; t, y; u, z)$) such that $x(0)$ and $x(T)$ are distributed according to μ_0 and μ_T, respectively.

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General observation:
Let $q(s, y; t, x)$ be a transition kernel associated with the following diffusion process $x(t)$

$$dx(t) = b(t, x(t))dt + \sigma(t, x(t))dW(t), \ 0 \leq t \leq T.$$

Then, constructing **reciprocal processes** corresponds to a change of measure on the path-space.

Important observation:
Consider the following controlled-diffusion process $x^u(t)$, $0 \leq t \leq T$

$$dx^u(t) = \big(b(t, x^u(t)) + a(t, x^u(t))u(t)\big)dt + \sigma(t, x^u(t))dW(t),$$

where $a(t, x^u(t)) = \sigma^T(t, x^u(t))\sigma(t, x^u(t))$.

Suppose that we are given two probability measures μ_0 and μ_T. Then, what is the **admissible minimum energy** control u^* for which $x^{u^*}(t)$ evolves from μ_0 to μ_T?

Introduction . . .

Remark

In this talk, our main purpose is to throw some light on the structure of controlled-diffusion processes pertaining to a chain of distributed systems.

At the same time, we also touch some related questions concerning entropy minimization subject to an initial distribution and a final attainable distribution for such controlled-diffusion processes.

Preliminaries

Consider the following distributed system

$$\left.\begin{aligned}
dx_t^1 &= m_1\left(t, x_t^1, \ldots, x_t^n\right)dt + \sigma\left(t, x_t^1, \ldots, x_t^n\right)dW(t) \\
dx_t^2 &= m_2\left(t, x_t^1, \ldots, x_t^n\right)dt \\
dx_t^3 &= m_3\left(t, x_t^2, \ldots, x_t^n\right)dt \\
&\;\;\vdots \\
dx_t^n &= m_n\left(t, x_t^{n-1}, x_t^n\right)dt, \quad 0 \le t \le T
\end{aligned}\right\}, \qquad (1)$$

where

- x^i is an $\mathbb{R}^d$-valued state for the ith subsystem, with $i \in \{1, 2, \ldots, n\}$,
- the functions $m_1 \colon (0, \infty) \times \mathbb{R}^{nd} \to \mathbb{R}^d$ and $m_i \colon (0, \infty) \times \mathbb{R}^{(n-i+1)d} \to \mathbb{R}^d$ for $i = 2, \ldots, n$ are uniformly Lipschitz, with bounded first derivatives,

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- $\sigma\colon [0,\infty)\times\mathbb{R}^{nd}\to\mathbb{R}^{d\times d}$ is Lipschitz with the least eigenvalue of $\sigma\,\sigma^T$ uniformly bounded away from zero, i.e.,

  $$\sigma\big(t, x_t^1, \ldots, x_t^n\big)\,\sigma^T\big(t, x_t^1, \ldots, x_t^n\big) \geq \lambda I_d, \quad \forall (x_t^1, \ldots, x_t^n) \in \mathbb{R}^{nd},$$

  for all for $t \geq 0$ and some $\lambda > 0$,
- W (with $W(0) = 0$) is a d-dimensional standard Wiener process.

Notation:

- we use bold face letters to denote variables in $\mathbb{R}^{nd}$,
- for any $t \geq 0$, the solution $\big(x_t^1, x_t^2, \ldots, x_t^n\big)$ is denoted by $\mathbf{x}_t$,
- for $\big(t, (x^{j-1}, \ldots, x^n)\big) \in (0,\infty)\times\mathbb{R}^{(n-j+1)d}$, $j = 2, \ldots, n$, the function $x^j \mapsto m_j\big(t, x^{j-1}, \ldots, x^n\big)$ is continuously differentiable w.r.t. x^j and its derivative denoted by $\big(t, x^{j-1}, \ldots, x^n\big) \mapsto D_{x^j} m_j\big(t, x^{j-1}, \ldots, x^n\big)$.

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Then, we can write (1) as follow

$$d\mathbf{x}_t = \mathbf{M}(t, \mathbf{x}_t)dt + G\,\sigma(t, \mathbf{x}_t)dW_t, \tag{2}$$

where $\mathbf{M} = \left[m_1, m_2, \dots, m_n\right]$ is an $\mathbb{R}^{nd}$-valued function and $G = \left[I_d, 0, \dots, 0\right]^T$ stands for an $(nd \times d)$ matrix that embeds $\mathbb{R}^d$ into $\mathbb{R}^{nd}$. Moreover, the **infinitesimal generator** associated with (2) is given by

$$\mathcal{L}_{t,\mathbf{x}} = \frac{1}{2}\operatorname{tr}\left(a(t, \mathbf{x})D^2_{x^1}\right) + m_1(t, \mathbf{x})D_{x^1} + \sum\nolimits_{j=2}^{n} m_j(t, \mathbf{x}^{j-1})D_{x^j},$$

where $a(t, \mathbf{x}) = \sigma(t, \mathbf{x})\sigma^T(t, \mathbf{x})$.

