

CASH FLOW MATCHING WITH RISKS CONTROLLED BY bPOE AND CVaR

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Abstract

Bond immunization is an important topic in portfolio management. This paper demonstrates a scenario based optimization framework for solving a cash flow matching problem where the time horizon of the liabilities is longer than the maturities of the available bonds and the interest rates are uncertain. Bond purchase decisions are made each period to generate cash flow for covering the obligations in future. Cash flows depend upon uncertain future prices of bonds. We use Buffered Probability of Exceedance (bPOE) and Conditional Value-at-Risk (CVaR) to control risk of shortfalls. The initial cost of the hedging portfolio of bonds is minimized and optimal positions of bonds are calculated at all time periods. We used Gurobi and Portfolio Safeguard (PSG) optimization packages to solve the optimization problems.

1 Introduction

Bond immunization, including duration matching and cash flow matching, is an important portfolio optimization problem. Given the stream of liabilities, the objective

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of cash flow matching problem is to match the future cash flow stream of liabilities with some asset cash flows.

Let p_0 be the vector of initial prices of the bonds available in the market, x_0 be the vector of initial positions of bonds, l_t and c_t be the liability and payment vectors at period t , respectively. The classic cash flow matching problem minimizes an initial value of the portfolio, under condition that liabilities are covered by the cash flows generated by the portfolio of bonds at all time moments:

$$\begin{aligned} & \min_{x_0} p_0^T x_0 \\ & \text{subject to} \\ & l_t - c_t^T x_0 \leq 0, \quad t = 1, \dots, N, \\ & -x_0 \leq 0. \end{aligned} \tag{1}$$

In this cash flow matching problem (1), the liabilities have a shorter (or equal) time horizon compared to the maturities of available bonds. So, the resulting portfolio can be truly immunized.

Hiller and Eckstein [3], Zenios [12], and Consigli and Dempster [2] proposed stochastic programming approaches for the cash flow matching problem.

Iyengar and Ma [5] used Conditional Value-at-Risk (CVaR)[10] to constraint risks in the bond cash matching problem. Considered bonds of various maturities pay coupons as well as face values at different time periods. But unlike classic cash flow matching problem, the liabilities have a longer time horizon than maturities of the currently available bonds. Hence some bond purchases are made in future periods. Moreover, it is considered that future prices of the bonds are uncertain. Therefore, the resulting portfolio cannot be truly immunized to changes of interest rates. The objective of the model constructed in the article is to design a portfolio providing needed cash flows with a high probability and to minimize an initial portfolio cost. This paper follows the general setting suggested by Iyengar and Ma [5].

This paper presents new variants of cash matching problem with risks controlled by Buffered Probability of Exceedance (bPOE), see [6, 7, 11]. bPOE, is a function similar to the Probability of Exceedance (POE), which is the chance that the liability is higher than the cash flow of portfolio of bonds, at least at one time moment. The value of bPOE is about two times larger than POE, see [6]. We compared optimization problem statement with risks controlled by bPOE and by CVaR. We conducted a case study demonstrating that the cash matching problems with bPOE functions can be efficiently solved with convex and linear programming algorithms, similar to the problems with CVaR risk function. Optimization was done with the optimization packages Gurobi and Portfolio Safeguard (PSG) [1]. PSG provides compact and intuitive problem formulations and codes for risk management problems.

Section 2 describes the mathematical cash matching problem formulations with risks controlled by CVaR and bPOE functions. Section 3 presents an approach for minimization of bPOE. Section 4 describes a case study.

2 Controlling Risks by CVaR and bPOE

The considered cash flow matching problem has a longer duration than the maturities of bonds available at initial time. Hence bonds purchased at initial period do not generate a cash flow with a duration long enough to cover the stream of liabilities. Bonds should be purchased in later periods and prices of these bonds are uncertain. We simulated future prices of bonds using interest rate scenarios generated by Prof. Ken Kortanek with the Hull and White [4] interest rate model. Rebonato [8] provides a comprehensive introduction to the calibration of interest rate models.

