Monotone performance measures based on reward-to-variability ratios

Mikhail Zhitlukhin

Steklov Mathematical Institute, Moscow, Russia

Gainesville, Novermber 9, 2015

Introduction

A performance measure assigns score $\rho(X)$ to future returns $X \in L^p$:

$$\rho\colon L^p \to \mathbb{R}$$

This talk is about

$$\text{performance} \;\sim\; \text{trade-off} \quad \frac{\text{profit}}{\text{uncertainty}}$$

Motivation – the Sharpe ratio

The Sharpe Ratio is a well-known performance measure:

$$\mathrm{S}(X) = \frac{\mathrm{E}X}{\sqrt{\mathrm{Var}\, X}} \qquad X \in L^2 \text{ – return over benchmark.}$$

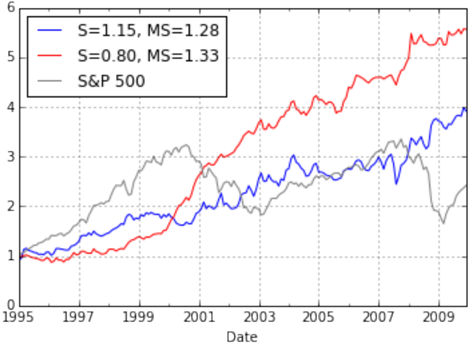
Advantages:

- Simple interpretation ($\sim t$-statistic)
- Easy to compute and optimize ($\sum_i \lambda_i X_i \to \max$)

Disadvantages:

- Not monotone: $X \geqslant Y \not\Rightarrow \mathrm{S}(X) \geqslant \mathrm{S}(Y)$
- Symmetric
- Not flexible: only $\mathrm{E}X$ and $\mathrm{Var}\, X$

Example – non-monotonicity of the Sharpe ratio



	Sharpe ratio	"Monotone" Sharpe ratio	Skew
Fund 1	0.80	1.35	1.28
Fund 2	1.15	1.28	0.43

Outline of the talk

1. Performance measures axioms, ideas to improve the Sharpe ratio

2. Main results: abstract reward-to-variability measures, their properties and representation theorems

3. Examples of new performance measures

4. Applications to data

Modifications of the Sharpe ratio

Two basic ideas to improve the Sharpe ratio:

- Monotone modification

$$\rho(X) = \sup_{Y \leqslant X} \frac{\mathrm{E} Y}{\sqrt{\operatorname{Var} Y}}$$

- Arbitrary measures of profit μ and uncertainty δ

$$\rho(X) = \sup_{Y \leqslant X} \frac{\mu(Y)}{\delta(Y)}$$

Axioms for performance measures

Cherny and Madan (2009, *Rev. Financ. Stud.*)

1. Quasi-concavity
   $\{X : \rho(X) \geqslant C\}$ is convex for any C
2. Upper semi-continuity
   $\{X : \rho(X) \geqslant C\}$ is closed for any C
3. Monotonicity
   $X \geqslant Y \implies \rho(X) \geqslant \rho(Y)$
4. Scale invariance
   $\rho(\lambda X) = \rho(X)$ for any $\lambda > 0$

Literature: other performance measures

Treynor ratio, Sortino ratio, Downside symmetric ratio, Omega measure, Gain-to-loss measure, Distortion measures, ...

Surveys:

- Le Sourd (2007) – 50 measures
- Cogneau & Hubner (2009) – 101 measure

This paper: have a simple interpretation & satisfy the axioms

Auxiliary objects: coherent utility and deviation measures

A coherent utility measure (measure of profit) is a functional $\mu\colon L^p \to \mathbb{R}$ which is

1. Concave
2. Positively homogeneous: $\mu(\lambda X) = \lambda\mu(X)$ for $\lambda \geqslant 0$
3. $\mu(C) = C$ for constants
4. Upper semi-continuous

Dual representation as "the worst scenario expectation":

$$\mu(X) = \inf_{Q\in\mathcal{Q}} \mathrm{E}[QX], \qquad \mathcal{Q} \subset L^q,\ \ \mathrm{E}Q = 1 \text{ for } Q \in \mathcal{Q}$$

(Artzner, Delbaen, Eber, Heath, 1997)

(Rockafellar, Uryasev, Zabarankin, 2002-2006)

A coherent deviation measure (measure of uncertainty) $\delta\colon L^p \to \mathbb{R}$

1. Convex
2. Positively homogeneous
3. $\mu(C) = 0$ for constants (and also $\mu(X) > 0$ for non-constants)
4. Lower semi-continuous

Dual representation:

$$\delta(X) = \inf_{R \in \mathcal{R}} \mathrm{E}[RX], \qquad \mathcal{R} \subset L^q,\ \ \mathrm{E}R = 0 \text{ for } R \in \mathcal{R}$$

Connection:

$$\mu(X) \text{ “=” } \mathrm{E}X - \lambda \cdot \delta(X) \qquad \text{profit = expectation - uncertainty}$$

