Appendix I: Mathematical Definition of Functions

Note:
To change the product logo for your own print manual or PDF, click "Tools > Manual Designer" and modify the print manual template.

Table of Contents

1 Appendix I: Mathematical Definition of Functions

1.1 Deterministic Function

1.1.1 Nonlinear Group

The Nonlinear Group consists of the following functions:

- Polynomial Absolute (software notation: polymon_abs...) (section **Polynomial Absolute)**
- Relative Entropy Function (software notation: entropyr_...) (section **Relative Entropy Function)**
- CVaR Component Positive (software notation: cvar_comp_pos_...) (section **CVaR Component Positive)**
- CVaR Component Negative (software notation: cvar_comp_neg_...) (section **CVaR Component Negative)**
- VaR Component Positive (software notation: var_comp_pos_...) (section **VaR Component Positive)**
- VaR Component Negative (software notation: var_comp_neg_...) (section **VaR Component Negative)**
- Maximum Component Positive (software notation: max_comp_pos_...) (section **Maximum Component Positive)**
- Maximum Component Negative (software notation: max_comp_neg_...) (section **Maximum Component Negative)**
- Quadratic function (software notation: quadratic_...) (section **Quadratic function)**
- Logarithms Sum (software notation: log_sum_...) (section **Logarithms Sum)**
- Logarithms Exponents Sum (software notation: logexp_sum_...) (section **Logarithms Exponents Sum)**

For more details about the Properties of this Group see the section **Properties of Nonlinear Group**.

The Polynomial Absolute function is the generalization of the Sum of Absolute Values function.

The Relative Entropy function is defined on some Point $\vec{x} = (x_1, x_2, \dots, x_I)$ and matrix of scenarios with one row.

Functions depend on the parameter w (threshold value) and are defined on some Point , $\vec{x} = (x_1, x_2, \dots, x_I)$, and the Matrix of Scenarios (in regular Matrix or packed in Pmatrix format) or Simmetric Matrix (Smatrix).

1.1.1.1 Polynomial Absolute (polynom_abs)

Given the following special type of matrix of scenarios with three rows

$$\begin{pmatrix} id & name1 & name2 & \dots & nameI & scenario\_benchmark \\ 1 & \eta_1 & \eta_2 & \cdots & \eta_I & \eta_0 \\ 2 & y_1 & y_2 & \cdots & y_I & y_0 \\ 3 & q_1 & q_2 & \cdots & q_I & q_0 \end{pmatrix}$$

The **Polynomial Absolute function** is calculated as follows:

$$\mathbf{polynom\_abs}(\vec{x}) = \eta_0 + \sum_{i=1}^{I} \eta_i \left| x_i - y_i \right|^{q_i},$$

where

$$q_i \geq 1,\ i = 1,\dots,I.$$

The first row (id = 1) can not be empty. If the second row is empty (id = 2), then set $y_i = 0$, $i = 1,\dots I$. If the second row is empty (id = 2), then the third row (id = 3) must be empty; if the third row is empty (id = 3), then set q_i=1, $i = 1,\dots I$. The column "scenario_benchmark" may be included in the matrix or omitted. If it is included in the matrix, only the value η_0 (if any) is used for calculating **polynom_abs.** Other values in the column "scenario_benchmark" (if any) are ignored.

1.1.1.2 Relative Entropy (entropyr)

Given the matrix of scenarios

$$\begin{pmatrix} id & scenario\_probability & scenario\_benchmark & name1 & name2 & \dots nameI \\ 1 & & & \theta_1 & \theta_2 & \cdots \ \theta_I \end{pmatrix}$$

with positive components, the **Relative Entropy** is calculated as follows:

$$\text{entropyr}(\vec{x}) = \sum_{i=1}^{I} x_i \ln\left(\frac{x_i}{\theta_i}\right)$$

where $x_i > 0,\ i = 1,\dots,I$.

This function is usually used with additional constraint

$$\sum_{i=1}^{I} x_i = 1$$

The Relative Entropy function can be used in linear combination with any other function that do not belong to the probability group. However, if you wish to accelerate optimization process with Relative Entropy function in objective, this function should be stand alone and linearized (see the section "**Problems Mode in Shell Environment**", subsection "**Add Function to Objective in Problems Mode**"). In this case number of decision variables may go up to 1,0000,0000, and BULDOZER solver is recommended.

1.1.1.3 CVaR Component Positive (cvar_comp_pos)

Given the following matrix with one row containing positive values

$$\begin{pmatrix} id & name1 & name2 & \ldots & nameI \\ 1 & \eta_1 & \eta_2 & \ldots & \eta_I \end{pmatrix}$$

and a point $\vec{x}=(x_1,\ldots,x_I)$,

arrange values $\{\eta_i x_i\}_{i=1,\ldots,I}$ in ascending order:

$\eta_{i_1} x_{i_1} \le \eta_{i_2} x_{i_2} \le \cdots \le \eta_{i_I} x_{i_I}$.

Let $\frac{1}{I} \le \alpha \le \frac{I-1}{I}$ be a confidence level.

Let us denote by $l(\alpha)$ an index such that $l(\alpha) > \alpha \cdot I$ and $l(\alpha)-1 \le \alpha \cdot I$. Let $l^* = \min l : \eta_{i_{l+1}} x_{i_{l+1}} = \eta_{i_I} x_{i_I}$. If the index $l(\alpha)$ is such that the confidence level $\alpha \le \frac{l^*-1}{I}$ and $\frac{l(\alpha)-1}{I} = \alpha$, then, **CVaR Component Positive** equals

$$\text{cvar\_comp\_pos}_\alpha(\vec{x}) = \frac{1}{I(1-\alpha)} \sum_{l(\alpha)\le l\le I} \eta_{i_l} x_{i_l}.$$

If $\frac{l^*-1}{I} < \alpha$, then $\text{cvar\_comp\_pos}_\alpha(\vec{x}) = \text{max\_comp\_pos}(\vec{x}) = \eta_{i_I} x_{i_I}$.

Let $\frac{1}{I} \le \alpha \le \frac{l^*-1}{I}$.

If $\frac{l(\alpha)-1}{I} < \alpha$ then CVaR Component Positive equals linear interpolation between CVaRs Component Positive with confidence levels $\underline{\alpha} = \frac{l(\alpha)-1}{I}$ and $\bar{\alpha} = \frac{l(\alpha)}{I}$, i.e.

$$\text{cvar\_comp\_pos}_\alpha(\vec{x}) = \frac{\bar{\alpha}-\alpha}{\bar{\alpha}-\underline{\alpha}} \cdot \frac{1-\underline{\alpha}}{1-\alpha} \text{cvar\_comp\_pos}_{\underline{\alpha}}(\vec{x}) + \frac{\alpha-\underline{\alpha}}{\bar{\alpha}-\underline{\alpha}} \cdot \frac{1-\bar{\alpha}}{1-\alpha} \text{cvar\_comp\_pos}_{\bar{\alpha}}(\vec{x})$$

1.1.1.4 CVaR Component Negative (cvar_comp_neg)

Given the following matrix with one row containing positive values

$$\begin{pmatrix} id & name1 & name2 & \ldots & nameI \\ 1 & \eta_1 & \eta_2 & \ldots & \eta_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \ldots, x_I)$,
CVaR Component Negative is calculated as follows:

$$\mathrm{cvar\_comp\_neg}_{\alpha}(\vec{x}) = \mathrm{cvar\_comp\_pos}_{\alpha}(-\vec{x})$$

1.1.1.5 VaR Component Positive (var_comp_pos)

Given the following matrix with one row containing positive values

$$\begin{pmatrix} id & name1 & name2 & \ldots & nameI \\ 1 & \eta_1 & \eta_2 & \cdots & \eta_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \ldots, x_I)$,
arrange values $\{\eta_i x_i\}_{i=1,\ldots I}$ in ascending order:
$\eta_{i_1} x_{i_1} \le \eta_{i_2} x_{i_2} \le \cdots \le \eta_{i_I} x_{i_I}$.