Suppose random variable L is future loss (or return with minus sign) of some investment. By definition, Value-at-Risk at level α is the α -quantile of L :

$$\text{VaR}_\alpha(L) = \inf\{z | F_L(z) > \alpha\},$$

where F_L denotes Cumulative Distributions Function (CDF) of the random variable L .

Conditional Value-at-Risk (CVaR) for continuous distributions equals the expected loss exceeding VaR (see, e.g., Rockafellar and Uryasev [9]):

$$\text{CVaR}_\alpha(L) = E[L | L \geq \text{VaR}_\alpha(L)].$$

This formula justifies the name of CVaR, as a conditional expectation. For the general case, the definition is more complicated, and can be found in Rockafellar and Uryasev [10].

There are two probability measures associated with VaR and CVaR [7, 6, 11]. The first measure is the *Probability of Exceedance (POE)*, which equals 1 minus CDF:

$$p_z(L) = P(L > z) = 1 - F_L(z).$$

By definition, CDF is an inverse function to VaR. The second probability measure is called *Buffered Probability of Exceedance (bPOE)*. There are two slightly different variants of bPOE, so called *Upper bPOE* and *Lower bPOE*. Lower bPOE is defined as follows

$$\bar{p}_z^{\text{Lower}}(L) = \begin{cases} 0, & \text{for } z \geq \sup L; \\ \{1 - \alpha | \text{CVaR}_\alpha(L) = z\}, & \text{for } EL < z < \sup L; \\ 1, & \text{for } z \leq EL. \end{cases} \quad (2)$$

So, at the interval $EL < z < \sup L$, bPOE equals $1 - \alpha$, where α is an inverse function of CVaR, i.e., a unique solution of the equation

$$\text{CVaR}_\alpha(L) = z. \quad (3)$$

Therefore, bPOE equals probability, $1 - \alpha$, of the tail such that CVaR for this tail is equal to z . At the point $z = \sup L$, the solution of equation (3) may not be unique, where $\sup L$ is the essential extremum of the random value L . The largest solution equals $\alpha = 1$, which corresponds to the *lowest* value of $1 - \alpha = 0$, i.e.

$$0 = \min\{1 - \alpha | \text{CVaR}_\alpha(L) = z = \sup L\}.$$

Therefore, bPOE defined in (2), which equals zero at points $z = \sup L$, is called Lower bPOE. For $z = \sup L$, the smallest solution of equation (3) equals $1 - P(z = \sup L)$, and, consequently

$$P(z = \sup L) = \max\{1 - \alpha \mid CVaR_\alpha(L) = z = \sup L\}.$$

This leads to the definition of Upper bPOE, which differs from Lower bPOE only at one point $z = \sup L$:

$$\bar{p}_z^{Upper}(L) = \begin{cases} P(z = \sup L), & \text{for } z = \sup L; \\ \bar{p}_z^{Lower}(L), & \text{otherwise .} \end{cases} \quad (4)$$

For continuous distributions of L , we have $P(z = \sup L) = 0$, and consequently, $\bar{p}_z^{Lower}(L) = \bar{p}_z^{Upper}(L)$. The paper [[7]] proves that the Upper bPOE can be, also, calculated as follows

$$\bar{p}_z^{Upper}(L) = \begin{cases} \max\{1 - \alpha \mid CVaR_\alpha(L) \geq z\}, & \text{for } z \leq \sup L; \\ 0, & \text{otherwise .} \end{cases} \quad (5)$$

Further to simplify notations, we will call Upper bPOE by just bPOE and denote $\bar{p}_z^{Upper}(L)$ by just $\bar{p}_z(L)$. The Lower bPOE will still be denoted by $\bar{p}_z^{Lower}(L)$.