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Assumption (1)

(a) *The functions* $m_1(t, \mathbf{x})$ *and* $m_j(t, \mathbf{x}^{j-1})$ *for* $j = 2, \ldots, n$ *satisfy Hölder conditions with respect to* $\mathbf{x}$ *and* $\mathbf{x}^{j-1}$*, respectively. Moreover,* $a(t, \mathbf{x})$ *is a bounded* $C^{\infty}\left([0, T] \times \mathbb{R}^{nd}\right)$*-function;* $a(t, \mathbf{x})$ *and* $D_{x^i} a(t, \mathbf{x})$ *are bounded and satisfy Hölder conditions with respect to both* $\mathbf{x}$ *and* t *(e.g., Hörmander (1967)).*

(b) *The infinitesimal generator* $\mathcal{L}_{t,\mathbf{x}}$ *is hypoelliptic.*

Remark

The hypoellipticity assumption is related to a strong accessibility property of controllable nonlinear systems that are driven by white noise. Note that the hypoellipticity assumption also implies that the diffusion process $\mathbf{x}_t$ *has a transition kernel* $q(s, \mathbf{y}, t, \mathbf{x})$ *with a strong Feller property.*

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Note that, from **Assumption (1)**, the parabolic PDE

$$\frac{\partial f}{\partial t} + \mathcal{L}_{t,\mathbf{x}} f = 0 \quad \text{in} \quad [0, T) \times \mathbb{R}^{nd}$$

has a fundamental solution $q(s, \mathbf{y}, t, \mathbf{x})$ which is twice continuously differentiable with respect to $\mathbf{y}$ and continuously differentiable with respect to s.

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Moreover, for any positive measurable function $g(\mathbf{x})$ such that

$$\int_{\mathbb{R}^{nd}} q(0, \mathbf{x}, T, \mathbf{z}) g(\mathbf{z}) d\mathbf{z} < +\infty \quad \text{for some} \quad \mathbf{x} \in \mathbb{R}^{nd}.$$

Then, the function

$$h(t, \mathbf{x}) = \int_{\mathbb{R}^{nd}} q(t, \mathbf{x}, T, \mathbf{z}) g(\mathbf{z}) d\mathbf{z}$$

belongs $C_b^{1,2}\left([0, T] \times \mathbb{R}^{nd}\right)$.

Remark

Note that the function $h(t, \mathbf{x})$ is the kernel of the operator $\left(\partial/\partial t + \mathcal{L}_{t,\mathbf{x}}\right)$, i.e., $\left(\partial h/\partial t + \mathcal{L}_{t,\mathbf{x}} h\right) = 0$.

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Remark

Note that a continuous change of measure on the path-space is related to changing the drift of the diffusion process associated with (2)*.*

Then, we have the following results.

Proposition (1)

Suppose that $\mathbf{x}_t$ *is a weak solution of* (2)*. Let the function* $h(t, \mathbf{x}) \in C_b^{1,2}\big([0, T] \times \mathbb{R}^{nd}\big)$ *be a strictly positive solution to following*

$$\frac{\partial h}{\partial t} + \mathcal{L}_{t,\mathbf{x}} h = 0 \quad in \quad [0, T) \times \mathbb{R}^{nd}$$

such that $\mathbb{E}\big\{h(t, \mathbf{x})\big\} < +\infty$ *and*

$$h(s, \mathbf{x}) = \mathbb{E}_{s,\mathbf{x}}\big\{h(t, \mathbf{x})\big\}, \quad 0 \leq s < t < T.$$

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Then, the following SDE

$$d\mathbf{x}_t^h = \Big(\mathbf{M}(t, \mathbf{x}_t^h) + G\, a(t, \mathbf{x}_t^h) D_{x^1} \log h(t, \mathbf{x}_t^h)\Big) dt + G\, \sigma(t, \mathbf{x}_t^h) dW_t$$

admits a weak solution in $[0, T)$ *and, moreover, its transition kernel is given by*

$$q^h(s, \mathbf{y}, t, \mathbf{x}) = \frac{q(s, \mathbf{y}, t, \mathbf{x}) h(t, \mathbf{x})}{h(s, \mathbf{y})}.$$

Proof.

