# Risk-averse optimization via multivariate stochastic order constraints

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# Outline

# Motivation

## Risk-Averse Optimization Models

Choose a decision $z \in Z$, which results in a random outcome $G(z) \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathbb{P})$ with “good" characteristics; special attention to low probability-high impact events.

- Utility models apply a nonlinear transformation to the realizations of $G(z)$ (expected utility) or to the probability of events (rank dependent utility/distortion). Expected utility models optimize $\mathbb{E}[u(G(z))]$
- Probabilistic / chance constraints impose prescribed probability on some events: $P[G(z) \geq \eta]$
- Mean–risk models optimize a composite objective of the expected performance and a scalar measure of undesirable realizations $\mathbb{E}[G(z)] - \varrho[G(z)]$ (risk/ deviation measures)
- Stochastic-order constraints compare random outcomes using stochastic orders and random benchmarks

# Integral Univariate Stochastic Orders

For $X, Y \in \mathcal{L}_1(\Omega, \mathcal{F}, P)$

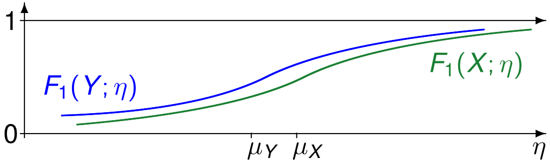
$$X \succeq_{\mathcal{U}} Y \Leftrightarrow \int_\Omega u(X(\omega))\, P(d\omega) \geq \int_\Omega u(Y(\omega))\, P(d\omega) \quad \forall\, u(\cdot) \in \mathcal{U}$$

Collection of functions $\mathcal{U}$ is the generator of the order.

**Generators:**

- $\mathcal{U}_1 = \{\text{nondecreasing functions } u : \mathbb{R} \to \mathbb{R}\}$ generates the usual stochastic order or first order stochastic dominance ($X \succeq_{(1)} Y$)
  Mann and Whitney (1947), Blackwell (1953), Lehmann (1955)
- $\mathcal{U}_2 = \{\text{nondecreasing concave } u : \mathbb{R} \to \mathbb{R}\}$ generates the second order stochastic dominance relation ($X \succeq_{(2)} Y$)
  Quirk and Saposnik (1962), Fishburn (1964), Hadar and Russell (1969)
- $\bar{\mathcal{U}}_2 = \{\text{nondecreasing convex } u : \mathbb{R} \to \mathbb{R}\}$ generates the increasing convex order ($X \preceq_{\text{ic}} Y$)

# First Order Stochastic Dominance




Distribution function $F(X;\eta) = \int_{-\infty}^{\eta} P_X(dt) = P\{X \le \eta\}$, $\eta \in \mathbb{R}$

Quantile function $F^{(-1)}(X;p) = \inf\{\eta : F(X;\eta) \ge p\}$, $p \in (0,1)$

Survival function $\bar{F}(X;\eta) = 1 - F(X;\eta) = P\{X > \eta\}$, $\eta \in \mathbb{R}$

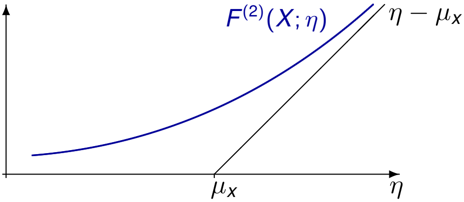
### The usual stochastic order

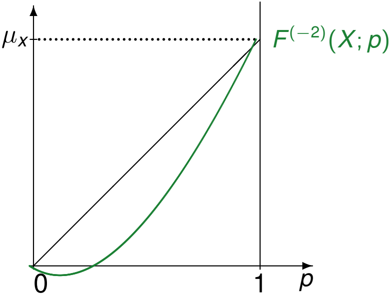
$$
\begin{aligned}
X \succeq_{(1)} Y \quad &\Leftrightarrow \quad F(X;\eta) \le F(Y;\eta) \quad \text{for all } \eta \in \mathbb{R}\\
&\Leftrightarrow \quad F^{(-1)}(X;p) \ge F^{(-1)}(Y;p) \quad \text{for all} \quad 0 < p < 1.\\
&\Leftrightarrow \quad \bar{F}(X;\eta) \ge \bar{F}(Y;\eta) \quad \text{for all} \quad \eta \in \mathbb{R}.
\end{aligned}
$$