Examples

1. The simplest measure of profit: the expectation

$$\mu(X) = \mathrm{E}X \qquad \mathcal{Q} = \{1\}$$

2. L^p deviation:

$$\delta(X) = \|X - \mathrm{E}X\|_p \qquad \mathcal{R} = \{R : \mathrm{E}R = 0,\ \|R\|_q = 1\}$$

3. Minus Average Value at Risk, $\lambda \in (0,1)$:

$$\mu(X) = -\mathsf{AVaR}_\lambda(X) \text{ ``$=$'' } \mathrm{E}(X \mid X \leqslant q_\lambda(X))$$
$$\mathcal{Q} = \{Q : Q \in [0, \lambda^{-1}], \mathrm{E}Q = 1\}$$

4. Worst-case return and deviation:

$$\mu(X) = \operatorname{ess\,inf}(X), \qquad \delta(X) = \mathrm{E}X - \operatorname{ess\,inf} X$$

5. Range

$$\delta(X) = \operatorname{ess\,sup}(X) - \operatorname{ess\,inf}(X)$$

6. AVaR range:

$$\delta(X) = AVaR_\lambda(X) - AVaR_\lambda(-X)$$

7. Log-exponential utility, for $\mathrm{E}X > 0$

$$\mu(X) = \mathrm{E}X \cdot \log \mathrm{E} \exp(X/\mathrm{E}X)$$

Main results

Definition of a monotone profit-to-uncertainty ratio:

$$\rho(X) = \sup_{Y \leqslant X} \frac{\mu(Y)}{\delta(Y)}, \qquad X \in L^p$$

Theorem 1. Properties of ρ:

1. the smallest monotone functional not less than the ratio μ/δ
2. quasi-concave
3. scale invariant
4. upper semi-continuous

A dual representation for ρ

Problem: $\rho(X)$ involves the double optimization problems:

$$\rho(X) = \sup_{Y \leqslant X} \inf_{Q,R} \frac{\mathrm{E}QY}{\mathrm{E}RY}$$

The next part:

1. A theorem reducing general $\rho(X)$ to a single optimization problem
2. Particular examples, where $\rho(X)$ reduces to optimization over $a \in R$

Theorem 2. A measure ρ can be represented in the form

$$\rho(X) = \inf_{R,Q} \left\{ \frac{\mathrm{E}QX}{\mathrm{E}RX} \;\middle|\; Q \cdot \mathrm{E}RX \geqslant R \cdot \mathrm{E}QX \text{ a.s.} \right\}$$

where $\inf$ is over $R \in \mathcal{R}_\delta$, $Q \in \mathcal{Q}_\mu$ satisfying the condition on the right.

Remark. If $\mu(\cdot) = \mathrm{E}(\cdot)$, then

$$\rho(X) = \mathrm{E}X \cdot \left(\sup_R \big\{ \mathrm{E}RX \;\big|\; \mathrm{E}RX \geqslant R \cdot \mathrm{E}X \text{ a.s.} \big\} \right)^{-1}$$

Technical assumptions needed for Theorem 2

A1 (Finiteness of μ, δ)

For $X \in L^p$, $p \in [1, \infty)$: $\mathcal{R}, \mathcal{Q}$ are bounded in L^q
For $X \in L^\infty$: $\mathcal{R}, \mathcal{Q}$ are uniformly integrable

A2 (Consistency of δ and L^p)

If $X \leqslant 0$ and $\sup_{R \in \mathcal{R}} \mathrm{E} RX < \infty$, then $X \in L^p$.

Examples

Example 1. Monotone Sharpe ratio

$$\mathrm{MS}(X) = \sup_{Y \leqslant X} \frac{\mathrm{E} Y}{\sqrt{\operatorname{Var} Y}}, \qquad X \in L^2$$

Representation

For any $X \in L^2$, $\mathrm{E}X > 0$, $\mathrm{P}(X < 0) > 0$ we have

$$\frac{1}{\mathrm{MS}(X)^2 + 1} = \inf_{x \geqslant 0} \mathrm{E}((1 - xX)^+)^2$$

Application to portfolio optimization

The problem of portfolio optimization

$$\mathrm{MS}\big(\textstyle\sum_i \lambda_i X_i\big) \to \max \qquad \text{over } \overline{\lambda} \in \Lambda$$

is equivalent to

$$\mathrm{E}\Big(\big(1 - x\textstyle\sum_i \lambda_i X_i\big)^+\Big)^2 \to \min \qquad \text{over } \overline{\lambda} \in \Lambda,\ x \geqslant 0$$

Equivalent representation

$$\mathrm{MS}(X) = \sqrt{\frac{x^*}{\mathrm{E}(x^* - X)^+} - 1}$$

where x^* is the unique root of the function

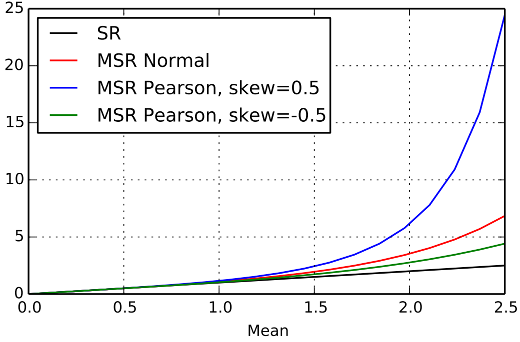
$$f(x) = \mathrm{E}(X \cdot (x - X)^+)$$

Properties of $g(x)$:

- $f(x)$ is continuous
- If X has density, then $f(x)$ is continuously differentiable
- If X is discrete, then $f(x)$ is piecewise linear

Behavior of MSR for skewed distributions

Example:
SR and MSR for Normal and Pearson distributions with variance 1.