The proof involves introducing a martingale process $z(t) = h(t, \mathbf{x})/h(0, \mathbf{x}_0)$. Then, using change of measure $\frac{dQ}{dP} = z(T - \epsilon)$, with $\epsilon > 0$, we can show that

$$\mathbb{E}_{s,\mathbf{x}}^Q\{f(\mathbf{x})\} = \int q^h(s, \mathbf{x}, t, \mathbf{y}) f(\mathbf{y}) d\mathbf{y} < +\infty$$

for any test function $f \in C_0^\infty(\mathbb{R}^{nd})$. $\square$

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Recall the following:

Definition (2)

Assume that μ and ν are σ-finite measures defined in the same measure space. Then, the *relative entropy* of μ w.r.t. ν is defined by

$$H(\mu|\nu) = \begin{cases} \int \log\left(\dfrac{d\mu}{d\nu}\right) d\mu, & \text{if} \quad \mu \ll \nu, \\ +\infty & \text{otherwise.} \end{cases}$$

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Proposition (2)

Let μ_0 and μ_T be two probability measures on $\mathbb{R}^{nd}$ and the transition probability density $q(s, \mathbf{y}, t, \mathbf{x})$. Then, there exists a unique pair of σ-finite measures (ν_0, ν_T) on $\mathbb{R}^{nd}$ such that the measure μ on $\mathbb{R}^{nd} \times \mathbb{R}^{nd}$ defined by

$$\mu(E) = \int_E q(0, \mathbf{y}, T, \mathbf{x}) \nu_0(d\mathbf{y}) \nu_T(d\mathbf{x})$$

has marginals μ_0 and μ_T. Furthermore, $\nu_0 \ll \mu_0$ and $\nu_T \ll \mu_T$ (i.e., they are mutually absolutely continuous measures).

Statement of the problem

Consider the following controlled-diffusion process

$$d\mathbf{x}_t^u = \big(\mathbf{M}(t, \mathbf{x}_t^u) + G\,\mathbf{u}_t\big)dt + G\,\sigma(t, \mathbf{x}_t^u)dW_t, \tag{3}$$

where $\mathbf{u}_t$ is an admissible control that satisfies

(i) $\mathbf{u}_t$ is $\sigma\{\mathbf{x}_t^u\}$-measurable;

(ii) (3) admits a weak solution in [0, T); and

(iii) $\mathbb{E}\int_0^T \|\mathbf{u}_t\|_{a^{-1}}^2 dt < +\infty$, where $\|\mathbf{u}_t\|_{a^{-1}}^2 \triangleq \|\sigma^{-1}(t, \mathbf{x}_t^u)\mathbf{u}_t\|^2$.

Statement of the problem . . .

Assume that we are given two probability measures μ_0 and μ_T, then we consider the following problem:

Problem (P)

Find an optimal admissible control $\mathbf{u}_t^*$ *such that*

(1) $\mathbf{x}_0^{u^*}$ *is distributed according to* μ_0, *and* $\mathbf{x}_T^{u^*}$ *according* μ_T; *and*

(2) *the optimal admissible control* $\mathbf{u}_t^*$ *(among all admissible controls satisfying condition (i)) minimizes the following cost functional*

$$J(\mathbf{u}_t) = \mathbb{E} \int_0^T \frac{1}{2} \|\mathbf{u}_t\|_{a^{-1}}^2 dt.$$

Statement of the problem . . .

Assume that $\mathbf{x}_t$ is a weak solution in $[0, T]$ to the following

$$d\mathbf{x}_t = \mathbf{M}(t, \mathbf{x}_t)dt + G\sigma(t, \mathbf{x}_t)dW_t, \quad \mathbf{x}_0 = \xi,$$

where ξ is distributed according to μ_0, with $\mathbb{E}|\xi|^2 < +\infty$.

Let $\mathcal{S}_t$ be an operator, acting on the set of σ-finite measures on $\mathbb{R}^{nd}$, defined by

$$\frac{d\mathcal{S}_t\mu}{d\lambda}(\mathbf{x}_t) = \int q(0, \mathbf{y}, t, \mathbf{x})\mu(d\mathbf{y}),$$

where $d\mathcal{S}_t\mu/d\lambda$ is the Radon-Nikodym derivative w.r.t. the Lebesque measure λ.

Main results - Connection with stochastic control problems

In what follows, using the **logarithmic transformations** approach from Fleming (e.g., Fleming (1978/782)), we provide a condition on the existence of an optimal admissible control for **Problem (P)**.