# Second-Order Stochastic Dominance

Shortfall function $F^{(2)}(X,\eta) = \int_{-\infty}^{\eta} F(X,t)\,dt = \mathbb{E}[(\eta - X)_+] \quad \eta \in \mathbb{R}.$

Lorenz function: $F^{(-2)}(X;p) = \int_0^p F^{(-1)}(X;t)\,dt \quad p \in (0,1].$



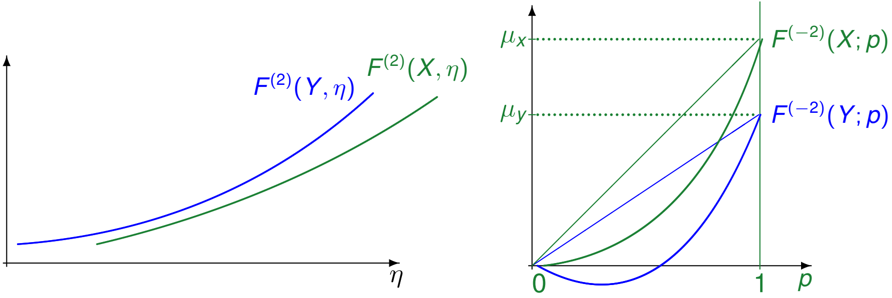

Fenchel conjugate function $\quad F^*(p) = \sup_u \{pu - F(u)\}.$

**Ogryczak-Ruszczyński 2002**

$F^{(-2)}(X;\cdot) = [F^{(2)}(X;\cdot)]^* \quad \text{and} \quad F^{(2)}(X;\cdot) = [F^{(-2)}(X;\cdot)]^*$

# Second-Order Stochastic Dominance

$$
\begin{aligned}
X \succeq_{(2)} Y &\Leftrightarrow \mathbb{E}[(\eta - X)_+] \le \mathbb{E}[(\eta - Y)_+] \\
&\Leftrightarrow F^{(-2)}(X;p) \ge F^{(-2)}(Y;p) \quad \forall p \in [0,1].
\end{aligned}
$$



# Increasing convex order

### Characterization by the integrated survival function

For $X, Y \in \mathcal{L}_1$, the relation $X \preceq_{\text{ic}} Y$ holds if and only if

$$\int_\eta^\infty P(X > t)\, dt \le \int_\eta^\infty P(Y > t)\, dt \quad \text{for all } \eta \in \mathbb{R}.$$

### The excess function and its Fenchel conjugate

$$H(Z,\eta) = \int_\eta^\infty \overline{F}(Z,t)\, dt = \mathbb{E}(Z-\eta)_+$$

$$L(Z,q) = -\int_{1+q}^1 F^{(-1)}(Z,t)\, dt \quad \text{for } -1 \le q < 0,$$

$$L(Z,0) = 0,\ L(Z,q) = \infty \text{ for } q \notin [-1,0]$$

### Increasing convex order vs. Second order dominance

$$X \preceq_{\text{ic}} Y \quad \Leftrightarrow \quad -X \succeq_{(2)} -Y.$$

# Multivariate Orders

Consider $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_m)$ in $\mathcal{L}_1^m(\Omega, \mathcal{F}, P)$.