Let α be a confidence level. If $1 \ge \alpha > \frac{I-1}{I}$, then VaR Component Positive equals

$$\mathrm{var\_comp\_pos}_{\alpha}(\vec{x}) = \eta_{i_I} x_{i_I} \ .$$

Let $\frac{1}{I} \le \alpha \le \frac{I-1}{I}$ be a confidence level.
Let us denote by $l(\alpha)$ an index such that $l(\alpha) > \alpha \cdot I$ and $l(\alpha) - 1 \le \alpha \cdot I$. If the index $l(\alpha)$ >1 is such that the confidence level α equals $\frac{l(\alpha)-1}{I} = \alpha$, then, discrete VaR Component Positive equals

$$\mathrm{var\_comp\_pos}_{\alpha}(\vec{x}) = \eta_{i_{l(\alpha)-1}} x_{i_{l(\alpha)-1}}$$

If $l(\alpha)$ =1 then $\mathrm{var\_comp\_pos}_{\alpha}(\vec{x}) = \eta_{i_1} x_{i_1}$.

If $\frac{l(\alpha)-1}{I} < \alpha$ then VaR Component Positive equals linear interpolation between VaRs Component Positive with confidence levels $\underline{\alpha} = \frac{l(\alpha)-1}{I}$ and $\bar{\alpha} = \frac{l(\alpha)}{I}$, i.e.,

$$\mathrm{var\_comp\_pos}_{\alpha}(\vec{x}) = \frac{\bar{\alpha} - \alpha}{\bar{\alpha} - \underline{\alpha}} \mathrm{var\_comp\_pos}_{\underline{\alpha}}(\vec{x}) + \frac{\alpha - \underline{\alpha}}{\bar{\alpha} - \underline{\alpha}} \mathrm{var\_comp\_pos}_{\bar{\alpha}}(\vec{x})$$

1.1.1.6 VaR Component Negative (var_comp_neg)

Given the following matrix with one row containing positive values

$$\begin{pmatrix} id & name1 & name2 & \ldots & nameI \\ 1 & \eta_1 & \eta_2 & \ldots & \eta_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \ldots, x_I)$,

VaR Component Negative equals

$$\text{var\_comp\_neg}_{\alpha}(\vec{x}) = \text{var\_comp\_pos}_{\alpha}(-\vec{x})$$

1.1.1.7 Maximum Component Positive (max_comp_pos)

Given the following matrix with one row containing positive values

$$\begin{pmatrix} id & name1 & name2 & \ldots & nameI \\ 1 & \eta_1 & \eta_2 & \ldots & \eta_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \ldots, x_I)$,

arrange values $\{\eta_i x_i\}_{i=1,\ldots I}$ in ascending order:

$\eta_{i_1} x_{i_1} \le \eta_{i_2} x_{i_2} \le \cdots \le \eta_{i_I} x_{i_I}$.

The **Maximum Component Positive** function is calculated as follows:

$$\text{max\_comp\_pos}(\vec{x}) = \eta_{i_I} x_{i_I} \ .$$

1.1.1.8 Maximum Component Negative (max_comp_neg)

Given the following matrix with one row containing positive values

$$\begin{pmatrix} id & name1 & name2 & \ldots & nameI \\ 1 & \eta_1 & \eta_2 & \ldots & \eta_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \ldots, x_I)$,

arrange values $\{\eta_i x_i\}_{i=1,\ldots I}$ in ascending order:

$\eta_{i_1} x_{i_1} \le \eta_{i_2} x_{i_2} \le \cdots \le \eta_{i_I} x_{i_I}$.

The **Maximum Component Negative** function is calculated as follows:

$$\text{max\_comp\_neg}(\vec{x}) = -\eta_{i_1} x_{i_1} \ .$$

1.1.1.9 Quadratic function (quadratic)

Given the matrix

$$A = \begin{pmatrix} id & name1 & \dots & nameI & \textbf{benchmark\_scenario} \\ 1 & a_{11} & \cdots & a_{1I} & a_{01} \\ \dots & \dots & \dots & \dots & \dots \\ I & a_{I1} & \cdots & a_{II} & a_{0I} \end{pmatrix}$$

and a point $\vec{x} = (x_1, \dots, x_I)$, the **quadratic** function is calculated as follows:

$$\text{quadratic}(\vec{x}) = \sum_{i=1}^{I} a_{0i} x_i + \sum_{i=1}^{I} \sum_{k=1}^{I} a_{ik} x_i x_k \ .$$

1.1.1.10 Logarithms Sum (log_sum)

Given the following matrix with one row containing positive values

$$\begin{pmatrix} id & name1 & name2 & \dots & nameI \\ 1 & \eta_1 & \eta_2 & \dots & \eta_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \dots, x_I)$ with strictly positive components,

the **Logarithms Sum** function is calculated as follows:

$$\text{log\_sum}(\vec{x}) = \sum_{i=1}^{I} \eta_i \, ln(x_i)$$

1.1.1.11 Logarithms Exponents Sum (logexp_sum)

Given the following matrix of scenarios

$$\begin{pmatrix} id & name1 & name2 & \dots & nameI & \textbf{probability\_scenario} & \textbf{benchmark\_scenario} \\ 1 & \theta_{11} & \theta_{12} & \cdots & \theta_{1I} & p_1 & \theta_{10} \\ 2 & \theta_{21} & \theta_{22} & \cdots & \theta_{2I} & p_2 & \theta_{20} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ J & \theta_{J1} & \theta_{J2} & \cdots & \theta_{JI} & p_J & \theta_{J0} \end{pmatrix},$$

where $\theta_{j0} = \{0, 1\}$, and a point $\vec{x} = (x_1, \dots, x_I)$,

the **Logarithms Exponents Sum** function is calculated as follows:

$$\textbf{logexp\_sum}(\vec{x}) = -\sum_{j=1}^{J} p_j \left[-\theta_{j0} \boldsymbol{X}_j + ln\left\{ 1 + \exp\left(\boldsymbol{X}_j \right) \right\} \right],$$

where

$$\boldsymbol{X}_j(\vec{x}) = \sum_{i=1}^{I} \theta_{ji} x_i$$

.

1.1.1.12 Properties of Nonlinear Group

Functions from the Nonlinear group are calculated with double precision. The name of any function from this group may contain up to 128 symbols. The name of the Polynomial Absolute function should begin with the string "polynom_abs_". The name of the Relative Entropy function should begin with the string "entropyr_". The name of the CVaR Component Positive function should begin with the string "cvar_comp_pos_". The name of the CVaR Component Negative function should begin with the string "cvar_comp_neg_". The name of the VaR Component Positive function should begin with the string "var_comp_pos_". The name of the VaR Component Negative function should begin with the string "var_comp_neg_". The name of the Maximum Component Positive function should begin with the string "max_comp_pos_". The name of the Maximum Component Negative function should begin with the string "max_comp_neg_". The name of the Quadratic function function should begin with the string "quadratic_". The name of the Log_sum function should begin with the string "log_sum_". The name of the Logexp_sum function function should begin with the string "logexp_sum_". The name of these functions may include only alphabetic characters, numbers, and the underscore sign, "_".
The names of these functions are "insensitive" to the case, i.e. there is no difference between low case and upper case in these names.