Remark

The result mainly relies on the interpretation of $\log h(t, \mathbf{x})$ *as a* **value function** *for a stochastic control problem associated with the distributed systems, which is amounted to changing the drift term by a certain perturbation suggested by Jamison in the context of reciprocal processes (cf. Proposition (2)).*

Specifically, we consider the following conditions:

- with a deterministic initial condition; and
- with a random initial condition.

Connection with stochastic control problems . . .

First, consider **Problem (P)** with a deterministic initial condition, i.e., when μ_0 assumes a Dirac measure that is concentrated at a point $\xi \in \mathbb{R}^{nd}$.

Proposition (3)

Suppose that μ_0 is a Dirac measure which is concentrated at a point $\xi \in \mathbb{R}^{nd}$. Further, assume that

$$H(\mu_T | \mathcal{S}_T \mu_0) < +\infty$$

and let $h(t, \mathbf{x})$ be given by

$$h(t, \mathbf{x}) = \int q(t, \mathbf{x}, T, \mathbf{z}) \log \frac{d\mu_T}{d\mathcal{S}_T \mu_0}(\mathbf{z}) d\mathbf{z}. \tag{4}$$

Then, $\mathbf{u}_t^* = a(t, \mathbf{x}_t) D_{x^1} \log h(t, \mathbf{x}_t)$ solves **Problem (P)** *with an optimal value of*

$$J(\mathbf{u}_t^*) = H(\mu_T | \mathcal{S}_T \mu_0).$$

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Remark

Note that the "energy" $J(\mathbf{u}_t) = \mathbb{E}\int_0^T \frac{1}{2}\|\mathbf{u}_t\|^2_{a^{-1}} dt$ has an interpretation in terms of the relative entropy.

Suppose that $P_{\mathbf{x}_t}$ and $P_{\mathbf{x}_t^u}$ are measures generated by $\mathbf{x}_t$ and $\mathbf{x}_t^u$ on the path-space $C^2([0,T];\mathbb{R}^{nd})$. Then, using Girsanov transformation, we have the following

$$
\begin{aligned}
H(P_{\mathbf{x}_t^u}|P_{\mathbf{x}_t}) &= \int \log \frac{dP_{\mathbf{x}_t^u}}{dP_{\mathbf{x}_t}} dP_{\mathbf{x}_t^u} \\
&= \mathbb{E}\left\{ \int_0^T \sigma^{-1}(t,\mathbf{x}_t^u)\mathbf{u}_t dW_t - \int_0^T \frac{1}{2}\|\mathbf{u}_t\|^2_{a^{-1}} dt \right\} \\
&\equiv J(\mathbf{u}_t).
\end{aligned}
$$

Connection with stochastic control problems . . .

Remark

Moreover, under the optimality condition, we have the following

$$H(P_{\mathbf{x}_t^u}|P_{\mathbf{x}_t}) = H(\mu_T|\mathcal{S}_T\nu_0)$$

that implies the **global relative entropy** *is exactly equal to the* **relative entropy** *between the final densities.*

Connection with stochastic control problems ...

Assume that ξ is distributed according to a measure μ_0. Note that, from Proposition (2), for μ_0 and μ_T (with $\mu_T \ll \mathcal{S}_T \mu_0$), there exist two σ-finite measures ν_0 and ν_T such that

$$\begin{aligned} \frac{d\mu_T}{d\lambda} &= \rho_T(\mathbf{x}) \int q(0, \mathbf{y}, T, \mathbf{x}) \nu_0(d\mathbf{y}) \\ &\triangleq \pi_T(\mathbf{x}) \end{aligned}$$

and

$$\frac{d\mu_0}{d\nu_0} = \int q(0, \mathbf{x}, T, \mathbf{z}) \rho_T(\mathbf{z}) d\mathbf{z},$$

where $\rho_T(\mathbf{x}_t) = d\nu_T / d\nu_0$.

Connection with stochastic control problems . . .

Then, for any initial random variable $\mathbf{x}_0 = \xi$ distributed according to ν_0 and satisfying $\mathbb{E}|\xi|^2 < +\infty$, we have the following result which is a generalization of Proposition (3).

