Multidimensional Buffered Probability of Exceedance: Dual Representations and Multivariate Stochastic Dominance

Alexander Mafusalov

University of Florida,
Industrial and Systems Engineering department

November 2015

Outline

1. Conservative distribution approximations
2. Buffered probability of exceedance (bPOE)
3. Multidimensional bPOE (M-bPOE)
4. (Multivariate) conservative distribution approximations

Problem Description

- Sample distribution of deviations from a desired value.
- Robust approximation is needed (preferences: continuous/grid, closed form).
- Distribution is one of the many in a system. Deviations from central values add up. Risk for the system.
- Conservative approximation needed: risks of high deviations must not be underestimated.

Optimization Problem

$$
\begin{aligned}
\max_{Y} \quad & \mathcal{H}(Y) \\
s.t. \quad & X \preceq'_2 Y, -X \preceq'_2 -Y, \\
& \sigma^2(Y) \le \sigma^2(X) + \Delta\sigma^2.
\end{aligned}
$$

1. Why Second-Order Stochastic Dominance?

   Risk averse decision maker

2. Why Entropy Maximization?

   Smooth and robust solution

3. Why Variance Constraint?

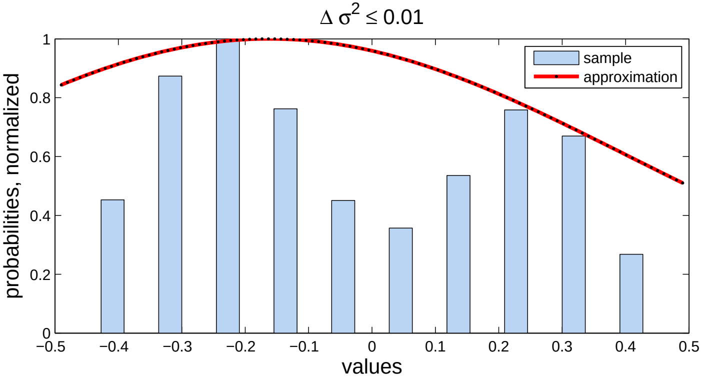
   Bounded feasible set

Method properties

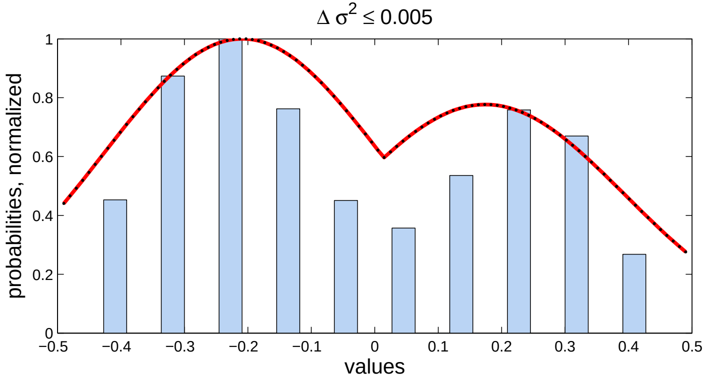
- Discussed dominance constraint guarantees "fatter" right and left tails compared to sample X
- When $\Delta\sigma^2 \to 0$, then optimal solution converges in distribution to the sample distribution
- Optimal solution is a maximum-of-Gaussians distribution

$$P(Y) = \max_{i=1,\ldots,m} C_i \exp\{-(Y-\mu_i)^2/2\sigma^2\}$$

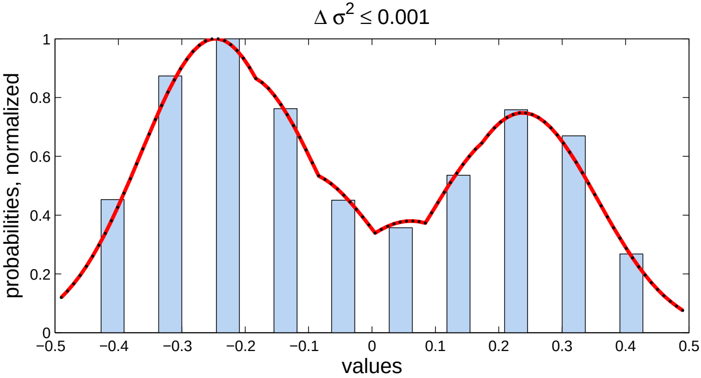
Convergence Example 1: $\Delta\sigma^2 \leq 0.01$



