CASE STUDY: Support Vector Machine Based on Tail Risk Measures (quadratic, cvar_risk, max_cvar_risk, var_risk, max_var_risk, cvar_comp_abs_norm, polynom_abs, max_comp_abs)

Background

This case study illustrates the application of the risk management framework to the Support Vector Machine (SVM) classification problem.

Given a training data $(\xi_1 y_{1)}, (\xi_2 y_2), \ldots, (\xi_J y_J)$, where $\xi_j \in R^n$ are features and $y_j \in \{-1,1\}$ are class labels, the basic idea of SVM is to find an optimal separating hyper-plane in the features space which maximizes the margin between two classes. Cortes et al. (1995) proposed to solve SVM classification problem as a quadratic programming. An alternative formulation, known as nu-SVM, was suggested by Scholkopf, et al. (2000). Takeda and Sugiyama (2008) proposed to use the CVaR risk measure in classification and formulated the SVM learning problem as a CVaR minimization problem. Wang (2009) proposed robust nu-Support Vector Machine based on worst-case CVaR Minimization. Tsyurmasto, and Uryasev (2012) proposed Support Vector Machines based on Value-at-Risk (VaR) measure. They obtained new SVM classifiers based on VaR risk measure for the following CVaR-based SVMs: Nu-SVM, Extended Nu-SVM, Robust Nu-SVM. Tsyurmasto, Zabarankin, and Uryasev (2013) proposed three closely related formulations of a new robust version of Support Vector Machine (SVM) based on value-at-risk (VaR) measure referred to as VaR-SVM and established relationships between those VaR-SVM formulations. Gotoh, and Uryasev (July 2013) present a unified scheme for SVMs which employ arbitrary convex function for empirical risk and arbitrary norm for regularizer, on the basis of convex analysis. Wide class of norms (CVaR, L_1, L_∞, the convex combination of the L_1- and L_∞-norms and their duals) was investigated by Gotoh, and Uryasev (May 2013) and Pavlikov, and Uryasev (2013).

Case study contains the following problem formulations: 1a) regularized CVaR, 1b) regularized VaR, 2a) CVaR minimization with unity constraint, 2b) VaR minimization with unity constraint, 3a) regularized robust CVaR minimization, 3b) regularized robust VaR minimization, 3c) regularized weighted difference of CVaRs. Problems 1a,1b,3a,3b,3c include additional quadratic regularization term.

Problems 4a, 4b, 5a, 5b, 6a, 6b employ arbitrary convex function for empirical risk and arbitrary norm for regularizer. Problems 4a (Primal) minimizes CVaR with constraint on L_∞ norm; 4b (Dual) maximizes - L_1 norm with CVaR envelop set of constraints; 5a (Primal) minimizes CVaR with constraint on Mixture of L_1-, and L_∞- norms (Deltoidal norm); 5b (Dual) maximizes - norm dual to mixture of L_1-, and L_∞- norms (Dual deltoidal norm) with CVaR envelop set of constraints; 6a (Primal) minimizes CVaR with constraint on CVaR_norm; 6b (Dual) maximizes - Dual CVaR Norm with CVaR envelop set of constraints.

References

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Notations

J = number of observations (interpreted as scenarios having equal probabilities);

j = index of observation points $\{1,\ldots, J\}$;

$(\xi_1 y_{1)}, (\xi_2 y_2), \ldots, (\xi_J y_J)$ = training data, where $\xi_j \in R^n$ is the feature column vector and $y_j \in \{-1,1\}$ is class label for observation j, ξ_j^T is transposed vector ξ_j;

$G^T = (\xi_1 y_{1,} \xi_2 y_2, \ldots, \xi_J y_J)$ = $n \times J$ matrix whose j-th column is $\xi_j y_j$;

$y^T = (y_1, y_{2,} \ldots, y_J)$ = vector of observations;

$\lambda^T = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ = vector of dual variables;

$\Lambda_{CVaR_\alpha} = \left\{\lambda: 0 \le \lambda_i \le \frac{1}{n(1-\alpha)},\ i = 1,2,\cdots,n,\ \sum_{i=1}^n \lambda_i = 1\right\}$ = CVaR envelope set of constraints;

$x = (x', x_0)$ = vector of decision variables;

$L(x)$ = random loss function given by observations (scenarios)

$L_j(x) = -y_j(\xi_j^T x' + x^0)$, $j=1,\ldots, J$;