We consider the setting similar to the bond-matching problem in Iyengar and Ma [5]. At every period, $t = \{1, \dots, N\}$, there is a liability l_t . We assume that the same set of bonds is available for investment for each, $t \in \{0, \dots, N\}$. We denote by p_0 a deterministic price vector of the bonds at time 0, and denote by p_t a random price vector of bonds at time $t \in \{1, \dots, N\}$. We denote by $c_{t,n}^j$ a cash flow at time moment $n \in \{1, \dots, N\}$ from bond $j \in \{1, \dots, M\}$ purchased at time moment $t \in \{0, \dots, n-1\}$, and $c_t = (c_t^1, \dots, c_t^M)^T$ is the corresponding column vector.

Suppose x_t^j is the number of shares of bond $j \in \{1, \dots, M\}$ purchased at time step $t \in \{0, \dots, N\}$, and $x_t = (x_t^1, \dots, x_t^M)^T$ is the corresponding column vector. An optimal value of this vector is determined by solving an optimization problem.

With these notations, the underperformance L_t of replicating portfolio of bonds versus the liability at the end of time period t equals,

$$L_t = l_t + p_t^T x_t - \sum_{s=0}^{t-1} c_{s,t}^T x_s, \quad t = 1, \dots, N.$$

Let us denote by $L(x_0, \dots, x_N)$ the random maximum loss over all time periods

$$L(x_0, \dots, x_N) = \max_{0 \leq t \leq N} L_t.$$

Further on, for simplicity, we will skip the argument in this maximum loss function. We consider the following cash flow matching problem with the constraint on CVaR of

the maximum loss:

$$\begin{aligned}
& \min_{x_0, \dots, x_N} p_0^T x_0 \\
& \text{subject to} \\
& CVaR_\alpha(L) \leq z, \\
& x_t \geq 0, \quad t = 0, \dots, N.
\end{aligned} \tag{6}$$

Iyengar and Ma [5] suggested a problem (6) with parameter $z = 0$. They find an optimal investment strategy by minimizing the cost of an initial portfolio of bonds. The maximum over time reinvestment risk related to the uncertainties in the bond prices is controlled by CVaR. The level of the protection from the reinvestment risk monotonically increases as a function of the confidence level α in CVaR.

Alternatively, we can place bPOE in constraint instead of CVaR

$$\begin{aligned}
& \min_{x_0, \dots, x_N} p_0^T x_0 \\
& \text{subject to} \\
& \bar{p}_z(L) \leq 1 - \alpha, \\
& x_t \geq 0, \quad t = 0, \dots, N.
\end{aligned} \tag{7}$$

By exchanging the objective and constraint in problem (6) we got the problem which minimizes risk subject to the constraint on the initial budget d :

$$\begin{aligned}
& \min_{x_0, \dots, x_N} CVaR_\alpha(L) \\
& \text{subject to} \\
& p_0^T x_0 \leq d, \\
& x_t \geq 0, \quad t = 0, \dots, N.
\end{aligned} \tag{8}$$

The considered problem (8) minimizes CVaR risk, but it does not directly minimize the probability that loss, L , exceeds some threshold z . bPOE is an upper bound for POE. Therefore, to lower POE, we can minimize bPOE (instead of CVaR):

$$\begin{aligned}
& \min_{x_0, \dots, x_N} \bar{p}_z(L) \\
& \text{subject to} \\
& p_0^T x_0 \leq d, \\
& x_t \geq 0, \quad t = 0, \dots, N.
\end{aligned} \tag{9}$$

Mafusalov and Uryasev [6] showed equivalence of constraints on CVaR and Lower bPOE for general distributions (including discrete distributions considered in this paper). Therefore, optimization problems (6) and (7) are equivalent with Lower bPOE in (7) for general distributions of L . This equivalence is a generalization of equivalence in a special case with $z = 0$, which was originally stated by Rockafellar and Royset []

for continuous distributions. For Upper bPOE, with discrete distribution considered in this paper, constraints on CVaR and on Upper bPOE are not equivalent. However, we show further that constraint on CVaR and Upper bPOE are “nearly” equivalent. Since this equivalence is an important fact for understanding relation of CVaR and bPOE, we prove this “near” equivalence with the following two statements.

Statement 1. Constraint on bPOE implies constraint on CVaR.