Proposition (4)

Suppose that $H(\mu_T|\mathcal{S}_T\nu_0) < +\infty$, $\int (d\mu_0/d\nu_0) d\mu_0 < +\infty$. Let $h(t, \mathbf{x}_t)$ be given by

$$h(t, \mathbf{x}_t) = \int q(t, \mathbf{x}, T, \mathbf{z})\rho_T(\mathbf{z})d\mathbf{z}.$$

Then, $\mathbf{u}_t^* = a(t, \mathbf{x}_t)D_{x^1} \log h(t, \mathbf{x}_t)$ solves **Problem (P)** *with an optimal value of*

$$\begin{aligned} J(\mathbf{u}_t^*) &= \mathbb{E}\int_0^T \frac{1}{2}\|\mathbf{u}_t^*\|^2_{a^{-1}} dt \\ &= H(\mu_T|\mathcal{S}_T\nu_0) - H(\mu_0|\nu_0). \end{aligned}$$

Connection with stochastic control problems ...

Remark

Note that the conditions under which Proposition (4) holds are rather difficult to meet. However, when μ_0 has compact support, we can replace them with suitable conditions.

Proposition (5)

Suppose that μ_0 has compact support and $H(\mu_T|\mathcal{S}_T\mu_0) < +\infty$. Then, we have

$$H(\mu_T|\mathcal{S}_T\nu_0) < +\infty \quad \textit{and} \quad \int \frac{d\mu_0}{d\nu_0} d\mu_0 < +\infty.$$

On the invariance property of path-space measure

If $\mathbf{x}_t$ is a weak solution of (2) with $\mathbf{M}(t,\mathbf{x}) \in C_b^2([0,T] \times \mathbb{R}^{nd}; \mathbb{R}^{nd})$ and $\sigma(t,\mathbf{x}) \in C_b^2([0,T] \times \mathbb{R}^{nd}; \mathbb{R}^{d\times d})$. Then, we can consider determining an asymptotic estimate for the probability of a small tube around $C^2([0,T];\mathbb{R}^{nd})$-function.

Note that, for a given $\varphi \in C^2([0,T];\mathbb{R}^{nd})$ and small $\varepsilon > 0$, we have the following asymptotic estimate

$$\mathbb{P}\left\{\left\|\mathbf{x}. - \varphi\right\| < \varepsilon\right\} \approx \kappa_\varepsilon \exp\left\{-\int_0^T L(t,\varphi,\dot{\varphi})dt\right\},$$

where $L(t,\varphi,\dot{\varphi}) = \frac{1}{2}\left\|\mathbf{M}(t,\varphi) - \dot{\varphi}\right\|_{a^{-1}}^2$.

Moreover, such an asymptotic estimate justifies the definition of **most probable path** that minimizes the functional $\int_0^T L(t,\varphi,\dot{\varphi})dt$.

On the invariance property of path-space measure ...

Then, we the following result.

Proposition (6)

Assume $\mathbf{M}(t, \mathbf{x}_t) \in C_b^2([0, T] \times \mathbb{R}^{nd}; \mathbb{R}^{nd})$ *and* $\sigma(t, \mathbf{x}_t) \in C_b^2([0, T] \times \mathbb{R}^{nd}; \mathbb{R}^{d \times d})$*. Consider the following two diffusion processes* $\hat{\mathbf{x}}_t$ *and* $\tilde{\mathbf{x}}_t$ *with the same diffusion term* $\sigma(t, \mathbf{x})$ *and whose drifts are*

$$\mathbf{M}(t, \mathbf{x}) \quad \text{and} \quad \mathbf{M}(t, \mathbf{x}) + G\, a(t, \mathbf{x}_t) D_{x^1} \log h(t, \mathbf{x}),$$

respectively, where $h(t, \mathbf{x})$ *is a strictly positive function and a kernel of the operator* $\left(\partial/\partial t + \mathcal{L}_{t,\mathbf{x}}\right)$*. Then,* $\hat{\mathbf{x}}_t$ *and* $\tilde{\mathbf{x}}_t$ *have the same extremal trajectories.*

Further remarks

Is there any meaningful extension that can be included?

For example, we can include a state dependent term in the cost functional

$$J(\mathbf{x}_t^u, \mathbf{u}_t) = \mathbb{E} \int_0^T \left(\frac{1}{2} \|\mathbf{u}_t\|_{a^{-1}}^2 + V(\mathbf{x}_t^u) \right) dt, \quad V \geq 0.$$

Then, if we take $h(t, \mathbf{x}_t)$ in the kernel of the operator $\left(\partial/\partial t + \mathcal{L}_{t,\mathbf{x}} - V\right)$ and $q(s, \mathbf{y}, t, \mathbf{x})$ as the fundamental solution of $(\partial/\partial t) f + \mathcal{L}_{t,\mathbf{x}} f - V f = 0$.

Remark

Observe that $q(s, \mathbf{y}, t, \mathbf{x})$ is the transition kernel of the killed diffusion process with killing rate V.