**Definition**

Given a closed convex set $C \in \mathbb{R}_+^m$ and a mapping $\mathfrak{M} : c \mapsto \mathcal{U}_c$, $c \in C$, where $\mathcal{U}_c$ are univariate generators, a random vector $X \in \mathcal{L}_1^m$ is stochastically larger than a random vector $Y \in \mathcal{L}_1^m$ with respect to $\mathfrak{M}$ and $C$ if

$$c^\top X \succeq_{\mathcal{U}_c} c^\top Y \text{ for all } c \in C.$$

**Example**

- Set $\mathfrak{M}(c) \subset \mathbb{R}$ and $S = \{c \in \mathbb{R}_+^m : \|c\|_1 = 1\}$. For $X, Y \in \mathcal{L}_1^m$,

$$X \succeq_{(2)}^{\mathfrak{M}} Y \Leftrightarrow \mathbb{E}[(c^\top X - \eta)_+] \leq \mathbb{E}[(c^\top Y - \eta)_+] \ \ \forall (c, \eta) \in \operatorname{graph} \mathfrak{M}.$$

- If $\mathfrak{M}(c) = [a, b]$, the order is known as linear second order dominance.

Other definitions: A. Müller, D. Stoyan, Homem-de-Mello and Mehrotra: linear dominance with $C$ a polyhedron, or a compact convex set.

# Multivariate Stochastic Dominance: Generator of the order

The set $\Psi(\mathfrak{M})$ contains all mappings $\phi : c \in C \mapsto \mathcal{U}_2(\mathfrak{M}(c))$ such that $(c, x) \to [\phi(c)](c^\top x)$ is Lebesgue measurable on $C \times \mathbb{R}^m$.

$\mathcal{M}(C)$ is the space of regular countably additive measures on $C$ with finite variation; $\mathcal{M}_+(C)$ is its subset of nonnegative measures.

With every mapping $\phi \in \Psi(\mathfrak{M})$ and every finite measure $\mu \in \mathcal{M}_+(C)$ we associate a function $\varphi_{\phi,\mu} : \mathbb{R}^m \to \mathbb{R}$ as follows:

$$\varphi_{\phi,\mu}(x) = \int_C [\phi(c)](c^\top x)\, \mu(dc).$$

Define $\mathcal{U}^m_{\mathfrak{M}} = \{\varphi_{\phi,\mu} : \phi \in \Psi(\mathfrak{M}), \mu \in \mathcal{M}_+(C)\}$.

## Theorem

For each $X, Y \in \mathcal{L}^m_1$ the relation $X \succeq^{\mathfrak{M}}_{(2)} Y$ is equivalent to

$$\mathbb{E}[\varphi(X)] \geq \mathbb{E}[\varphi(Y)] \quad \text{for all } \varphi \in U^m_{\mathfrak{M}}.$$

Moreover, $\mathcal{U}^m_{\mathfrak{M}}$ is the maximal generator of the order $\succeq^{\mathfrak{M}}_{(2)}$.

# Rank Dependent Utility Functions/Distortions

$\mathcal{W}_1 = \{w : [0,1] \to \mathbb{R} : \ w \text{ is continuous nondecreasing}\}$.
$\mathcal{W}_2 = \{w \in \mathcal{W}_1 : w \text{ concave subdifferentiable at } 0\}$.

**Theorem** [DD, A. Ruszczyński, 2006]

For all random variables $X, Y \in \mathcal{L}_1$, the relation $X \succeq_{(i)} Y$, $i = 1, 2$ holds if and only if for all $w \in \mathcal{W}_i$

$$\int_0^1 F^{(-1)}(X;p)\, dw(p) \geq \int_0^1 F^{(-1)}(Y;p)\, dw(p). \tag{1}$$

**Corollary:** $X \preceq_{\text{ic}} Y \ \Leftrightarrow$ (1) for all convex functions $w$.

Let $\mathfrak{M}^- : C \rightrightarrows (0,1)$ have closed convex images.