1.1.2 Cardinality Group

Functions from this group are defined on some Point , $\vec{x} = (x_1, x_2, \ldots, x_I)$, and a special types of the Matrix of Scenarios .

This group includes the following functions:

- Cardinality Positive (software notation: cardn_pos _...) (section **Cardinality Positive**)
- Cardinality Negative (software notation: cardn_neg _...) (section **Cardinality Negative**)
- Cardinality (software notation: cardn_...) (section **Cardinality**)
- Buyin Positive (software notation: buyin_pos_...) (section **Buyin Positive**)
- Buyin Negative (software notation: buyin_neg_...) (section **Buyin Negative**)
- Buyin (software notation: buyin_...) (section **Buyin**)
- Fixed Charge Positive (software notation: fxchg_pos_...) (section **Fixed Charge Positive**)
- Fixed Charge Negative (software notation: fxchg_neg_...) (section **Fixed Charge Negative**)
- Fixed Charge (software notation: fxchg_...) (section **Fixed Charge**)

For more details about the Properties of this Group see the section **Properties of Cardinality Group**.

1.1.2.1 Cardinality Positive (cardn_pos)

Given the following matrix with one row containing positive values

$$\begin{pmatrix} id & name1 & name2 & \dots & nameI \\ 1 & \eta_1 & \eta_2 & \dots & \eta_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \dots x_I)$,

the **Cardinality Positive** function is calculated as follows:

$$\text{cardn\_pos}(\vec{x}, w) = \sum_{i=1}^{I} g\big((\eta_i \cdot x_i), w\big)$$

where

$$g(y, w) = \begin{cases} 1, & \text{if } y \geq w \\ 0, & \text{otherwise} \end{cases};$$

w is a threshold value $(w \geq 0)$.

It is recommended to select the multiplier row equal to the unit vector and $w = 10^{-6}$ * **(units of decision variables)**. For instance if decision variables are measured in thousands than $w = 10^{-6}$ *(1,000) = 10^{-3}.

1.1.2.2 Cardinality Negative (cardn_neg)

Given the following matrix with one row containing positive values

$$\begin{pmatrix} id & name1 & name2 & \dots & nameI \\ 1 & \eta_1 & \eta_2 & \dots & \eta_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \dots x_I)$,

the **Cardinality Negative** function is calculated as follows:

$$\text{cardn\_neg}(\vec{x}, w) = \sum_{i=1}^{I} h((\eta_i \cdot x_i), w)$$

where

$$h(y, w) = \begin{cases} 1, & \text{if } y \leq -w \\ 0, & \text{otherwise} \end{cases};$$

w is a threshold value $(w \geq 0)$.

It is recommended to select the multiplier row equal to the unit vector and $w = 10^{-6}$ * **(units of decision variables)**. For instance if decision variables are measured in thousands than $w = 10^{-6}$ *(1,000)= 10^{-3}.

1.1.2.3 Cardinality (cardn)

Given the following matrix with two rows containing positive values

$$\begin{pmatrix} id & name1 & name2 & \ldots & nameI \\ 1 & a_1 & a_2 & \ldots & a_I \\ 2 & b_1 & b_2 & \ldots & b_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \ldots x_I)$,

the **Cardinality** function is calculated as follows:

$$\mathbf{cardn}(\vec{x}, w) = \sum_{i=1}^{I} g((a_i \cdot x_i), w) + \sum_{i=1}^{I} h((b_i \cdot x_i), w),$$

where

$$g(y, w) = \begin{cases} 1, & \text{if } y \geq w \\ 0, & \text{otherwise} \end{cases};$$

$$h(y, w) = \begin{cases} 1, & \text{if } y \leq -w \\ 0, & \text{otherwise} \end{cases};$$

w is a threshold value $(w \geq 0)$.

The second row in the matrix is optional. If it is not available, then by default it coincides with the first row.

It is recommended to select the multiplier rows equal to the unit vector and $w = 10^{-6}$ *** (units of decision variables)**. For instance if decision variables are measured in thousands than $w = 10^{-6}$ ***(1,000)=** 10^{-3}**.**

1.1.2.4 Buyin Positive (buyin_pos)

Given the following matrix with two rows containing positive values

$$\begin{pmatrix} id & name1 & name2 & \ldots & nameI \\ 1 & a_1 & a_2 & \ldots & a_I \\ 2 & \eta_1 & \eta_2 & \ldots & \eta_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \ldots x_I)$,

the Buyin Positive function is calculated as follows:

$$\mathbf{buyin\_pos}(\vec{x}, w) = \sum_{i=1}^{I} g((\eta_i \cdot x_i), a_i, w),$$

where

$$g(y,a,w)=\begin{cases}1, & \textbf{if } w \le y \le a\\ 0, & \textbf{otherwise}\end{cases};$$

w is a threshold value $(w \ge 0)$.

The second row in the matrix is optional. If it is not available, then by default all values of $\eta_1, \eta_2, ..., \eta_I$ are equal to 1.

It is recommended to select the multiplier row equal to the unit vector and $w = 10^{-6}$ * **(units of decision variables)**. For instance if decision variables are measured in thousands than $w = 10^{-6}$ *(1,000)= 10^{-3}.

1.1.2.5 Buyin Negative (buyin_neg)

Given the following matrix with two rows containing positive values

$$\begin{pmatrix} id & name1 & name2 & \ldots & nameI \\ 1 & a_1 & a_2 & \ldots & a_I \\ 2 & \eta_1 & \eta_2 & \ldots & \eta_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \ldots x_I)$,

the Buyin Negative function is calculated as follows:

$$\textbf{buyin\_neg}(\vec{x}, w) = \sum_{i=1}^{I} h((\eta_i \cdot x_i), a_i, w),$$

where

$$h(y,a,w)=\begin{cases}1, & \textbf{if } \text{-}a \le y \le -w\\ 0, & \textbf{otherwise}\end{cases};$$

w is a threshold value $(w \ge 0)$.

The second row in the matrix is optional. If it is not available, then by default all values of $\eta_1, \eta_2, ..., \eta_I$ are equal to 1.

It is recommended to select the multiplier row equal to the unit vector and $w = 10^{-6}$ * **(units of decision variables)**. For instance if decision variables are measured in thousands than $w = 10^{-6}$ *(1,000)= 10^{-3}.

1.1.2.6 Buyin (buyin)

Given the following matrix with four rows containing positive values

$$\begin{pmatrix} id & name1 & name2 & \ldots & nameI \\ 1 & a_1 & a_2 & \ldots & a_I \\ 2 & b_1 & b_2 & \ldots & b_I \\ 3 & \eta_1 & \eta_2 & \ldots & \eta_I \\ 4 & \gamma_1 & \gamma_2 & \ldots & \gamma_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \ldots x_I)$,

the Buyin function is calculated as follows:

$$\textbf{buyin}(\vec{x}, w) = \sum_{i=1}^{I} g((\eta_i \cdot x_i), a_i, w) + \sum_{i=1}^{I} h((\gamma_i \cdot x_i), b_i, w),$$

where

$$g(y, a, w) = \begin{cases} 1, & \text{if } w \le y \le a \\ 0, & \text{otherwise} \end{cases};$$

$$h(y, b, w) = \begin{cases} 1, & \text{if } -b \le y \le -w \\ 0, & \text{otherwise} \end{cases};$$

w is a threshold value $(w \geq 0)$.