Let $0 < \alpha < 1$, then $\bar{p}_z(L) \leq 1 - \alpha$ implies $CVaR_\alpha(L) \leq z$.

Proof.

To begin with, we give a quick schematic proof (without details) using known facts from paper [6]. Since, $\bar{p}_z(L) \geq \bar{p}_z^{Lower}(L)$, we have

$$\bar{p}_z(L) \leq 1 - \alpha \Rightarrow \bar{p}_z^{Lower}(L) \leq 1 - \alpha \Leftrightarrow CVaR_\alpha(L) \leq z.$$

Now, to explain the relation between bPOE and CVaR, we give the same proof in a more detailed format, without using paper [6]. According to Norton and Uryasev [7] and Mafusalov and Uryasev [6], bPOE can be calculated as follows

$$\bar{p}_z(L) = \min_{\lambda \geq 0} E[\lambda(L - z) + 1]^+. \quad (10)$$

Inequality $\bar{p}_z(L) < 1$ implies that an optimal λ^* in this minimization formula (10) is strictly positive, i.e., $\lambda^* > 0$. Therefore,

$$\bar{p}_z(L) = E[\lambda^*(L - z) + 1]^+ \leq 1 - \alpha,$$

which can be rearranged as follows

$$-\frac{1}{\lambda^*} + \frac{1}{1 - \alpha} E \left[(L - z) + \frac{1}{\lambda^*} \right]^+ \leq 0.$$

By changing variable $1/\lambda^* = z - \mu^*$ in the last inequality we get

$$\mu^* + \frac{1}{1 - \alpha} E[L - \mu^*]^+ \leq z. \quad (11)$$

Therefore, using CVaR minimal representation (see, [10]), we get

$$CVaR_\alpha(L) = \min_{\mu} \left\{ \mu + \frac{1}{1 - \alpha} E[L - \mu]^+ \right\} \leq z. \quad (12)$$

Statement 2. Constraint on CVaR implies constraint on bPOE.

Let $0 < \alpha < 1$ and $y > z$, then inequality $CVaR_\alpha(L) \leq z$ implies $\bar{p}_y(L) < 1 - \alpha$.

Proof.

To begin with, we give a quick schematic proof (without details) using known facts from paper [6]. Since

$$\bar{p}_z^{Lower}(L) \geq \bar{p}_y,$$

we have

$$CVaR_\alpha(L) \leq z \iff \bar{p}_z^{Lower}(L) \leq 1 - \alpha \implies \bar{p}_y(L) < 1 - \alpha.$$

Now, we give the same proof in a more detailed format, without using paper [6]. According to the CVaR minimization representation [10],

$$CVaR_\alpha(L) = \mu^* + \frac{1}{1 - \alpha} E[L - \mu^*]^+, \quad (13)$$

where $\mu^* = VaR_\alpha(L)$. Consequently,

$$\mu^* \leq \mu^* + \frac{1}{1 - \alpha} E[L - \mu^*]^+ \leq z < y. \quad (14)$$

Since $\mu^* < y$, we can define a change of variables $\mu^* = y - 1/\lambda^*$, where $\lambda^* > 0$. By changing variables in inequality

$$\mu^* + \frac{1}{1 - \alpha} E[L - \mu^*]^+ < y$$

we get

$$E[\lambda^*(L - y) + 1]^+ < 1 - \alpha.$$

Consequently,

$$\bar{p}_y(L) = \min_{\lambda \geq 0} E[\lambda(L - y) + 1]^+ \leq E[\lambda^*(L - y) + 1]^+ < 1 - \alpha,$$

and Statement 2 is proved.

Mafusalov and Uryasev [6] also showed equivalence of CVaR minimization problem (8) and pBOE minimization problem (9) with a convex set of constraints in the following sense:

- for every parameter value α in problem (8) and an optimal objective value z of this problem, the optimization problem (9) with parameter z has the optimal objective value $1 - \alpha$;
- for every parameter value z in problem (9) and an optimal objective value $1 - \alpha$ of this problem, the optimization problem (8) with parameter α has the optimal objective value z .