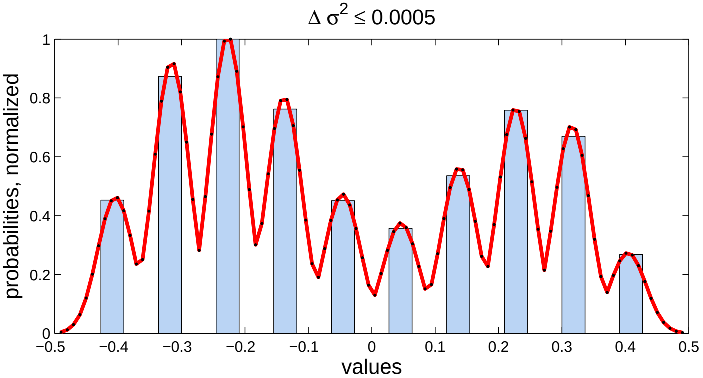
Convergence Example 2: $\Delta\sigma^2 \leq 0.005$



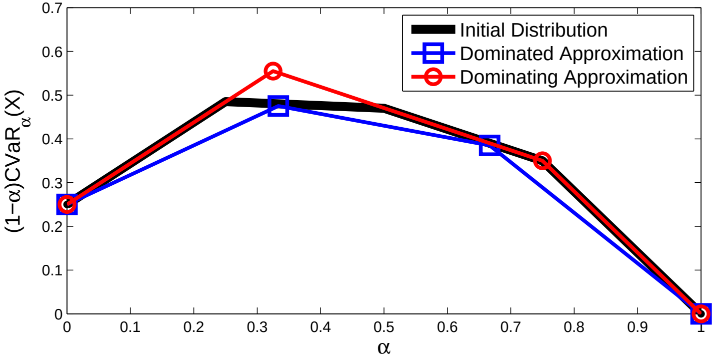
Convergence Example 3: $\Delta\sigma^2 \leq 0.001$



Convergence Example 4: $\Delta\sigma^2 \leq 0.0005$



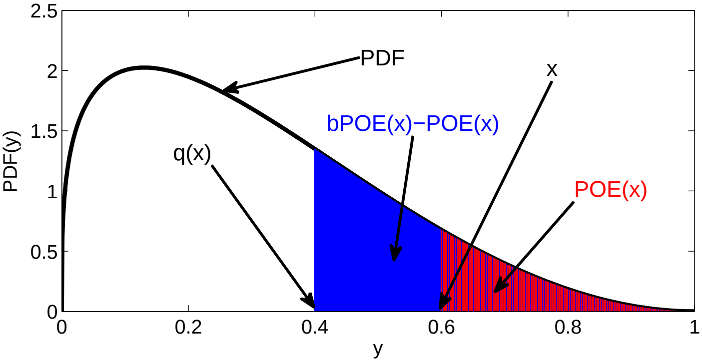
For discrete distribution the scaled superquantile is
$$(1-\alpha_i)\bar{Q}_{\alpha_i}(X) = \sum_{j=i+1}^{m} p_i x_i,\ \alpha_0 = 0,\ \alpha_i = \sum_{j=1}^{i} p_i$$



$\bar{Q}_\alpha(Y) \geq \bar{Q}_\alpha(X)$ for $\alpha = \alpha_i$ is sufficient for SOSD

bPOE explanation (continuous case)

$P(X > x)$: probability of exceedance
$P(X > q(x))$: *buffered* probability of exceedance
$q(x) : E[X|X > q(x)] = x$, $x - q(x)$: "buffer"



Buffered Probability of Exceedance

$$P(X \geq x) \equiv p_x(X) \leq \bar{p}_x(X)$$

$\bar{p}_x(X)$ is the only smallest **quasi-convex** and **law-invariant** upper bound for $p_x(X)$

$$\bar{p}_x(X) = \inf_{a \geq 0} E[a(X - x) + 1]^+$$

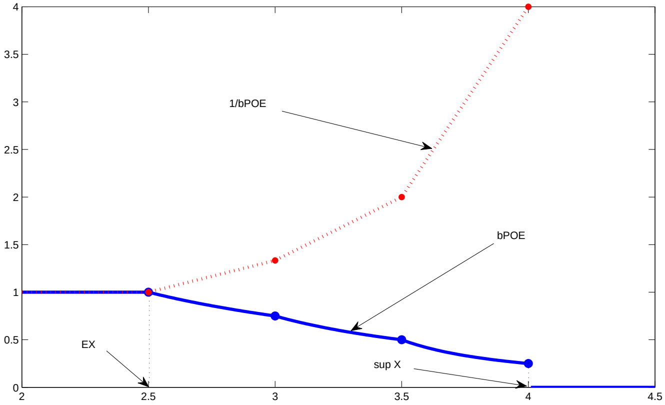
where $[x]^+ = \max\{0, x\}$

$$\bar{p}_x(X) = 1 \text{ for } x < EX \qquad \bar{p}_x(X) = 0 \text{ for } x > \sup X$$