The matrix rows 2, 3, and 4 are optional. If the second row is not present, then by default it is equal to the first row. If the fourth row is not present, then by default it consists of ones. If the third row is not present, then by default it consists of ones.

It is recommended to select the last two rows equal to the unit vector and $w = 10^{-6}$ *** (units of decision variables)**. For instance if decision variables are measured in thousands than $w = 10^{-6}$ ***(1,000)=** 10^{-3}.

1.1.2.7 Fixed Charge Positive (fxchg_pos)

Given the following matrix with two rows containing positive values

$$\begin{pmatrix} id & name1 & name2 & \ldots & nameI \\ 1 & \lambda_1 & \lambda_2 & \ldots & \lambda_I \\ 2 & \eta_1 & \eta_2 & \ldots & \eta_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \ldots x_I)$,

the Fixed Charge Positive function is calculated as follows:

$$\text{fxchg\_pos}(\vec{x}, w) = \sum_{i=1}^{I} \lambda_i g((\eta_i \cdot x_i), w),$$

where

$$g(y, w) = \begin{cases} 1, & \text{if } y \ge w \\ 0, & \text{otherwise} \end{cases};$$

w is a threshold value $(w \geq 0)$.

The second row in the matrix is optional. If it is not specified, then by default all values of $\eta_1, \eta_2, \ldots, \eta_I$ are equal to 1.

It is recommended to select the last row equal to the unit vector and $w = 10^{-6}$ *(units of decision variables). For instance if decision variables are measured in thousands than $w = 10^{-6}$ *(1,000)= 10^{-3}.

1.1.2.8 Fixed Charge Negative (fxchg_neg)

Given the following matrix with two rows containing positive values

$$\begin{pmatrix} id & name1 & name2 & \ldots & nameI \\ 1 & \gamma_1 & \gamma_2 & \cdots & \gamma_I \\ 2 & \eta_1 & \eta_2 & \cdots & \eta_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \ldots x_I)$,

the Fixed Charge Negative function is calculated as follows:

$$\text{fxchg\_neg}(\vec{x}, w) = \sum_{i=1}^{I} \gamma_i h((\eta_i \cdot x_i), w),$$

where

$$h(y, w) = \begin{cases} 1, & \text{if } y \le -w \\ 0, & \text{otherwise} \end{cases};$$

w is a threshold value $(w \ge 0)$.

The second row in the matrix is optional. If it is not available, then by default all values of $\eta_1, \eta_2, \ldots, \eta_I$ are equal to 1.

It is recommended to select the last row equal to the unit vector and $w = 10^{-6}$ *(units of decision variables). For instance if decision variables are measured in thousands than $w = 10^{-6}$ *(1,000)= 10^{-3}.

1.1.2.9 Fixed Charge (fxchg)

Given the following matrix with five rows containing positive values

$$\begin{pmatrix} id & name1 & name2 & \ldots & nameI \\ 1 & \lambda_1 & \lambda_2 & \cdots & \lambda_I \\ 2 & \gamma_1 & \gamma_2 & \cdots & \gamma_I \\ 3 & a_1 & a_2 & \cdots & a_I \\ 4 & b_1 & b_2 & \cdots & b_I \end{pmatrix}$$

and a point $\vec{x} = (x_1, \ldots x_I)$,

the Fixed Charge function is calculated as follows:

$$\mathrm{fxchg}(\vec{x}, w) = \sum_{i=1}^{I} \lambda_i g((a_i \cdot x_i), w) + \sum_{i=1}^{I} \gamma_i h((b_i \cdot x_i), w),$$

where

$$g(y, w) = \begin{cases} 1, & \text{if } y \geq w \\ 0, & \text{otherwise} \end{cases};$$

$$h(y, w) = \begin{cases} 1, & \text{if } y \leq -w \\ 0, & \text{otherwise} \end{cases};$$

w is a threshold value $(w \geq 0)$.

The matrix rows 2, 3, and 4 are optional. If the second row is not present, then by default it is equal to the first row. If the third row is not present, then by default it consists of ones. If the fourth row is not present, then by default it consists of ones.

It is recommended to select two last rows equal to the unit vector and $w = 10^{-6}$ *** (units of decision variables)**. For instance if decision variables are measured in thousands than $w = 10^{-6}$ ***(1,000)=** 10^{-3}**.**

1.1.2.10 Properties of Cardinality Group

Functions from Cardinality group are calculated with double precision. The name of any function from this group may contain up to 128 symbols. The name of the Cardinality Positive function should begin with the string “cardn_pos_”, the name of the Cardinality Negative function should begin with the string “cardn_neg_”, the name of the Cardinality function should begin with the string “cardn_”, the name of the Buyin Positive function should begin with the string “buyin_pos_”, the name of the Buyin Negative function should begin with the string “buyin_neg_”, the name of the Buyin function should begin with the string “buyin_”, the name of the Fixed Charge Positive function should begin with the string “fxchg_pos_”, the name of the Fixed Charge Negative function should begin with the string “fxchg_neg_”, the name of the Fixed Charge function should begin with the string “fxchg_”. The names of these functions are "insensitive" to the case, i.e. there is no difference between low case and upper case in these names.

1.2 Risk Functions Defined by Matrix of Scenarios

In the discrete case, i.e., when models are based on a finite number of probabilistic scenarios, the Risk Function is defined on a Matrix of Scenarios. A Risk Function is some statistical characteristic calculated with a probability distribution of the Loss (Gain) Function. Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector, for instance, it is a vector of portfolio exposures. Given a Matrix of Scenarios, a Loss Function which is a random value is defined as

$$L(\vec{x}, \vec{\theta}) = L(\vec{x}, \theta_0, \theta_1 \dots, \theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

The loss function has J scenarios, $L(\vec{x}, \theta_{10}, \theta_{11}, \cdots \theta_{1I}), \dots, L(\vec{x}, \theta_{J0}, \theta_{J1}, \cdots \theta_{JI})$,with probabilities, $p_j, \ j = 1, \dots, J$.

It is supposed that the performance of a model is described by a linear function

$$\sum_{i=1}^{I}\theta_i x_i$$

and a benchmark performance is described by θ_0. The loss function can be interpreted as an underperformance of the outcome

$$\sum_{i=1}^{I}\theta_i x_i \ ,$$

compared to the benchmark θ_0.

For instance, suppose that a portfolio contains ***i*=1,...,*I*** instruments. Component, θ_i, in this case, denotes a random return of the ***i***-th instrument and their possible realizations are given by ***J*** scenarios, $\theta_{1i},\ldots,\theta_{Ji}$.

The component, θ_0, denotes the return of some benchmark (e.g., return of an index). The loss function, in this case, is an underperformance of the portfolio return,

$$\sum_{i=1}^{I}\theta_i x_i \ ,$$

compared to the benchmark, θ_0. Discrete scenarios can be used to approximate the continuous case. If the original problems can be described by a continuous distribution, it is assumed that the random vector, $\vec{\theta}=(\theta_0,\theta_1,\ldots\theta_I)$,has a smooth probability density, $p(\vec{\theta})$, inducing the following probability distribution function of the loss $L(\vec{x},\vec{\theta})$

$$\psi(\vec{x},\zeta)=\int_{L(\vec{x},\vec{\theta})\le\zeta} p(\vec{\theta})d\vec{\theta} \quad .$$

ψ is the cumulative distribution function with the parameter ζ for the loss associated with the decision $\vec{x}$.