Example 2. Sharpe ratio with L^p deviation

$$\mathrm{MS}_p(X) = \sup_{Y \leqslant X} \frac{\mathrm{E}Y}{\|Y - \mathrm{E}Y\|_{L_p}}$$

Representation

$$\frac{(\mathrm{MS}_p(X))^q}{q-1} = \max_{x,y \geqslant 0} \Big\{ p(y-1) - \mathrm{E}\big(|f(X,x,y)|^p - pf(X,x,y)\big) \Big\}$$

where

$$f(X,x,y) = 1 - (y - xX)^+$$

Example 3. AVaR deviation ratio

$$\rho(X) = \sup_{Y \leqslant X} \frac{\mathrm{E}Y}{\mathrm{E}Y + \mathsf{AVaR}_\lambda(Y)}$$

Representation

If $\mathsf{AVaR}(X) \geqslant 0$ then

$$\rho(X) = \frac{\mathrm{E}X}{\mathrm{E}X + \mathsf{AVaR}_\lambda(X)}$$

If $\mathsf{AVaR}(X) < 0$ then

$$\frac{\lambda}{(1-\lambda)\rho(X) + \lambda} = \min_x \mathrm{E}(1 + xX)^+$$

Applications to investment funds performance data

Normalization

To compare $\mathrm{MS}(X)$ with $\mathrm{S}(X)$, define

$$f := (x \mapsto \mathrm{MS}(N(x,1)))^{-1}$$

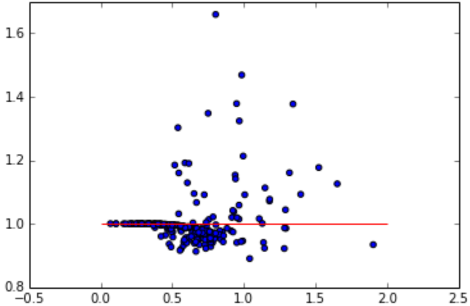
The normalized monotone Sharpe ratio:

$$\overline{\mathrm{MS}}(X) = f(\mathrm{MS}(X))$$

"SR of a Normal r.v. with the same MSR", $\qquad \overline{\mathrm{MS}}(N(\mu,\sigma^2)) = \dfrac{\mu}{\sigma}$

The Morningstar Database

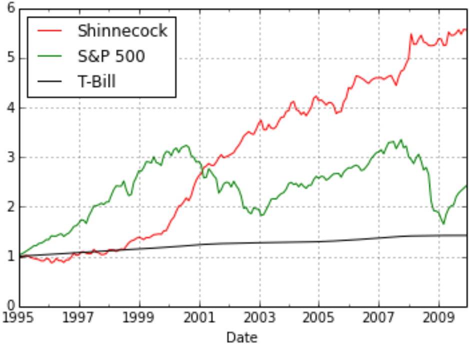
$\mathrm{S}(X)$ and $\overline{\mathrm{MS}}(X)$ for different funds in 1995–2009.



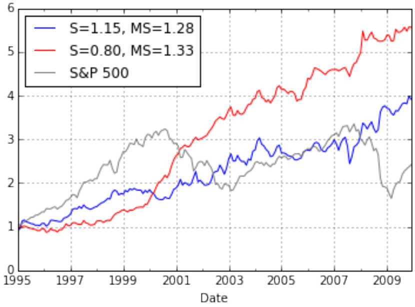
x-axis: $\mathrm{S}(X)$, $\quad$ y-axis: $\overline{\mathrm{MS}}(X)/\mathrm{S}(X)$

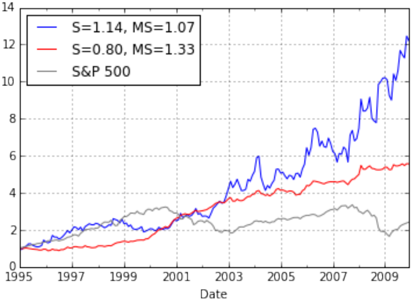
The highest $\overline{\mathrm{MS}}/\mathrm{S}$ ratio ("Shinnecock Futures Fund")

$\mathrm{S} = 0.80$, $\overline{\mathrm{MS}} = 1.33$



Comparison with some two other funds





Conclusion

- We proposed a new class of performance measures

$$\rho(X) = \sup_{Y \leqslant X} \frac{\mu(Y)}{\delta(Y)}$$

- They are monotone and satisfy additional “nice” properties
- The general dual representation theorem can be used to reduce to a simpler optimization problem
- Efficient representations are obtained for particular cases