**Multivariate distortions**

For $X, Y \in \mathcal{L}_1^m$, the relation $X \succeq_{(i)} Y$, $i = 1, 2$ holds if and only if for all measurable $\vartheta : c \in C \to \mathcal{W}_i(\mathfrak{M}^-(c))$ and all $\mu \in \mathcal{M}_+(C)$

$$\int_C \int_0^1 F^{(-1)}(c^\top X;p)\, d\vartheta_c(p)\, d\mu(c) \geq \int_C \int_0^1 F^{(-1)}(c^\top Y;p)\, d\vartheta_c(p)\, d\mu(c) \tag{2}$$

# Dominance Relation as Constraints in Optimization

$$\begin{aligned} &\min f(z) \\ &\text{subject to } G_i(z) \succeq_{\mathcal{U}(i)} Y_i, \quad i = 1..m \\ &z \in Z \end{aligned}$$

$Y_i$ - benchmark random outcome

**The dominance constraints reflect risk aversion**

$G_i(z)$ is preferred over $Y_i$ by all risk-averse decision makers with utility functions in the generator $\mathcal{U}(i)$.

$G(z) \succeq_{(1)} Y \equiv$ continuum of chance constraints;
$G(z) \succeq_{(2)} Y \equiv$ continuum of AV@R constraints.

Introduced by Dentcheva and Ruszczyński (2003)

# Second Order Dominance Constraints

Given $Y \in \mathcal{L}_1^m$ - benchmark random vector
Let $\mathfrak{M} : C \rightrightarrows \mathbb{R}$ and $\mathfrak{M}^- : C \rightrightarrows (0,1)$ have compact convex images.

### Direct Stochastic Order Constraints

$$
(\mathscr{P}) \qquad \begin{aligned} & \min f(z) \\ & \text{s. t. } \mathbb{E}[(\eta - c^\top G(z))_+] \le \mathbb{E}[(\eta - c^\top Y)_+] \\ & \qquad \forall\, (c,\eta) \in \operatorname{graph} \mathfrak{M}, \\ & z \in Z. \end{aligned}
$$

### Inverse Stochastic Order Constraints

$$
(\mathscr{Q}) \qquad \begin{aligned} & \min f(z) \\ & \text{s. t. } F^{(-2)}(c^\top G(z); p) \ge F^{(-2)}(c^\top Y; p) \\ & \qquad \forall\, (c,p) \in \operatorname{graph} \mathfrak{M}^-, \\ & z \in Z. \end{aligned}
$$

$Z$ is a closed subset of a Banach space $\mathcal{Z}$; $\mathfrak{M}(c)$, resp. $\mathfrak{M}^-(c)$ are compact, $G : \mathcal{Z} \to \mathcal{L}_1^m$ is continuous and for $P$-almost all $\omega \in \Omega$ the functions $[G_i(\cdot)](\omega)$ are concave and continuous. $f : \mathcal{Z} \to \mathbb{R}$ is convex and continuous.

# Optimality Conditions Using von Neumann Utility Functions

**The Lagrangian-like functional** $L : \mathcal{Z} \times \mathcal{U}^m_{\mathfrak{M}} \to \mathbb{R}$

$$L(z, u) = f(z) + \mathbb{E}\big[u(Y) - u(G(z))\big]$$

**Uniform Dominance Condition (UDC) for problem** $(\mathscr{P})$

$$\exists \tilde{z} \in Z : \inf_{(\eta,c)\in \operatorname{graph} \mathfrak{M}} \left\{ F^{(2)}(c^\top Y; \eta) - F^{(2)}(c^\top G(\tilde{z}); \eta) \right\} > 0.$$

**Theorem**

Assume UDC. If $\hat{z}$ is an optimal solution of $(\mathscr{P})$ then $\hat{u} \in \mathcal{U}^m_{\mathfrak{M}}$ exists:

$$L(\hat{z}, \hat{u}) = \min_{z\in Z} L(z, \hat{u}) \qquad (3)$$

$$\mathbb{E}\big[\hat{u}(G(\hat{z})\big] = \mathbb{E}\big[\hat{u}(Y)\big] \qquad (4)$$

If for some $\hat{u} \in \mathcal{U}^m_{\mathfrak{M}}$ an optimal solution $\hat{z}$ of (3) satisfies the dominance constraints and (4), then $\hat{z}$ solves $(\mathscr{P})$.