The Gain Function is defined as

$$G(\vec{x},\vec{\theta})=L(\vec{x},-\vec{\theta})=-L(\vec{x},\vec{\theta})=-\theta_0+\sum_{i=1}^{I}\theta_i x_i \quad .$$

Portfolio management methodologies rely on some measures of risk impacting allocation of instruments in the portfolio. Financial risks involve variability of returns leading to potentially worse or better than expected returns. The classical Markowitz portfolio theory identifies risk with the volatility (Standard Deviation) of a portfolio. However, many applications involve other measures of deviation such as the downside standard deviation, mean absolute deviation and the CVaR deviation. Moreover, probability and quantile (percentile) functions are commonly used for analysis of models.

The following notations are used in PSG to identify risk and deviation measures for probabilistic distributions:

- We use the notion Risk for Loss for probabilistic characteristics measuring the magnitude of losses $L(\vec{x},\vec{\theta})$. For a specific characteristic we say “<name of characteristic> Risk for Loss” and use the software notation “<name of characteristic>_RISK”. For example, if the CVaR of the probability distribution is considered as a measure of risk then it is called the CVaR Risk for Loss and is denoted by CVaR_RISK.
- We use the notion Deviation for Loss for probabilistic characteristics measuring the width of the distribution of the losses. For all the deviations of losses considered in PSG (except CDaR deviation, Drawdown Deviation Maximum, and Drawdown Deviation Average) the deviations are calculations through the distance of the loss function from it’s average. Here is the formula for the distance from the average depending upon a random outcome:

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I}(\theta_i - E[\theta_i])x_i \quad .$$

For a specific characteristic we say "<name of characteristic> Deviation for Loss" and use the software notation "<name of characteristic>_DEV". For example, if the Maximum distance of the loss from it's mean value is considered as a measure of the deviation, then the appropriate deviation characteristics is called the Maximum Deviation for Loss and is denoted by MAX_DEV.

- We use the notion Risk for Gain for probabilistic characteristics measuring the magnitude of the gains $G(\vec{x},\vec{\theta})$. For a specific characteristic we say "<name of characteristic> Risk for Gain" and use the software notation "<name of characteristic>_RISK_G". For example, for VaR the appropriated gain measure is called VaR Risk for Gain and is denoted by VaR_RISK_G.
- We use the notion Deviation for Gain for probabilistic characteristics measuring the width of the distribution of the gains. For all deviations of gains considered in PSG, (except CDaR deviation, Drawdown Deviation Maximum, and Drawdown Deviation Average) the deviations are calculated through the distance of the gain function from it's average. Here is the formula for the distance from the average depending upon a random outcome:

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = -(\theta_0 - E[\theta_0]) + \sum_{i=1}^{I}(\theta_i - E[\theta_i])x_i \quad .$$

For a specific characteristic we say "<name of characteristic> Deviation for Gain" and use the software notation "<name of characteristic>_DEV." For example, if the VaR characteristic of the probability distribution is considered as a measure of the deviation then it is called the VaR Deviation for Gain and is denoted VaR_DEV_G.

References

1. Rockafellar, R. T., Uryasev, S. and M. Zabarankin (2006): Generalized Deviations in Risk Analysis. *Finance and Stochastics*, Vol. 10, pp. 51-74.
2. Uryasev, S. (2000): Introduction to the Theory of Probabilistic Functions and Percentiles (Value-at-Risk). Uryasev, S. (Ed.) Probabilistic Constrained Optimization: Methodology and Applications. Kluwer Academic Publishers, pp. 1-25.

Risk and Deviation functions are divided into the following groups:

- **Average Group**
- **CVaR Group**
- **VaR Group**
- **Maximum Group**
- **Mean Abs Group**
- **Partial Moment Group**
- **Probability Group**
- **CDaR Group**
- **Standard Group**
- **Utilities Group**

1.2.1 Average Group (avg avg_g)

Functions from this group are used for calculating the average of the probability distribution of the loss (gain) function. The Average Group includes two functions:

- Average Loss function (software notation: avg_...)
- Average Gain function (software notation: avg_g_...)

These functions are defined on some Point,
$\vec{x} = (x_1, x_2, \ldots, x_I)$, and the Matrix of Scenarios as follows:

$$\mathbf{avg}\left(L\left(\vec{x},\vec{\theta}\right)\right) = E\left[L\left(\vec{x},\vec{\theta}\right)\right] = \sum_{j=1}^{J} p_j L(\vec{x},\vec{\theta}_j),$$

$$\mathbf{avg\_g}\left(G\left(\vec{x},\vec{\theta}\right)\right) = E\left[G\left(\vec{x},\vec{\theta}\right)\right] = \sum_{j=1}^{J} p_j G(\vec{x},\vec{\theta}_j) = -\sum_{j=1}^{J} p_j L(\vec{x},\vec{\theta}_j),$$

here:

E denotes the expectation sign;

random vector $\vec{\theta}$ has components $(\theta_0, \theta_1, \ldots \theta_I)$ and J vector scenarios, $\{\vec{\theta}_1, \ldots, \vec{\theta}_J\}$;

random value θ_i, which is the i-th component of the random vector, $\vec{\theta}$, has J discrete scenarios $\{\theta_{1i}, \ldots, \theta_{Ji}\}$;

p_j is probability of the scenario $\vec{\theta}_j$, $j = 1, \ldots, J$;

$L(\vec{x},\vec{\theta}) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i$ is the Loss Function (section **Risk Functions Defined by Matrix of Scenarios**);

$G(\vec{x},\vec{\theta}) = -L(\vec{x},\vec{\theta})$ is the Gain Function (section **Risk Functions Defined by Matrix of Scenarios**).

For more details about the Properties of this Group see section **Properties of Average Group**.

1.2.1.1 Properties of Average Group

The name of the Average for Loss Function may contain up to 128 symbols and should begin with the string “avg_”. The name of the Average for Gain Function may contain up to 128 symbols and should begin with the string “avg_g_”. The names of these functions may include only alphabetic characters, numbers, and the underscore sign, “_”. The names of these functions are "insensitive" to the case, i.e. there is no difference between low case and upper case in these names. Functions from the Average Group are calculated with double precision.

1.2.2 CVaR Group

Functions from this group are used for the calculation of CVaR-based measures of risk and deviation.
The term CVaR, which is an abbreviation for Conditional Value-at-Risk, is introduced in Rockafellar and Uryasev (2000).
CVaR-based functions depend on the confidence level parameter α satisfying the following condition: $0 \le \alpha \le 1$; typical values for α are 0.9, 0.95, 0.99.
For continuous distributions, the CVaR is defined as the conditional expectation of outcomes under the condition that the outcomes exceed the α-Value-at-Risk (VaR).
This definition for continuous distributions gave the motivation for the name “Conditional Value-at-Risk.”

In other words, for continuous distributions, CVaR is defined as an average of the $(1-\alpha) * 100\%$ largest outcomes.