Also, it can be shown that optimization problems (6), (7), (8), (9) generate coinciding parts of the efficient frontiers. Similar results were presented in Norton and Uryasev [7]. By definition, an efficient frontier is a set of Pareto-optimal solutions in a two criteria optimization problem. In this case, the first criteria is an initial investment, $p_0^T x_0$, and the second criteria is CVaR or bPOE.

Firstly, let us fix parameter α in the considered optimization problems. Some coinciding parts of the efficient frontiers can be generated with:

- problem (6) by varying parameter z ;
- problem (7) by varying parameter z ;
- problem (8) by varying parameter d .

Now, let us fix parameter z in the considered optimization problems. Some coinciding parts of efficient frontiers can be generated with:

- problem (6) by varying parameter α ;
- problem (7) by varying parameter α ;
- problem (9) by varying parameter d .

3 Approaches to bPOE Minimization

According to Mafusalov and Uryasev [6] and Norton and Uryasev [7], Lower bPOE can be calculated as follows

$$\bar{p}_z(X) = \min_{\lambda \geq 0} E[\lambda(X - z) + 1]^+. \quad (15)$$

The maximum loss function, $L = \max_{0 \leq t \leq N} L_t$, depends upon a set of decision vectors, x_1, \dots, x_N . Let us combine these vectors in one decision vector $\vec{x} = (x_1, \dots, x_N)$.

Let us consider a general linear loss function $L_t(\vec{x}) = (\vec{a}_t)^T \vec{x} + b_t$ with random coefficients \vec{a}_t . We suppose that these coefficients, \vec{a}_t , are random vectors with finite discrete distribution, and the random loss function $L_t^k(\vec{x}) = (\vec{a}_t^k)^T \vec{x} + b_t^k$, has scenarios $k = 1, \dots, K$ with probabilities $p_k = 1/K$. Then, (15) implies

$$\bar{p}_z(L) = \min_{\lambda \geq 0} E[\lambda(\max_{0 \leq t \leq N} L_t - z) + 1]^+ = \min_{\lambda \geq 0} \sum_{k=1}^K p_k [\lambda(\max_{0 \leq t \leq N} \{(\vec{a}_t^k)^T \vec{x} + b_t^k\} - z) + 1]^+.$$

The minimization problem for bPOE w.r.t. \vec{x} can be written as follows

$$\begin{aligned} \min_{\vec{x}} \bar{p}_z(L) &= \min_{\vec{x}, \lambda \geq 0} \sum_{k=1}^K p_k [\lambda(\max_{0 \leq t \leq N} \{(\vec{a}_t^k)^T \vec{x} + b_t^k\} - z) + 1]^+ \\ &= \min_{\vec{x}, \lambda \geq 0} \sum_{k=1}^K p_k [\max_{0 \leq t \leq N} \{(\vec{a}_t^k)^T \lambda \vec{x} + \lambda(b_t^k - z)\} + 1]^+. \end{aligned}$$

By changing variables $\vec{y} = \lambda \vec{x}$ in the last equation and substituting the objective in problem (9) we obtain

$$\min_{\vec{y}, \lambda} \sum_{k=1}^K p_k [\max_{0 \leq t \leq N} \{(\vec{a}_t^k)^T \vec{y} + \lambda(b_t^k - z)\} + 1]^+ \quad (16)$$

subject to

$$\begin{aligned} p_0^T y_0 - d\lambda &\leq 0, \\ \lambda &\geq 0, \quad y_t \geq 0, \quad t = 0, \dots, N. \end{aligned}$$

