CASH FLOW MATCHING WITH RISKS CONTROLLED BY bPOE AND CVaR

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Outline

1. Cash-Flow Matching: Many Periods and Rebalancing
2. Buffered Probability of Exceedance bPOE
3. Portfolio Optimization with bPOE and CVaR
4. bPOE Minimization => Convex Programming => LP
5. Case Study using Portfolio Safeguard (PSG)

Deterministic Cash Flow Matching Problem

$\{1, \ldots, M\}$ = set of considered bonds
$p_0 = (p_0^1, \ldots, p_0^M)$ = vector of initial prices of bonds
$x_0 = (x_0^1, \ldots, x_0^M)$ = vector of bond positions
$c_t = (c_t^1, \ldots, c_t^M)$ = cash flow vector at time t
l_t = liability at time t
$S_t(x_0)$ = shortage at time t

Minimizing initial cost s.t. constraints for shortages

$$\min_{x_0} \, (p_0, x_0)$$

subject to

$$S_t(x_0) = l_t - (c_t, x_0) \leq 0, \quad t = 1, \ldots, T,$$

$$x_0 \geq 0.$$

Stochastic Cash Flow Matching Problem

Iyengar and Ma (2009)

p_t = random price vector for bonds at time $t = 1, ..., T$
x_t = vector of bond positions purchased at time t
$c_{s,t}$ = c.f. vector at time t from bonds purchased at time s
$S_t(x_0, ..., x_t)$ = random shortage at time t:

$$S_t(x_0, ..., x_t) = (p_t, x_t) + l_t - \sum_{s=0}^{t-1}(c_{s,t}, x_s), \quad t = 1, ..., T.$$

$L(x_0, ..., x_T)$ = random maximum shortage (Loss) over all time periods:

$$L(x_0, ..., x_T) = \max_{0 \le t \le T} S_t(x_0, ..., x_t).$$

Stochastic Cash Flow Matching Problem

Iyengar and Ma (2009) - CVaR in Constraint with different α

$$\min_{x_0,\ldots,x_T} \ (p_0, x_0)$$

subject to

$$CVaR_\alpha(L(x_0, \ldots, x_T)) \leq 0,$$

$$x_t \geq 0, \quad t = 0, \ldots, T.$$

Problem statement with absract Risk function

$$\min_{x_0,\ldots,x_T} \ (p_0, x_0)$$

subject to

Risk controled by Risk function depending on Loss $L(x_0, \ldots, x_T)$

$$x_t \geq 0, \quad t = 0, \ldots, T.$$

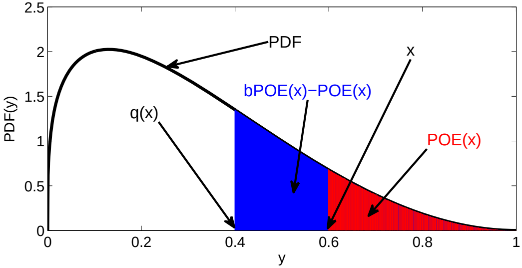
Buffered Probability of Exceedance (bPOE)

$\underline{\text{POE} = \ P(X > x)}$

Let $q(x)$ is defined by $E[X|X > q(x))] = x$

$\underline{\text{bPOE} = P(X > q(x))}$, "Buffer"$= x - q(x)$

$CVaR_\alpha(L) \geq VaR_\alpha(L); \quad bPOE_z(L) \geq POE_z(L)$



bPOE: (Mafusalov & U and Norton & U, 2014)

Lower bPOE

$$bPOE_z^-(L) = \begin{cases} 0, & \text{for } z \geq \sup L; \\ 1 - \{\alpha \,|\, CVaR_\alpha(L) = z\}, & \text{for } E[L] < z < \sup L; \\ 1, & \text{for } z \leq E[L]. \end{cases}$$

Upper bPOE

$$bPOE_z^+(L) = \begin{cases} 0, & \text{for } z > \sup L; \\ 1 - \{\alpha \,|\, CVaR_\alpha(L) = z\}, & \text{for } E[L] < z \leq \sup L; \\ 1, & \text{for } z \leq E[L]. \end{cases}$$

$$bPOE_z^+(L) = \min_{\lambda \geq 0} E[\lambda(L - z) + 1]^+ .$$

Example: CVaR and bPOE for 3 scenarios

We work with Upper bPOE and just write $bPOE_z(L)$

$$CVaR_\alpha(L) \geq VaR_\alpha(L); \quad bPOE_z(L) \geq POE_z(L)$$

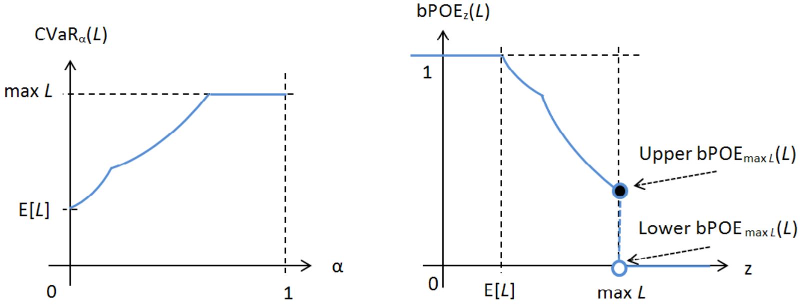


Figure: CVaR and bPOE for random Loss L with 3 scenarios.