# Optimality Conditions Using Rank Dependent Utility Function

## Lagrangian-like functional

$$\Lambda(z,\vartheta,\mu) = f(z) + \int_C \int_0^1 F^{(-1)}(c^\top Y; p) - F^{(-1)}(c^\top G(z); p)\; d\vartheta_c(p)\, d\mu(c)$$

## Uniform inverse dominance condition (UIDC) for $(\mathscr{Q})$

$$\exists \tilde{z} \in Z \quad \inf_{(p,c)\in \operatorname{graph} \mathfrak{M}^-} \left\{ F^{(-2)}(c^\top G(\tilde{z}); p) - F^{(-2)}(c^\top Y; p) \right\} > 0.$$

## Theorem

Assume UIDC. If $\hat{z}$ is a solution of $(\mathscr{Q})$, then $\hat{\vartheta} : C \to \mathcal{W}_2(\mathfrak{M}^-)$ and $\hat{\mu} \in \mathcal{M}_+(C)$ exist:

$$\Lambda(\hat{z},\hat{\vartheta},\hat{\mu}) = \min_{z\in Z} \Lambda(z,\hat{\vartheta},\hat{\mu}) \tag{5}$$

$$\int_C \int_0^1 F^{(-1)}(c^\top G(\hat{z}); p)\, d\hat{\vartheta}_c(p)\, d\hat{\mu}(c) = \int_C \int_0^1 F^{(-1)}(c^\top Y; p)\, d\hat{\vartheta}_c(p)\, d\hat{\mu}(c) \tag{6}$$

If for some $\hat{\vartheta} : C \to \mathcal{W}(\mathfrak{M}^-)$, $\hat{\mu} \in \mathcal{M}_+(C)$ and a solution $\hat{z}$ of (5) the order constraint and (6) are satisfied, then $\hat{z}$ is a solution of $(\mathscr{Q})$.

# Duality Relations to Utility Theories

### The Dual Functionals

$$D(u) = \inf_{z \in Z} L(z,u) \qquad \Delta(\vartheta,\mu) = \inf_{z \in Z} \Lambda(z,\vartheta,\mu)$$

### The Dual Problems

$$(\mathscr{D}_2) \quad \max_{u \in \mathcal{U}^m_{\mathfrak{M}}} D(u) \qquad (\mathscr{D}_{-2}) \quad \max_{\vartheta,\mu} \Phi(\vartheta,\mu).$$

### Theorem

Under UDC/UIDC, if problem $(\mathscr{P})$ resp. $(\mathscr{Q})$ has an optimal solution, then the corresponding dual problem has an optimal solution and the same optimal value. The optimal solutions of the dual problem $(\mathscr{D}_2)$ are utility functions $\hat{u} \in \mathcal{U}^m_{\mathfrak{M}}$ satisfying (3)–(4) for an optimal solution $\hat{z}$ of problem $(\mathscr{P})$. The optimal solutions of $(\mathscr{D}_{-2})$ provide rank dependent utility functions $\hat{\vartheta}_c \in \mathcal{W}(\mathfrak{M}^-)$ and non-negative measures on $C$ satisfying (5)–(6) for an optimal solution $\hat{z}$ of problem $(\mathscr{Q})$.