The objective in (16) is a so called *partial moment* with threshold -1 of the random function $L = \max_{0 \leq t \leq N} \{(\bar{a}_t^k)^T \bar{y} + \lambda(b_t^k - z)\}$. This objective function is a piecewise linear convex function w.r.t. variables \bar{y}, λ . The problem (16) can be reformulated as a Linear Programming (LP) problem. Let us introduce an additional vector of decision variables $\vec{u} = (u_1, \dots, u_K)$. Problem (16) is equivalent to the following LP

$$\min_{\bar{y}, \lambda, \vec{u}} \sum_{k=1}^K p_k u_k \quad (17)$$

subject to

$$\begin{aligned} u_k &\geq (\bar{a}_t^k)^T \bar{y} + \lambda(b_t^k - z) + 1, \quad t = 0, \dots, N, \quad k = 1, \dots, K, \\ p_0^T y_0 - d\lambda &\leq 0, \\ \lambda &\geq 0, \quad y_t \geq 0, \quad t = 0, \dots, N, \quad u_k \geq 0, \quad k = 1, \dots, K. \end{aligned}$$

4 Case study

Data, codes, and solutions for this case study are posted at this link¹.

We considered 120 time steps with 0.5 year length. Table 1 shows the maturities, coupon rates and initial prices of the 11 available bonds.

We made $K = 200$ simulations of bond prices. Every simulation provides prices of all bonds for all time periods. $p_t^{k,j}$ is the price of bond $j \in \{1, \dots, M\}$ at time step $t \in \{1, \dots, N\}$ for simulation $k \in \{1, \dots, K\}$.

Similar to [5] we consider the following stream of liabilities:

$$l_t = \begin{cases} 100 & \text{if } t/2 = 0, \dots, 10, \\ 110 - 2.2 \times (\frac{t}{2} - 10) & \text{if } t/2 = 11, \dots, 60, \\ 0 & \text{otherwise.} \end{cases}$$

To solve optimization problems we used the Portfolio Safeguard (PSG) package containing preprogrammed functions, CVaR, Partial Moment, and bPOE functions. PSG codes are quite simple and transparent. Here is the PSG code for minimizing CVaR with problem (8):

¹http://www.ise.ufl.edu/uryasev/research/testproblems/financial_engineering/case-study-cash-matching-with-bpoe-and-cvar-functions/

Table 1: Details of Treasury Bonds.

Bond Index	Name	Maturity	Coupon Rate (%)	Current Price
1	T-bill	0.5	0	95.8561
2	T-note	1	4.5	96.1385
3	T-note	2	4.5	92.6873
4	T-note	3	4.5	89.5784
5	T-note	4	4.5	86.7610
6	T-note	5	4.5	84.1959
7	T-bond	10	5.0	77.5948
8	T-bond	15	5.0	71.9232
9	T-bond	20	5.0	68.1357
10	T-bond	25	5.0	65.5990
11	T-bond	30	5.0	63.8989

```

minimize
  CVaR(0.9,lmax(matrix_1L,...,matrix_120L))
Constraint:  <= 1172.368
  linear(matrix_0)
Box:  >= 0

```

PSG code for minimizing bPOE with problem (9):

```

minimize
  bPOE(0,lmax(matrix_1,...,matrix_120))
Constraint:  <= 1172.368
  linear(matrix_0)
Box:  >= 0

```

PSG code for minimizing bPOE with problem (16):

```

minimize
  pm_pen(-1,lmax(matrix_1L0,...,matrix_120L0))
Constraint:  <= 0
  linear(matrix_0)
  -1172.368*variable(lambda)
Box:  >= 0

```

Table 2 contains results of several optimization runs. The first line contains calculation results for Problem (6) with $\alpha = 0.9$. The optimal objective value equals 1172.368. The second line corresponds to Problem (8) with the right-hand side $d = 1172.368$ equals to the optimal objective value of Problem (6). As expected, the optimal objective value of Problem (8) equals zero, because CVaR is bounded by zero in Problem

Table 2: Calculation Results for Three Optimization Problems.

