CASE STUDY: Omega Portfolio Rebalancing (omega, pm_pen, avg)

Background

This case study demonstrates an Omega optimization setup for a portfolio optimization problem. A fund of funds blends the risk-return profiles of various hedge fund managers/strategies to meet investor requirements. The data for this case study are prepared with the Converter_Omega_Portfolio. To install this converter you should download installation file **"Converter_Omega_Portfolio_setup.zip"** in the Client Area (www.aorda.com) from the Download page by selecting the "Case Studies" downloading option.

References

- Keating, C., and W. Shadwick (2002): A Universal Performance Measure. *The Journal of Performance Measurement* v. 6 #3.
- Mauser, H., Saunders, D., and L. Seco (2006): Optimizing Omega. Risk, November, pp. 88-92.
- Avouyi-Dovi, S., Morin, A., and D. Neto (2004)" Optimal asset allocation with omega function, tech. report, Banque de France. Research Paper.
- Passow, A. (2005): Omega portfolio construction with Johnson distributions. Risk, April, pp. 85–90.

Notations

I = number of instruments in the portfolio; $i=\{1,\ldots,I\}$ index of instruments in the portfolio;

$y_i =$ exposure to instrument i -th (in currency);

$w_i = \dfrac{y_i}{\sum_{i=1}^{I} y_i} =$ weight of i -th instrument in the portfolio;

$w = (w_1,\ldots,w_I) =$ vector of weights of instruments in the portfolio;

r_i^t = rate of return of i -th instrument in time period t ;

r_h = hurdle rate of return;

$L(y,t) = \sum_{i=1}^{I}(r_h - r_i^t)y_i =$ loss function (underperformance of the portfolio) at time t as a function of y (in currency) ;

$L(w,t) = \dfrac{\sum_{i=1}^{I}(r_{Ih} - r_i^t)y_i}{\sum_{i=1}^{I} y_i} = \sum_{i=1}^{I}(r_{Ih} - r_i^t)\dfrac{y_i}{\sum_{i=1}^{I} y_i} = \sum_{i=1}^{I}(r_{Ih} - r_i^t)w_i =$ loss function (underperformance of the portfolio) at time t as a function of w (in fractions);

$\underline{\underline{\eta}}(y) = \dfrac{1}{T}\sum_{\substack{t=1 \\ L(t,y)\geq 0}}^{T} L(y,t) =$ low partial moment for loss function $L(y,t)$ with respect to benchmark 0 (in currency);

$\underline{\underline{\eta}}(w) = \frac{1}{T} \sum_{\substack{t=1 \\ L(t,w)\geq 0}}^{T} L(w,t) =$ low partial moment for loss function $L(w,t)$ with respect to benchmark 0 (in fractions);

$\bar{\eta}(y) = \frac{1}{T} \sum_{\substack{t=1 \\ L(t,y)<0}}^{T} -L(y,t) =$ upper partial moment for loss function $L(y,t)$ with respect to benchmark 0 (in currency);

$\bar{\eta}(w) = \frac{1}{T} \sum_{\substack{t=1 \\ L(t,w)<0}}^{T} -L(w,t) =$ upper partial moment for loss function $L(w,t)$ with respect to benchmark 0 (in fractions);

$q(y) = \frac{1}{T} \sum_{t=1}^{T} -L(y,t) =$ expected gain with respect to hurdle rate as a *function* of y (in currency);

$q(w) = \frac{1}{T} \sum_{t=1}^{T} -L(w,t) =$ expected gain with respect to hurdle rate as a *function* of w (in fractions);

$$\Omega(w) = \frac{\bar{\eta}(y)}{\underline{\underline{\eta}}(y)} = \frac{\dfrac{\bar{\eta}(y)}{\sum_{i=1}^{I} y_i}}{\dfrac{\underline{\underline{\eta}}(y)}{\sum_{i=1}^{I} y_i}} = \frac{\bar{\eta}(w)}{\underline{\underline{\eta}}(w)} = \frac{\bar{\eta}(w) - \underline{\underline{\eta}}(w) + \underline{\underline{\eta}}(w)}{\underline{\underline{\eta}}(w)} = \frac{q(w) + \underline{\underline{\eta}}(w)}{\underline{\underline{\eta}}(w)} = 1 + \frac{q(w)}{\underline{\underline{\eta}}(w)} = \textbf{ Omega function;}$$

$M =$ number of strategies;
J_m = set of managers using m -th strategy, $m = 1,\ldots,M$;
b_m^l = lower bound in the constraints on allocations to strategies, $m = 1,\ldots,M$;
b_m^u = upper bound in the constraints on allocations to strategies, $m = 1,\ldots,M$;
l_i = lower bound in the box constraints for individual positions, $i = 1, 2,\ldots,I$;
u_i = upper bound in the box constraints for individual positions, $i = 1, 2,\ldots,I$;