CVaR vs bPOE in Constraint

$$\min_{x_0,\ldots,x_T} (p_0, x_0)$$

subject to

$$CVaR_\alpha(L) \leq z,$$

$$x_t \geq 0, \quad t = 0, \ldots, T.$$

Direct controlling probability of maximum shortage with bPOE

$$\min_{x_0,\ldots,x_T} (p_0, x_0)$$

subject to

$$bPOE_z(L) \leq 1 - \alpha,$$

$$x_t \geq 0, \quad t = 0, \ldots, T.$$

CVaR vs bPOE in Objective

$$\min_{x_0,\ldots,x_T} CVaR_\alpha(L)$$
$$\text{subject to}$$
$$(p_0, x_0) \leq d,$$
$$x_t \geq 0, \quad t = 0, \ldots, T.$$

Direct minimizing probability of maximum shortage with bPOE

$$\min_{x_0,\ldots,x_T} bPOE_z(L)$$
$$\text{subject to}$$
$$(p_0, x_0) \leq d,$$
$$x_t \geq 0, \quad t = 0, \ldots, T.$$

Equivalence of CVaR abd bPOE Constraints

Problems with CVaR and Lower bPOE constraints are equivalent.

Problems with CVaR and (Upper) bPOE constraints are "near" equivalent:

Constraint on bPOE implies constraint on CVaR.

$$bPOE_z(L) \leq 1-\alpha \quad => \quad CVaR_\alpha(L) \leq z$$

Constraint on CVaR implies constraint on bPOE.

Let $y > z$, then $CVaR_\alpha(L) \leq z \quad => \quad bPOE_y(L) < 1-\alpha$

bPOE Minimization => Convex Programming

$S_t = (a_t, x) + b_t$ = random function of decision vectors x

$$\min_x bPOE_z(L) = \min_x \min_{\lambda \geq 0} E[\lambda(\max_{0 \leq t \leq T} S_t - z) + 1]^+$$

$$= \min_{\lambda \geq 0, x} E[\lambda(\max_{0 \leq t \leq T} \{(a_t, x) + b_t\} - z) + 1]^+$$

$$= \min_{\lambda \geq 0, x} E[\max_{0 \leq t \leq T} \{(a_t, \lambda x) + \lambda(b_t - z)\} + 1]^+.$$

Change of variables: $(\lambda x, \lambda) \rightarrow (y, \lambda)$

$$\min_x bPOE_z(L) = \min_{\lambda \geq 0, y} E[\max_{0 \leq t \leq T} D_t(\lambda, y, z) + 1]^+,$$

$D_t(\lambda, y, z) = (a_t, y) + \lambda(b_t - z)$ = random linear function of vector y and variable λ with parameter z.

bPOE Minimization => Convex Programming

Change of variables: $(\lambda x, \lambda) \to (y, \lambda)$

$$S(x) = (a, x) + b \quad \longrightarrow \quad D(\lambda, y, c) = (a, y) + \lambda(b - c)$$

Constraints transformation

Linear: $C_l \le Ax \le C_u \longrightarrow C_l\lambda \le Ay$ and $Ay \le C_u\lambda$

Quadratic: $(x, Qx) \le c \longrightarrow (y, Qy) \le c\lambda^2$

Cvar: $CVaR_\alpha(S(x)) \le c \longrightarrow CVaR_\alpha(D(\lambda, y, 0)) \le c\lambda$

Partial Moment: $PM_z(S(x)) \le c \longrightarrow PM_0(D(\lambda, y, z)) \le c\lambda$

Direct minimizing bPOE in new space using LP

$$\min_{\lambda \ge 0,\, y} E[\max_{0 \le t \le T} D_t(\lambda, y, z) + 1]^+ = \min_{\lambda, y} PM_{-1}(\max_{0 \le t \le T} D_t(\lambda, y, z))$$

$$(p_0, y_0) - d\lambda \le 0,$$

$$\lambda \ge 0, \quad y_t \ge 0, \; t = 0, \ldots, T.$$

bPOE Minimization => Linear Programming

LP with additional variables $u_k, k = 1, ..., K$ corresponding to scenarios of $L(x_0, ..., x_T)$:

$$\min_{\lambda, y, u} \sum_{k=1}^{K} p_k u_k$$

subject to

$$u_k \geq (a_t^k, y_t) + \lambda(b_t^k - z) + 1, \; t = 0, ..., T, \; k = 1, ..., K,$$

$$(p_0, y_0) - d\lambda \leq 0,$$

$$\lambda \geq 0, \;\; y_t \geq 0, \; t = 0, ..., T, \; u_k \geq 0, \; k = 1, ..., K\,.$$