$\bar{p}_x(X)$ is decreasing and continuous for $x \in [EX, \sup X)$

bPOE Properties w.r.t. Parameter x

$1/\bar{p}_x(X)$ is a convex nondecreasing function of x and piecewise-linear if X is discretely distributed



Primal and Dual formulations for bPOE

Primal

$$\bar{p}_x(X) = \min_{a \geq 0} E[a(X - x) + 1]^+$$

Dual

$$\begin{aligned} \bar{p}_x(X) = \max \quad & EW \\ s.t. \quad & EXW \geq xEW \\ & 0 \leq W \leq 1 \end{aligned}$$

Dual Multivariate

$$\begin{aligned} \bar{p}_{\mathbf{x}}(\mathbf{X}) = \max \quad & EW \\ s.t. \quad & EX_iW \geq x_iEW, \quad i = 1, \ldots, d \\ & 0 \leq W \leq 1 \end{aligned}$$

Multidimensional bPOE

Primal Multivariate

$$\bar{p}_{\mathbf{x}}(\mathbf{X}) = \min_{\mathbf{a} \geq 0} E[\mathbf{a}^T(\mathbf{X} - \mathbf{x}) + 1]^+$$

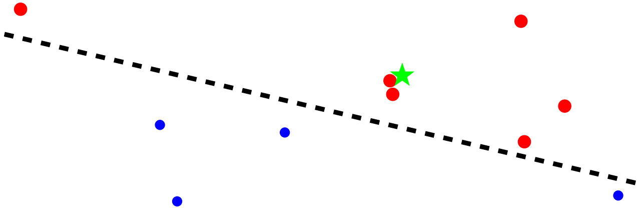
Suppose that instead of random variable X there is a random vector $\mathbf{X} = (X_1, \ldots, X_n)$.
If the law g for aggregating components of $\mathbf{X}$ into a value $g(\mathbf{X})$, determining undesired events, is known, then the problem is reduced to bPOE framework.
What if there is no known law? Let us quantify risks as follows:

$$F_{-\mathbf{X}}(-\mathbf{x}) = P(X_1 \geq x_1 \text{ and } \ldots \text{ and } X_n \geq x_n) \equiv p_{\mathbf{x}}(\mathbf{X}) \leq \bar{p}_{\mathbf{x}}(\mathbf{X})$$

Lift Zonoid (by Koshevoy & Mosler)

$$Z(\mathbf{X}) = \{(p, \mathbf{y}) \in \mathbb{R}^{d+1} | p = EW, y_i = EWX_i, 0 \le W \le 1\}$$

$Z(\mathbf{X})$ uniquely determines distribution of $\mathbf{X}$;
$Z(\mathbf{X})$ is a convex set; $(0, \mathbf{0}) \in Z(\mathbf{X})$; $(1, E\mathbf{X}) \in Z(\mathbf{X})$;
if $E|\mathbf{X}| < \infty$, then $Z(\mathbf{X})$ is compact;
if $\mathbf{X}$ is discretely distributed, then $Z(\mathbf{X})$ is polyhedral
extreme points $\leftarrow$ expectations within linear cut-offs



Lift Zonoid Transformation

Homography $(p, \mathbf{u}) \to (1/p, \mathbf{u}/p)$ applied to $Z(\mathbf{X})$
line $\to$ line & continuous in $\mathbb{R}_{++} \times \mathbb{R}^d|$ $\Rightarrow$ preserves convexity
Thransforms $Z(\mathbf{X})$ into $epi(1/f_{\mathsf{X}}(\mathsf{x}))$, where

$$f_{\mathsf{X}}(\mathsf{x}) = \sup_{W} \; EW \quad \text{s.t.} \quad EW\mathbf{X} = \mathsf{x}EW, \quad 0 \le W \le 1 \qquad = \inf_{\mathbf{a}} \; E[\mathsf{a}^T(\mathbf{X} - \mathsf{x}) + 1]^+$$

$1/f_{\mathsf{X}}(\mathsf{x})$ is convex, piecewise-linear, with knots of type

$$\bar{\mathsf{x}}(\lambda, b) = \sum_{\lambda^T \mathbf{x}^i \ge b} \mathsf{x}^i p^i \Big/ \sum_{\lambda^T \mathbf{x}^i \ge b} p^i, \qquad \mathcal{X}_{\mathsf{X}} := \bigcup_{\lambda, b} \bar{\mathsf{x}}(\lambda, b)$$

Stochastic Dominance Equivalence

Zonoid Dominance (Mosler & Koshevoy):

$\mathbf{X} \leq_Z \mathbf{Y} \Leftrightarrow Z(\mathbf{X}) \subseteq Z(\mathbf{Y})$

Hence, $\mathbf{X} \leq_Z \mathbf{Y} \Leftrightarrow f_{\mathbf{X}}(\mathbf{z}) \leq f_{\mathbf{Y}}(\mathbf{z})$ for $\mathbf{z} \in \mathcal{X}_{\mathbf{X}}$

$\mathbf{X} \leq_Z \mathbf{Y}$ constraint as a finite number of linear constraints!