	Constraint Setting	Optimal Objective Value
Problem (6)	$CVaR_{\alpha=0.9}(L) \leq 0$	$p_0^T x_0 = 1172.368$
Problem (8)	$p_0^T x_0 \leq d = 1172.368$	$CVaR_{\alpha=0.9} = 0$
Problem (9)	$p_0^T x_0 \leq d = 1172.368$	$bPOE = 0.1$
Problem (16)	$p_0^T y_0 - d\lambda \leq 0, d = 1172.368$	$bPOE = 0.1$

Table 3: Efficient Frontier.

d (bound on budget)	Minimal bPOE
1162	1
1163	0.9703236
1164	0.8577470
1165	0.7469944
1166	0.6404300
1167	0.5377273
1168	0.4457100
1169	0.3517119
1170	0.2727167
1171	0.2028206
1172	0.1286488
1172.368	0.1000000
1172.5	0.0905712
1172.75	1.091354E-10
1173	2.182642E-11
1174	1.359135E-12
1175	0.000000

(6). The third line in Table 2 contains solution of bPOE minimization problem (9) with $d = 1172.368$. The optimal bPOE=0.1, because $1 - \alpha = 1 - 0.9 = 0.1$. The fourth line in Table 2 contains solution of bPOE minimization problem (16) with the same parameter $d = 1172.368$. The optimal bPOE=0.1.

Finally, we build an efficient frontier by solving problem (16) for a series of values of parameter d . Table 3 contains calculations for this frontier (first column = budget d , second column = optimal bPOE), see Figure 1.

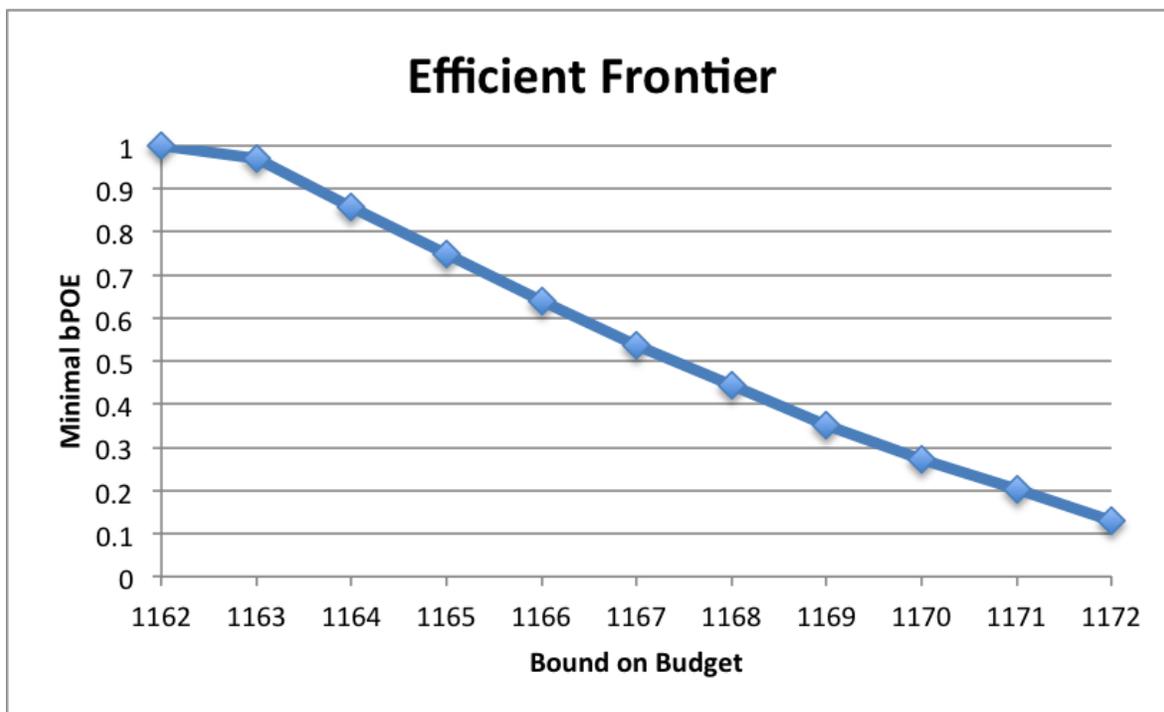


Figure 1: Efficient frontier: bound on budget d vs minimal bPOE.

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