# Finite localizations

If $z \in \mathbb{R}^n$, then at most $n+2$ target values $\eta_k$, scalarizations $c^k$ and shortfall levels $\mathbb{E}\big[(\eta_k - Y^\top c^k)_+\big]$, $k = 1, \dots, n+2$, exists such that problem $(\mathscr{P})$ is equivalent to

$$\begin{aligned} &\min f(z) \\ &\text{s.t. } \mathbb{E}\big[(\eta_k - \langle c^k, G(z)\rangle)_+\big] \le \mathbb{E}\big[(\eta_k - \langle c^k, Y\rangle)_+\big], \\ &\qquad k = 1, \dots, n+2, \\ &\quad z \in Z. \end{aligned}$$

**Corollary**

If $\hat{z}$ is an optimal solution of problem $(\mathscr{P})$, then a piecewise linear function $\hat{\varphi} \in \mathcal{U}^m_{\mathfrak{M}}$ exists with no more than $n+2$ pieces such that conditions (5)-(6) are satisfied.

Similar result is established for $(\mathscr{Q})$.

Homem de Mello– Merothra (2009), Noyan–Rudolf (2013) have shown that for a finite probability space $(c^j, \eta_j)$ are vertices of a particular polyhedron.

# Shortfall approximation method (with Eli Wolhagen)

Step 0: Set $k = 1$, $(\eta_1, c^1) \in [a, b] \times S$, and $J_1 = \{(\eta_1, c^1)\}$.

Step 1: Solve the master problem:

$$\begin{aligned} & \min f(z) \\ & \text{s.t. } \mathbb{E}\big[\big(\eta_j - \langle c^j, G(z)\rangle\big)_+\big] \le \mathbb{E}\big[\big(\eta_j - \langle c^j, Y\rangle\big)_+\big], \quad j \in J_k. \\ & \quad z \in Z. \end{aligned}$$

Let $z^k$ denote its solution and let $X^k = G(z^k)$.

Step 2: Calculate the quantity

$$\delta_k = \sup_{\eta, c} \big\{ \mathbb{E}\big[(\eta - c^\top X^k)_+ - (\eta - c^\top Y)_+\big] : \ (\eta, c) \in [a, b] \times S \big\}.$$

Step 3: If $\delta_k \le 0$, stop; otherwise, continue.

Step 4: Determine $(\eta_k, c^k)$ such that

$$\mathbb{E}\big[(\eta - c^\top X^k)_+ - (\eta - c^\top Y)_+\big] \ge \frac{\delta_k}{2}.$$

Step 5: Set $J_{k+1} = J_k \cup \{(\eta_k, c^k)\}$; $k \leftarrow k + 1$, and go to Step 1.

# Reduced Problems and Their Duals

Given a collection of point $(\eta_j, c^j)$, $j \in J_k$ for some index set $J_k$, $k \in \mathbb{N}$, the reduced problem is

$$\begin{aligned} &\min f(z) \\ &\text{s.t. } \mathbb{E}\big[\big(\eta_j - (c^j)^\top G(z)\big)_+\big] \le \mathbb{E}\big[\big(\eta_j - (c^j)^\top Y\big)_+\big], \quad j \in J_k. \\ &\quad z \in Z \end{aligned}$$

The Lagrangian of the reduced problem is defined on $Z \times \mathbb{R}_+^{|J_k|}$ is

$$L_k(z, \mu^k) = f(z) + \mathbb{E}\Big[\sum_{j \in J_k} \mu_j^k \big(\eta_j - (c^j)^\top G(z)\big)_+ - \sum_{j \in J_k} \mu_j^k \big(\eta_j - (c^j)^\top Y\big)_+\Big].$$

Extension $\tilde{\mu}^k$ of $\mu^k$ to the entire set $[a, b] \times S$ by setting

$$\tilde{\mu}^k(\mathcal{A}) = \sum_{j \in J_k} \mu_j^k \mathbf{1}_{\mathcal{A} \cap \{(\eta_j, c^j)\}}.$$

The reduced dual function $D_k \colon \mathbb{R}^{|J_k|} \to \mathbb{R}$ at the point $\mu^k$

$$D_k(\mu^k) = \min_{z \in Z} L_k(z, \mu^k) = D(\tilde{\mu}^k).$$

# Subgradients of the Reduced Dual Function

**The subgradients of $D_k(\mu)$**

$$\Gamma_j^k(\mu) = \mathbb{E}\big[\big(\eta_j - (c^j)^\top G(z_\mu^k)\big)_+ - \big(\eta_j - (c^j)^\top Y\big)_+\big], \quad j = 1, \dots, |J_k|,$$

where $z_\mu^k$ is such that $D_k(\mu) = L_k(z_\mu^k, \mu)$.