For instance, if α=0.9, then the CVaR is an average of 10% of the largest outcomes.
For continuous distributions, this risk measure is also known as the Expected Shortfall or Tail Value-at-Risk.
However, for general distributions including discrete distributions considered in PSG, CVaR is **NOT** equal to the conditional expectation for outcomes exceeding VaR.
For general distributions, the CVaR is defined as the expectation of the α-tail distribution or as a weighted average of the VaR and the expectation of losses strictly exceeding VaR, see Rockafellar and Uryasev (2002).
For discrete distributions, some scenarios may need to be split to take exactly the expectation of the α-tail distribution.
Without splitting scenarios the CVaR function, generally, is not convex with respect to the decision vector.
An alternative equivalent definition of the CVaR (which is called Expected Shorfall) is given by Acerbi (2004).
The Expected Shortfall is an average of the percentiles exceeding the VaR.
This package uses another constructive equivalent definition of CVaR for general distributions, which is a weighted average of the conditional expectation of outcomes including and exceeding VaR and the conditional expectation of outcomes strictly exceeding VaR.
CVaR is a Coherent Risk Measure, as it is defined by Artzner et al (1999).
CVaR is a Risk Measure since it measures the magnitude of outcomes versus zero.
In PSG, CVaR for Losses is called CVaR Risk. CVaR for Gains is called CVaR Risk Gain.

References

1. Rockafellar, R. T. and S. Uryasev (2000): Optimization of Conditional Value-At-Risk, *The Journal of Risk*, Vol. 2, No. 4, pp. 21-51.
2. Rockafellar, R.T. and S. Uryasev (2002): Conditional Value-at-Risk for General Loss Distributions, Journal *of Banking and Finance*, 27/7.
3. Acerbi, C. (2004): *Risk Measures for the 21st Century.* Szegö, G., ed. New York: John Wiley and Sons.
4. Artzner, P. et al (1999). Coherent Measures of Risk. *Mathematical Finance* 9, pp. 203-228.

The CVaR group includes the following functions:

- CVaR Risk for Loss (software notation: cvar_risk_...) (section **Calculation of CVaR Risk for Loss**)
- CVaR Risk for Loss Normal Independent (software notation: cvar_risk_ni_...) (section **Calculation of CVaR Risk for Loss Normal Independent (cvar_risk_ni)**)
- CVaR Risk for Loss Normal Dependent (software notation: cvar_risk_nd_...) (section **Calculation of CVaR Risk for Loss Normal Dependent (cvar_risk_nd)**)
- CVaR Risk for Gain (software notation: cvar_risk_g...) (section **Calculation of CVaR Risk for Gain**)
- CVaR Risk for Gain Normal Independent (software notation: cvar_risk_ni_g_...) (section **Calculation of CVaR Risk for Gain Normal Independent (cvar_risk_ni_g)**)
- CVaR Risk for Gain Normal Dependent (software notation: cvar_risk_nd_g_...) (section **Calculation of CVaR Risk for Gain Normal Dependent (cvar_risk_nd_g)**)
- CVaR Deviation for Loss (software notation: cvar_dev_...) (section **Calculation of CVaR Deviation for Loss**)
- CVaR Deviation for Loss Normal Independent (software notation: cvar_ni_dev_...) (section **Calculation of CVaR Deviation for Loss Normal Independent (cvar_ni_dev)**)
- CVaR Deviation for Loss Normal Dependent (software notation: cvar_nd_dev_...) (section **Calculation of CVaR Deviation for Loss Normal Dependent (cvar_nd_dev)**)
- CVaR Deviation for Gain (software notation: cvar_dev_g...) (section **Calculation of CVaR Deviation for Gain**)
- CVaR Deviation for Gain Normal Independent (software notation: cvar_ni_dev_g) (section **Calculation of CVaR Deviation for Gain Normal Independent (cvar_ni_dev_g)**)
- CVaR Deviation for Gain Normal Dependent (software notation: cvar_nd_dev_g) (section **Calculation of**

CVaR Deviation for Gain Normal Dependent (cvar_nd_dev_g))

CVaR group functions are defined on some **Point** , $\vec{x} = (x_1, x_2, \dots, x_I)$, and use **Matrix of Scenarios** or matrix with parameters of normal distribution..
For more details about Properties of this Group see the section **Properties CVaR Group**.

1.2.2.1 Calculation of CVaR Risk for Loss (cvar_risk)

For continuous distributions, given $\vec{x}$ and any specified probability level α in (0, 1) the α-CVaR Risk for Loss is

$$\mathbf{cvar\_risk}_\alpha\left(L\left(\vec{x},\vec{\theta}\right)\right) = (1-\alpha)^{-1} \int\limits_{L(\vec{x},\vec{\theta}) \geq VaR_\alpha(L(\vec{x},\vec{\theta}))} L(\vec{x},\vec{\theta})p(\vec{\theta})d\vec{\theta} \quad ,$$

where

$$VaR_\alpha(L(\vec{x},\vec{\theta})) = \mathbf{min}\{\zeta \in R : \psi(\vec{x},\zeta) \geq \alpha\} \quad ,$$

and $\psi(\vec{x},\zeta)$ is the probability distribution function of the loss $L(\vec{x},\vec{\theta})$, and $p(\vec{\theta})$ is a density of the random vector $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$.

For discrete distributions considered in PSG, when models are based on scenarios and finite sampling, calculation of the CVaR Risk for Loss includes the following steps:

1. Calculate values of the Loss function for all scenarios:

$$L(\vec{x},\vec{\theta}_j) = \theta_{j0} - \sum_{i=1}^{I} \theta_{ji} x_i \quad , \quad j = 1,\dots,J \quad .$$

2. Sort losses

$$L(\vec{x},\vec{\theta}_{j_1}) \leq L(\vec{x},\vec{\theta}_{j_2}) \leq \dots \leq L(\vec{x},\vec{\theta}_{j_J}) \quad .$$

If $\alpha = 0$, then

$$\mathbf{cvar\_risk}_\alpha\left(L\left(\vec{x},\vec{\theta}\right)\right) = \mathbf{avg}\left(L\left(\vec{x},\vec{\theta}\right)\right) = E\left[L\left(\vec{x},\vec{\theta}\right)\right] = \sum_{j=1}^{J} p_j L(\vec{x},\vec{\theta}_j) \quad .$$

Setting $\alpha = 0$ for CVaR is not recommended because PSG contains the Average Loss function (avg) (section Average Group) dedicated for this purpose. This function calculates the same value in a more efficient way.

Let $l^* = \mathbf{min}\, l : L(\vec{x},\vec{\theta}_{j_{l+1}}) = L(\vec{x},\vec{\theta}_{j_J}) \quad .$

If $\sum_{l=1}^{l^*-1} p_{jl} < \alpha \quad ,$ then

$$\mathbf{cvar\_risk}_\alpha\left(L\left(\vec{x},\vec{\theta}\right)\right) \;=\; \mathbf{max\_risk}\left(L\left(\vec{x},\vec{\theta}\right)\right) \;=\; \max_{1 \leq j \leq J} L(\vec{x},\vec{\theta}_j) \;.$$

Setting $\sum_{l=1}^{l^*-1} p_{jl} < \alpha \quad ,$ for CVaR is not recommended because PSG contains the function Maximum Risk for Loss (max_risk) (see section **Maximum Group**) dedicated to calculating the same in a more efficient way.