Linear Second Order Dominance (Dentcheva & Ruszczynski):

$\mathbf{X} \leq_2^{\text{lin}} \mathbf{Y} \Leftrightarrow \mu^T\mathbf{X} \leq_2 \mu^T\mathbf{Y}$ for all $\mu \geq 0$

$\mathbf{X} \leq_2^{\text{lin}} \mathbf{Y} \Leftrightarrow \bar{p}_{\mathbf{z}}(\mathbf{X}) \leq \bar{p}_{\mathbf{z}}(\mathbf{Y})$ for $\mathbf{z} \in \mathcal{X}_{\mathbf{X}}^+ := \cup_{\mu \geq 0, b}\ \bar{\mathbf{x}}(\mu, b)$

$\mathbf{X} \leq_2^{\text{lin}} \mathbf{Y}$ constraint as a finite number of linear constraints!

$\mathbf{X} \leq_Z \mathbf{Y}$ and $\mathbf{X} \leq_2^{\text{lin}} \mathbf{Y}$ are closely related as $f_{\mathbf{X}}(\mathrm{x})$ and $\bar{p}_{\mathrm{x}}(\mathbf{X})$:

$$f_{\mathbf{X}}(\mathrm{x}) = \inf_{\mathbf{a} \in \mathbb{R}^d} E[\mathrm{a}^T(\mathbf{X} - \mathrm{x}) + 1]^+ \qquad \bar{p}_{\mathrm{x}}(\mathbf{X}) = \inf_{\mathbf{a} \geq 0} E[\mathrm{a}^T(\mathbf{X} - \mathrm{x}) + 1]^+$$

Optimization problem formulation

Take $\gamma > 1$ (analogue of $\Delta\sigma^2$ in 1-dimensional version)

$$\begin{aligned} \max_{\mathbf{Y}} \quad & \mathcal{H}(\mathbf{Y}) \\ s.t. \quad & \mathbf{X} \leq_z \mathbf{Y} \qquad (\Leftrightarrow -\mathbf{X} \leq_z -\mathbf{Y}) \\ & \sigma^2(\mu^T \mathbf{Y}) \leq \gamma \cdot \sigma^2(\mu^T \mathbf{X}) \quad \text{for all } \mu \end{aligned}$$

Dominance constraint $\Leftrightarrow$ for all $\bar{\mathbf{x}}^i \in \mathcal{X}_{\mathbf{X}}$:

$$\begin{aligned} E_{V^i} 1 &\geq f_{\mathbf{X}}(\bar{\mathbf{x}}^i) \\ E_{V^i} \mathbf{Y} &= \bar{\mathbf{x}}^i E_{V^i} 1 \\ 0 \leq V^i &\leq P \end{aligned}$$

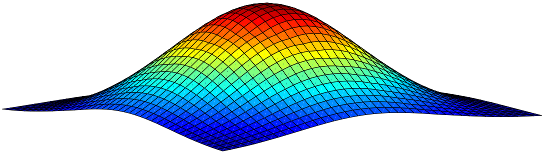
Variance constraint $\Leftrightarrow$ ($\Lambda \succ 0 \leftrightarrow \Lambda$ — PSD)

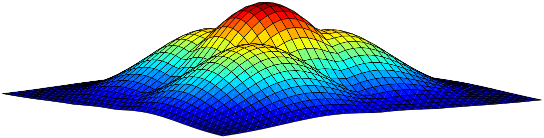
$$E_P \mathbf{Y}\mathbf{Y}^T - \gamma E\mathbf{X}\mathbf{X}^T + (\gamma - 1) E\mathbf{X} E\mathbf{X}^T \succ 0$$

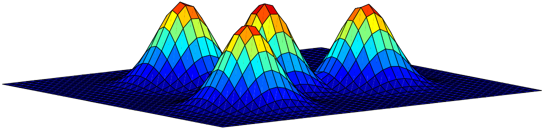
Maximum of Gaussians form of optimal solution

With $\gamma \to 1$, optimal solution $\to_F$ sample distribution
Optimal solution is a weighted maximum of Gaussian functions: $\Lambda \succ 0, \;\; \mu_\alpha \geq 0$

$$P(Y) = \exp\left\{ -\mathbf{Y}^T \Lambda \mathbf{Y} - \lambda_0^T \mathbf{Y} - \lambda_1 + \sum_i [\mu_\alpha^i + (\mathbf{Y} - \bar{\mathbf{x}}^i)^T \lambda_x^i]^+ \right\}$$







Thank you!