For atomic measure $\mu^\ell$ with atoms on the set $\{(\eta_j, c^j),\ j \in J_\ell\}$, we define an extension $\mu^{\ell,k} \in \mathbb{R}_+^{|J_k|}$ to the set $\{(\eta_j, c^j),\ j \in J_k \supset J_\ell\}$ by setting

$$\mu_j^{\ell,k} = \begin{cases} \mu_j^\ell & \text{if } j \in J_\ell \\ 0 & \text{if } j \notin J_\ell. \end{cases}$$

Note: the subgradients $\Gamma^{\ell,k}(\mu^{\ell,k})$ are <u>not</u> subgradients of the dual function.

**Piecewise linear model $\mathcal{D}^k : \mathbb{R}^{|J_k|} \to \mathbb{R}$ of the reduced dual function**

$$\mathcal{D}^k(\mu) = \min_{1 \le \ell \le k} \{D(\tilde{\mu}^\ell) + (\Gamma^{\ell,k})^\top(\mu - \mu^{\ell,k})\}.$$

# Dual Approximative Bundle Method (with Eli Wolhagen)

Step 1: Solve $\min_{x\in\mathcal{X}} L_k(x,\mu^k)$ and calculate new subgradients $\Gamma^{\ell,k}$ at $\mu^k$.

Step 2: If $k=1$ or if $\mathcal{D}(\mu^k) \geq (1-\gamma)\mathcal{D}(w^{k-1}) + \gamma\vartheta^{k-1}(\mu^k)$,
then set $w^k := \mu^k$; else set $w^k := w^{k-1,k}$.

Step 3: Calculate a solution $(\theta_{k+1},\mu^{k+1})$ of the master problem:

$$\max_{\mu\geq 0}\{\theta - \frac{\varrho}{2}\|\mu - w^k\|_2^2 \;:\; \mathcal{D}(\mu^\ell) + (\Gamma^{\ell,k})^\top(\mu-\mu^\ell) \geq \theta \;\; \ell = 1\dots k\}.$$

Set $\vartheta^k(\mu^{k+1}) = \theta_{k+1}$, $\tilde{x}^k = \sum_{\ell=1}^k \pi_\ell x^\ell$, and $\tilde{X}^k = G(\tilde{x}^k)$,
where $\pi$ is the optimal Lagrange multiplier.

Step 4: Calculate the quantities

$$\delta_k = \sup_{\eta,c}\Big\{\mathbb{E}\big[\big(\eta - c^\top\tilde{X}^k\big)_+ - \big(\eta - c^\top Y\big)_+\big] : (\eta,c)\in[a,b]\times S\Big\},$$

$$\delta_k' = \max_{j\in J_k}\Big(\mathbb{E}\Big[\big(\eta_j - (c^j)^\top\tilde{X}^k\big)_+ - \big(\eta_j - (c^j)^\top Y\big)_+\Big]\Big).$$

Step 5: If $\mathcal{D}(w^k) \geq \theta^{k+1} - \varepsilon$ and $\delta_k \leq \varepsilon$, then stop; otherwise continue.

Step 6: If $\delta_k > \varepsilon$ and $\delta_k' \leq \frac{\delta_k}{4}$, then determine $(\eta_*, c^*)$ such that

$$\mathbb{E}\big[(\eta_* - (c^*)^\top\tilde{X}^k)_+ - (\eta_* - (c^*)^\top Y)_+\big] \geq \tfrac{1}{2}\delta_k$$

and set $J_{k+1} = J_k \cup \{(\eta_*, c^*)\}$; else set $J_{k+1} = J_k$.
Set $k \leftarrow k+1$ and go to Step 1.