3. Let $0 < \alpha \le \sum_{l=1}^{l^*-1} p_{jl}$.

Determine an index $l(\alpha)$ such that

$$\sum_{l=1}^{l(\alpha)} p_{jl} > \alpha \quad \text{and} \quad \sum_{l=1}^{l(\alpha)-1} p_{jl} \le \alpha \quad .$$

4. If the index $l(\alpha)$ is such that the confidence level α equals

$$\sum_{l=1}^{l(\alpha)-1} p_{jl} = \alpha \quad ,$$

then the CVaR Risk for Loss equals

$$\mathbf{cvar\_risk}_{\alpha}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \frac{1}{1-\alpha}\sum_{l(\alpha)\le l\le J} p_{jl}\, L(\vec{x},\vec{\theta}_{jl}).$$

If $\sum_{l=1}^{l(\alpha)-1} p_{jl} < \alpha$, then the CVaR Risk for Loss equals the linear interpolation between CVaR Risks for Loss with confidence levels

$$\underline{\alpha} = \sum_{l=1}^{l(\alpha)-1} p_{jl}$$

and

$$\overline{\alpha} = \sum_{l=1}^{l(\alpha)} p_{jl} \quad ,$$

i.e.,

$$\mathbf{cvar\_risk}_{\alpha}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \frac{\overline{\alpha}-\alpha}{\overline{\alpha}-\underline{\alpha}}\cdot\frac{1-\underline{\alpha}}{1-\alpha}\mathbf{cvar\_risk}_{\underline{\alpha}}\left(L\left(\vec{x},\vec{\theta}\right)\right) +$$

$$+\frac{\alpha-\underline{\alpha}}{\overline{\alpha}-\alpha}\cdot\frac{1-\overline{\alpha}}{1-\alpha}\mathbf{cvar\_risk}_{\overline{\alpha}}\left(L\left(\vec{x},\vec{\theta}\right)\right).$$

If $\alpha = 0$, then

$$\mathbf{cvar\_risk}_{\underline{\alpha}}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \mathbf{avg}\left(L\left(\vec{x},\vec{\theta}\right)\right) = E\left[L\left(\vec{x},\vec{\theta}\right)\right] = \sum_{j=1}^{J} p_j L(\vec{x},\vec{\theta}_j).$$

1.2.2.2 Calculation of CVaR Risk for Loss Normal Independent (cvar_risk_ni)

The CVaR Risk for Loss Normal Independent is a special case of the **Calculation of CVaR Risk for Loss Normal Dependent (cvar_risk_nd)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\ldots,I.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;\ \ \sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2 \sigma_I^2.$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

$VaR_\alpha^{st} = \Phi^{-1}(\alpha).$

The CVaR Risk for Loss Normal Independent is calculated as follows:

$$\mathrm{cvar\_risk\_ni}_\alpha\left(L\left(\vec{x},\vec{\theta}\right)\right) = \sigma_L \mathrm{cvar}_\alpha^{st} + \mu_L,$$

where

$$\mathrm{cvar}_\alpha^{st} = \frac{1}{1-\alpha} \int_{VaR_\alpha^{st}}^{\infty} t\emptyset(t)\, dt = \frac{1}{1-\alpha} \emptyset(VaR_\alpha^{st}).$$

1.2.2.3 Calculation of CVaR Risk for Loss Normal Dependent (cvar_risk_nd)

The CVaR Risk for Loss Normal Dependent is a special case of the **Calculation of CVaR Risk for Loss (cvar_risk)** for continuous distributions when random coefficients in a loss function follow the multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma)$, where $\vec{\mu} = (\mu_0, \mu_1, \ldots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I$;
Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of means and covariance Smatrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L,\sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}}\,dy,$$

where

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \phi(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

$VaR_\alpha^{st} = \Phi^{-1}(\alpha)$.

The CVaR Risk for Loss Normal Dependent is calculated as follows:

$$\text{cvar\_risk\_nd}_\alpha\left(L\left(\vec{x},\vec{\theta}\right)\right) = \sigma_L \text{cvar}_\alpha^{st} + \mu_L,$$

where

$$\text{cvar}_\alpha^{st} = \frac{1}{1-\alpha}\int_{VaR_\alpha^{st}}^{\infty} t\phi(t)\,dt = \frac{1}{1-\alpha}\phi(VaR_\alpha^{st}).$$

1.2.2.4 Calculation of CVaR Risk for Gain (cvar_risk_g)

CVaR Risk for Gain equals

$$\textbf{cvar\_risk\_g}_{\boldsymbol{\alpha}}\left(G\left(\vec{x},\vec{\theta}\right)\right) = \textbf{cvar\_risk}_{\boldsymbol{\alpha}}\left(L\left(\vec{x},-\vec{\theta}\right)\right).$$

1.2.2.5 Calculation of CVaR Risk for Gain Normal Independent (cvar_risk_ni_g)

The CVaR Risk for Gain Normal Independent is a special case of the **Calculation of CVaR Risk for Gain Normal Dependent (cvar_risk_nd_g)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

Corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N\left(\mu_i, \sigma_i^2\right), i = 0,1,\ldots,I.$

Parameters of normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L,\sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}}\,dy,$$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;\ \sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2\sigma_I^2.$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

$VaR_\alpha^{st} = \Phi^{-1}(\alpha).$

CVaR Risk for Gain Normal Independent is calculated as follows:

$$\text{cvar\_risk\_ni\_g}_\alpha\left(G\left(\vec{x},\vec{\theta}\right)\right) = \text{cvar\_risk\_ni}_\alpha\left(L\left(\vec{x},-\vec{\theta}\right)\right) = \sigma_L \text{cvar}_\alpha^{st} - \mu_L,$$

where

$$\text{cvar}_\alpha^{st} = \frac{1}{1-\alpha}\int_{VaR_\alpha^{st}}^{\infty} t\emptyset(t)\,dt = \frac{1}{1-\alpha}\emptyset(VaR_\alpha^{st}).$$

1.2.2.6 Calculation of CVaR Risk for Gain Normal Dependent (cvar_risk_nd_g)

The CVaR Risk for Gain Normal Dependent is a special case of the **Calculation of CVaR Risk for Gain (cvar_risk_g)** for continuous distributions when coefficients in a gain function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

Corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma),$ where

$\vec{\mu} = \left(\mu_0,\mu_1,\ldots,\mu_I\right)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I;$

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of means and covariance Smatrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_0 & \mu_1 & \dots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L,\sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}}\,dy,$$

where

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

$$VaR_\alpha^{st} = \Phi^{-1}(\alpha).$$

CVaR Risk for Gain Normal Dependent is calculated as follows:

$$\text{cvar\_risk\_nd\_g}_\alpha\left(G\left(\vec{x},\vec{\theta}\right)\right) = \text{cvar\_risk\_nd}_\alpha\left(L\left(\vec{x},-\vec{\theta}\right)\right) = \sigma_L \text{cvar}_\alpha^{st} - \mu_L,$$

where

$$\text{cvar}_\alpha^{st} = \frac{1}{1-\alpha}\int_{VaR_\alpha^{st}}^{\infty} t\emptyset(t)\,dt = \frac{1}{1-\alpha}\emptyset(VaR_\alpha^{st}).$$

1.2.2.7 Calculation of CVaR Deviation for Loss (cvar_dev)

CVaR Deviation for Loss equals

$$\text{cvar\_dev}_{\alpha}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \text{cvar\_risk}_{\alpha}\left(f\left(\vec{x},\vec{\theta}\right)\right),$$

where

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] \quad .$$

1.2.2.8 Calculation of CVaR Deviation for Loss Normal Independent (cvar_ni_dev)

The CVaR Deviation for Loss Normal Independent is a special case of the **Calculation of CVaR Deviation for Loss Normal Dependent (cvar_nd_dev)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\ldots,I.$
Consider the random function