Omega Maximization Problem

$$\max \Omega(w) = 1 + \max \frac{q(w)}{\underline{\underline{\eta}}(w)} \tag{CS.1}$$

subject to

budget constraint

$$\sum_{i=1}^{I} w_i = 1 \tag{CS.2}$$

constraints on allocations to strategies

$$b \le \sum_{\in} w \le b \qquad m = \qquad \ldots M \tag{CS.3}$$

box constraints for individual positions

$$l_i \le w_i \le u_i, \; i = 1,2,\ldots I \tag{CS.4}$$

Reduction Omega Optimization to Partial Moment Optimization

Problem (CS.1)-(CS.4) can be reduced to a convex optimization programming problem by introducing an additional variable and restriction. Let $x_0 > 0$ such that

$$x_0 \underline{\underline{\eta}}(w) = \underline{\underline{\eta}}(x_0 w) = 1 . \tag{CS.5}$$

The following equality is valid

$$\frac{q(w)}{\underline{\underline{\eta}}(w)} = \frac{x_0 q(w)}{x_0 \underline{\underline{\eta}}(w)} = \frac{q(x_0 w)}{\underline{\underline{\eta}}(x_0 w)} . \tag{CS.6}$$

Denote $x_i = x_0 w_i, \; i = 1,2,\ldots,I, \quad x = (x_0, x_1, \ldots, x_I)$. The problem (CS.1)-(CS.4) can be reduced to the following problem.

Maximizing Expected Gain Subject to Constraint on Partial Moment

$$\max[1 + q(x)] = 1 + \max AVG\_GAIN(L(x,t)) \tag{CS.7}$$

subject to

downside loss constraint

$$\underline{\underline{\eta}}(x) = PM\_PEN(L(x,t)) \le 1, \tag{CS.8}$$

budget constraint

$$\sum_{i=1}^{I} x_i = x_0, \tag{CS.9}$$

constraints on allocations to strategies

$$b_m^l x_0 \le \sum_{j \in J_m} x_j \le b_m^u x_0, \quad m = 1,2,\ldots M, \tag{CS.10}$$

constraints on allocations to individual managers

$$x_0 l_i \le x_i \le x_0 u_i, \; i = 1,2,\ldots I. \tag{CS.11}$$

constraints on variables

$$x_0 > 0, \; x_i \ge 0, \quad i = 1,2,\ldots,I . \tag{CS.12}$$

Let us denote by $x^* = (x_0^*, x_1^*, \ldots, x_I^*)$ an optimal solution of problem (CS.7)-(CS.12). An optimal solution of the original problem (CS.1)-(CS.4) equals $w_i^* = \frac{x_i^*}{x_0^*}, i = 1,2,\ldots,I$.

With expression (CS.9) we excluded variable x_0^* from problem (CS.7)- (CS.12) and obtained the following simplified formulation:

Maximizing Expected Gain Subject to Constraint on Partial Moment (Simplified Formulation)

$$\max[1+q(x)] = 1 + \max AVG\_GAIN(L(x,t)) \tag{CS.13}$$

subject to

downside loss constraint

$$\underline{\underline{\eta}}(x) = PM\_PEN(L(x,t)) \le 1, \tag{CS.14}$$

constraints on allocations to strategies

$$b_m^l \sum_{i=1}^{I} x_i \le \sum_{j \in J_m} x_j \le b_m^u \sum_{i=1}^{I} x_i, \quad m = 1,2,\ldots M, \tag{CS.15}$$

constraints on allocations to individual managers

$$l_i \sum_{k=1}^{I} x_k \le x_i \le u_i \sum_{k=1}^{I} x_k, \; i = 1,2,\ldots I. \tag{CS.16}$$

constraints on variables

$$x_i \ge 0, \quad i = 1,2,\ldots,I. \tag{CS.17}$$

An optimal decision vector of the original problem (CS.1)-(CS.4) equals

$$w_i^* = \frac{x_i^*}{\sum_{i=1}^{I} x_i}, i = 1,2,\ldots,I. \tag{CS.18}$$

Constraints (CS.15) can be transformed to the following linear constraints

$$\sum_{i=1}^{I} (\delta_{im} - b_m^l) x_i \ge 0, \quad m = 1,2,\ldots M, \tag{CS.15-1}$$

$$\sum_{i=1}^{I} (\delta_{im} - b_m^u) x_i \le 0, \quad m = 1,2,\ldots M, \tag{CS.15-2}$$

where

$$\delta_{im} = \begin{cases} 1, \text{ if } i \in J_m \\ 0, \text{ otherwise} \end{cases}.$$

Constraints (CS.16) can be transformed to the following linear constraints

$$\sum_{k=1}^{I} (\tilde{\delta}_{ik} - l_i) x_k \ge 0, \; i = 1,2,\ldots I \tag{CS.16-1}$$

$$\sum_{k=1}^{I} (\tilde{\delta}_{ik} - u_i) x_k \le 0, \; i = 1,2,\ldots I, \tag{CS.16-2}$$

where

$$\tilde{\delta}_{ik} = \begin{cases} 1, \text{ if } i = k \\ 0, \text{ otherwise} \end{cases}.$$