M = number of data subsets;

J_m = number of observation in the data subset m ;

$cvar\_risk_\alpha(L(x))$ = CVaR risk with confidence level α for random loss function $L(x)$;

$var\_risk_\alpha(L(x))$ = VaR risk with confidence level α for random loss function $L(x)$;

$max\_cvar\_risk_\alpha(L(x))$ = maximum of CVaRs with confidence level α over set of data subsets, $m = 1,\ldots, M;$

$max\_var\_risk_\alpha(L(x))$ = maximum of VaRs with confidence level α over set of data subsets, $m = 1,\ldots, M.$

Optimization Problem 1a

Minimize sum of regularization term and CVaR

$$\min_x \quad (x')^T x' + cvar\_risk_\alpha(L(x)) \qquad \text{(CS.1a)}$$

Optimization Problem 1a'

2-fold cross-validation for Minimize sum of regularization term and CVaR

$$\min_x \quad (x')^T x' + cvar\_risk_\alpha(L(x))$$

Optimization Problem 1b

Minimize sum of regularization term and VaR

$$\min_x \quad (x')^T x' + var\_risk_\alpha(L(x)) \qquad \text{(CS.1b)}$$

Optimization Problem 2a

Minimize CVaR, subject to unity constraint on quadratic term

$$\min_x \quad cvar\_risk_\alpha(L(x)) \qquad \text{(CS.2a)}$$

$$s.t. \quad (x')^T x' = 1$$

Optimization Problem 2b

Minimize VaR, subject to unity constraint on quadratic term

$$\min_x \quad var\_risk_\alpha(L(x)) \qquad \text{(CS.2b)}$$

$$s.t. \quad (x')^T x' = 1$$

Optimization Problem 3a

Minimize sum of regularization term and maximum of CVaRs

$$\min_x \quad (x')^T x' + max\_cvar\_risk_\alpha(L(x)) \tag{CS.3a}$$

Optimization Problem 3b

Minimize sum of regularization term and maximum of VaRs

$$\min_x \quad (x')^T x' + max\_var\_risk_\alpha(L(x)) \tag{CS.3b}$$

Optimization Problem 3c

Minimize sum of regularization term and weighted difference of two CVaRs

$$\min_x \quad (x')^T x' + (1-\alpha_L)\ cvar\_risk_{\alpha_L}(L(x)) - (1-\alpha_U)\ cvar\_risk_{\alpha_U}(L(x)) \tag{CS.3c}$$

Optimization Problem 4a (Primal)

Minimize CVaR

$$\min_x \quad cvar\_risk_\alpha(L(x)) \tag{CS.4a}$$

$$s.t. \quad \| x' \|_\infty \leq 1$$

Optimization Problem 4b (Dual)

Maximize L_1 norm

$$max_\lambda \quad -\| G\lambda \|_1 \tag{CS.4b}$$

$$s.t. \quad y^T\lambda = 0, \lambda \in \Lambda_{CVaR_\alpha}$$

Optimization Problem 5a (Primal)

Minimize CVaR

$$\min_x \quad cvar\_risk_\alpha(L(x)) \tag{CS.5a}$$

$$s.t. \quad (1-\tau) \| x' \|_1 + \tau \| x' \|_\infty \leq 1$$

Optimization Problem 5b (Dual)

Maximize the norm

$$max_\lambda \quad \left[-max \left\{ \left|G\lambda_{(1)}\right|, \frac{|G\lambda_{(1)}|+|G\lambda_{(2)}|}{2-\tau}, \cdots, \frac{|G\lambda_{(1)}|+\cdots+|G\lambda_{(J)}|}{J-(J-1)\tau} \right\}\right] \tag{CS.5b}$$

s.t. $y^T \lambda = 0, \lambda \in \Lambda_{CVaR_\alpha}$

Optimization Problem 6a (Primal)

Minimize CVaR

$$\min_x \quad cvar\_risk_\alpha(L(x)) \tag{CS.6a}$$

s.t. $J(1-\beta)cvar_\beta(x') \leq 1$

Optimization Problem 6b (Dual)

Maximize the norm

$$max_\lambda \quad \left[-max\left\{\frac{1}{J(1-\beta)}\|G\lambda\|_1, \|G\lambda\|_\infty\right\}\right] \tag{CS.6b}$$

s.t. $y^T \lambda = 0, \lambda \in \Lambda_{CVaR_\alpha}$