# Convergence (with Eli Wolhagen)

Assume that problem ($\mathscr{P}$) has a feasible solution, $Z$ is compact convex set, the operator $G : \mathbb{R}^n \to \mathcal{L}_1^M(\Omega)$ is continuous, $f$ is a convex function; $[G_i(x)](\omega)$ is concave.

### Theorem

- The shortfall approximation method generates a sequence $\{z^k\}$ whose accumulation points are optimal solutions of problem ($\mathscr{P}$).
- The dual method stops after finitely many iterations with approximate optimal solutions $\tilde{x}^k$ and $w^k$ of the primal and the dual problems, respectively, and a computable bound of the approximation accuracy.

# Order-Verification Problem (with Eli Wolhagen)

$$\min\Big\{\mathbb{E}\big[\eta - c^\top Y\big]_+ - \mathbb{E}\big[\eta - c^\top X^k\big]_+ \,:\, \eta \in [a,b],\ c \in S\Big\} \geq 0?$$

- DC-optimization Linearization of the function $\mathbb{E}\big[\max(0, \eta - c^\top X^k)\big]$, $X^k = G(z^k)$, at each point $(c^i, \eta_i) \in J_k$ by subdifferentiation. Event
$$\mathcal{A}^{ik} = \{\omega \in \Omega : \langle c^i, X^k(\omega)\rangle \leq \eta_i\},$$
Subgradient of $h^k$ at $(c^i, \eta_i)$ by Strassen's Theorem:
$$\begin{pmatrix} \mathbb{P}(\mathcal{A}^{ik}) \\ -\mathbb{E}[X^k \mathbf{1}_{\mathcal{A}^{ik}}] \end{pmatrix} \in \partial h^k(\eta_i, c^i).$$
- Combinatorial methods for finite probability space.
Every event $\mathcal{A}$ is represented by $\alpha_i = \begin{cases} 1, & \text{if } i \in \mathcal{A}; \\ 0, & \text{if } i \notin \mathcal{A} \end{cases}$ for $i = 1, \dots, N$.
$$\min\ \mathbb{E}\big[(c^\top Z - \eta)_+\big] - \sum_{i=1}^{N} p_i \alpha_i (c^\top X^{k,i} - \eta)$$
$$\text{s. t. } (c, \eta) \in S \times [a,b],\ \alpha \in \{0,1\}^N.$$
We alternate between minimizing in $\alpha$ for a fixed $(c, \eta)$ and minimizing in $(c, \eta)$ keeping $\alpha$ fixed.

# Further Numerical Methods

- Cutting Surface Method and Sample Average Approximation Homem de Mello-Mehrotra 2009, 2012;
- Exact Penalty Method and Sample Average Approximation Meskarian-Fliege-Xu 2014;
- Strassen theorem representation Armbruster-Luedtke 2014;
- Augmented Lagrangian method Dentcheva-Martinez-Wolfhagen 2015.
- Methods based on Inverse formulation for univariate constraints Dentcheva-Ruszczyński 2011

# Further Research

- Two-stage problems with constraints on stochastic-order constraints on the recourse
  Drapkin-Schultz 2007, Gollmer -Neise-Schultz 2008,
  Dentcheva-Martinez 2012, Dentcheva-Wolfhagen 2015
- Multi-stage problem with stochastic order constraints
  Dentcheva-Ruszczyński 2008
- Markov decision processes
  Haskell-Jane 2012, Haskell-Shanthikumar-Shen 2015
- Stability, sensitivity and asymptotic behavior
  Dentcheva-Henrion-Ruszczyński 2007, Dentcheva-Römisch 2013,
  Klaus-Schultz 2015
- Shape optimization with stochastic-order constraints
  Gotzes-Schultz 2014
- Radiation therapy design
  Dentcheva-Ruszczyński-Vitt-Yue, 2015