Restoring x_t: If $\lambda^* > 0$ then $x_t^* = y_t^* / \lambda^*$;
If $\lambda^* = 0$ then $bPOE_z(L) = E[\lambda^*(L - z) + 1]^+ = 1$ for any feasible $x_0, ..., x_T$.

Case study: Data and Problems Size

$T = 120$ time periods with 0.5 year length;
$K = 200$ simulations (scenarios) of bond prices;
$M = 11$ number of available bonds;
Stream of liabilities (like in: Iyengar and Ma (2009))

$$l_t = \begin{cases} 100 & \text{if } t/2 = 0, ..., 10, \\ 110 - 2.2 \times (\frac{t}{2} - 10) & \text{if } t/2 = 11, ..., 60, \\ 0 & \text{otherwise.} \end{cases}$$

1331 = number of variables in optimization problems;
24201 = number of constraints in LP problem.

Case study: Details of Treasury Bonds

Bond	Name	Maturity	Coupon Rate (%)	Current Price
1	T-bill	0.5	0	95.8561
2	T-note	1	4.5	96.1385
3	T-note	2	4.5	92.6873
4	T-note	3	4.5	89.5784
5	T-note	4	4.5	86.7610
6	T-note	5	4.5	84.1959
7	T-bond	10	5.0	77.5948
8	T-bond	15	5.0	71.9232
9	T-bond	20	5.0	68.1357
10	T-bond	25	5.0	65.5990
11	T-bond	30	5.0	63.8989

Optimization using Portfolio Safeguard (PSG)

$$\min_{x_0,\dots,x_T} bPOE_z(L)$$

$$\text{subject to}$$

$$(p_0, x_0) \le d,$$

$$x_t \ge 0, \quad t = 0, \dots, T.$$

Problem statement in PSG

minimize

linearize =1

bPOE(0,Lmax(matrix_1,...,matrix_120))

Constraint: <= 1172.368

linear(matrix_0)

Box: >= 0

Solver: cargrb

Solution obtained by Portfolio Safeguard (PSG)

Solution_problem_cash_flow_matching_3.txt

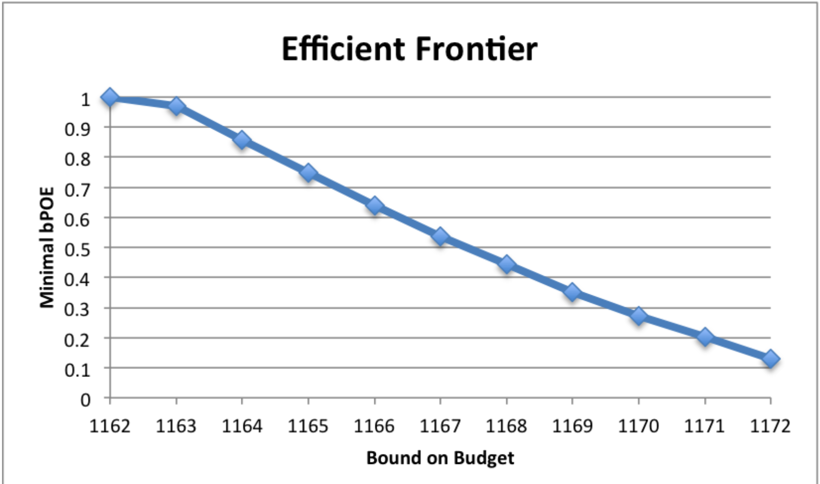
Problem: solution_status = optimal
Variables: optimal_point = point_problem_cash_flow_matching_3
Objective: = 0.100000000062
Constraint: constraint_1 = 0.0 [0.0]
Function: bpoe(0,lmax(matrix_1,...,matrix_120)) = 0.1000000000053
Function: linear(matrix_0) = 1172.368

Point_problem_cash_flow_matching_3.txt

Component_name	Value
x0_1	0.000000000000e+00
x0_2	1.837109350078e-01
x0_3	2.078663430491e-01
x0_4	2.272771609577e-01
....	
x120_10	0.000000000000e+00
x120_11	0.000000000000e+00

Efficient frontier

Four optimization problems generate "almost" the same efficient frontiers for $bPOE_0(L(x_0, ..., x_T)$ and (p_0, x_0).



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