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I}(\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the matrix of variances

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$f(\vec{x},\vec{\theta}) \sim N(0,\sigma_f^2), \text{ and } F(z) = P\{f(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_f\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}}dy,$$

where

$$\sigma_f^2 = \sigma_0^2 + \sum_{i=1}^{I} x_i^2\sigma_i^2.$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z}\emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

$$VaR_{\alpha}^{st} = \Phi^{-1}(\alpha).$$

The CVaR Deviation for Loss Normal Independent is calculated as follows:

$$\text{cvar\_ni\_dev}_\alpha\left(L\left(\vec{x},\vec{\theta}\right)\right) = \text{cvar\_risk\_ni}_\alpha\left(f\left(\vec{x},\vec{\theta}\right)\right) = \sigma_f \text{cvar}_\alpha^{st},$$

where

$$\text{cvar}_\alpha^{st} = \frac{1}{1-\alpha}\int_{VaR_\alpha^{st}}^{\infty} t\emptyset(t)\,dt = \frac{1}{1-\alpha}\emptyset(VaR_\alpha^{st}).$$

1.2.2.9 Calculation of CVaR Deviation for Loss Normal Dependent (cvar_nd_dev)

The CVaR Deviation for Loss Normal Dependent is a special case of **Calculation of CVaR Deviation for Loss (cvar_dev)** for continuous distributions when coefficients in a loss function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma)$, where $\vec{\mu} = \left(\mu_0, \mu_1, \ldots, \mu_I\right)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I$;
Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Consider the random function

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I}(\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the covariance Smatrix, which has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with properties of normal distribution,

$f(\vec{x},\vec{\theta}) \sim N(0,\sigma_f^2)$, and $F(z) = P\{f(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_f\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}} dy,$

where

$\sigma_f^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

$VaR_\alpha^{st} = \Phi^{-1}(\alpha).$

The CVaR Deviation for Loss Normal Dependent is calculated as follows:

$$\text{cvar\_nd\_dev}_\alpha\left(L\left(\vec{x},\vec{\theta}\right)\right) = \text{cvar\_risk\_nd}_\alpha\left(f\left(\vec{x},\vec{\theta}\right)\right) = \sigma_f \text{cvar}_\alpha^{st},$$

where

$$\text{cvar}_\alpha^{st} = \frac{1}{1-\alpha}\int_{VaR_\alpha^{st}}^{\infty} t\emptyset(t)\,dt = \frac{1}{1-\alpha}\emptyset(VaR_\alpha^{st}).$$

1.2.2.10 Calculation of CVaR Deviation for Gain (cvar_dev_g)

CVaR Deviation for Gain equals

$$\mathbf{cvar\_dev\_g}_\alpha\left(G\left(\vec{x},\vec{\theta}\right)\right) = \mathbf{cvar\_risk\_g}_\alpha\left(g\left(\vec{x},\vec{\theta}\right)\right),$$

where

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = L(\vec{x},-\vec{\theta}) - E[L(\vec{x},-\vec{\theta})] \quad .$$

1.2.2.11 Calculation of CVaR Deviation for Gain Normal Independent (cvar_ni_dev_g)

The CVaR Deviation for Gain Normal Independent is a special case of the **Calculation of CVaR Deviation for Gain Normal Dependent (cvar_nd_dev_g)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

Corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i,\sigma_i^2), i = 0,1,\ldots,I.$

Consider the random function

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = -(\theta_0 - E[\theta_0]) + \sum_{i=1}^{I}(\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the matrix of variances

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \dots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$\mathrm{g}\left(\vec{x},\vec{\theta}\right) \sim N(0,\sigma_{\mathrm{g}}^2), \text{ and } F(z) = P\{\mathrm{g}\left(\vec{x},\vec{\theta}\right) \le z\} = \frac{1}{\sigma_{\mathrm{g}}\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_{\mathrm{g}}^2}}dy,$$

where

$$\sigma_{\mathrm{g}}^2 = \sigma_0^2 + \sum_{i=1}^{I} x_i^2\sigma_i^2.$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z}\emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

$$VaR_\alpha^{st} = \Phi^{-1}(\alpha).$$

The CVaR Deviation for Gain Normal Independent is calculated as follows:

$$\mathrm{cvar\_ni\_dev\_g}_\alpha\left(G\left(\vec{x},\vec{\theta}\right)\right) = \mathrm{cvar\_risk\_ni\_g}_\alpha\left(\mathrm{g}\left(\vec{x},\vec{\theta}\right)\right) = \sigma_{\mathrm{g}}\mathrm{cvar}_\alpha^{st},$$

where

$$\mathrm{cvar}_\alpha^{st} = \frac{1}{1-\alpha}\int_{VaR_\alpha^{st}}^{\infty} t\emptyset(t)\,dt = \frac{1}{1-\alpha}\emptyset(VaR_\alpha^{st}).$$

1.2.2.12 Calculation of CVaR Deviation for Gain Normal Dependent (cvar_nd_dev_g)

The CVaR Deviation for Gain Normal Dependent is a special case of the **Calculation of CVaR Deviation for Gain (cvar_dev_g)** for continuous distributions when random coefficients in a gain function follow multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\dots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

Corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu}, \Sigma)$, where $\vec{\mu} = (\mu_0, \mu_1, \dots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1, \dots, I$;
Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Consider the random function

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = -(\theta_0 - E[\theta_0]) + \sum_{i=1}^{I} (\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the covariance Smatrix, which has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$\mathrm{g}(\vec{x},\vec{\theta}) \sim N(0,\sigma_{\mathrm{g}}^2), \text{ and } F(z) = P\{\mathrm{g}(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_{\mathrm{g}}\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_{\mathrm{g}}^2}} dy,$$

where

$$\sigma_{\mathrm{g}}^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k .$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

$$VaR_{\alpha}^{st} = \Phi^{-1}(\alpha).$$

The CVaR Deviation for Gain Normal Dependent is calculated as follows:

$$\mathrm{cvar\_nd\_dev\_g}_{\alpha}\left(G(\vec{x},\vec{\theta})\right) = \mathrm{cvar\_risk\_nd\_g}_{\alpha}\left(\mathrm{g}(\vec{x},\vec{\theta})\right) = \sigma_{\mathrm{g}} \mathrm{cvar}_{\alpha}^{st},$$

$$\mathrm{cvar}_{\alpha}^{st} = \frac{1}{1-\alpha} \int_{VaR_{\alpha}^{st}}^{\infty} t\emptyset(t)\, dt = \frac{1}{1-\alpha} \emptyset(VaR_{\alpha}^{st}).$$

1.2.2.13 Properties of CVaR Group

The confidence level, α, satisfies the following condition: $0 \le \alpha \le 1$.

Functions from the CVaR group are calculated with double precision. The name of any function from this group may contain up to 128 symbols. The name of the CVaR Risk for Loss function begins with the string "cvar_risk_", the name for CVaR Risk for Loss Normal Independent function begins with the string "cvar_risk_ni_", the name for CVaR Risk for Loss Normal Dependent function begins with the string "cvar_risk_nd_", the name for CVaR Risk for Gain function begins with the string "cvar_risk_g_", the name for CVaR Risk for Gain Normal Independent function begins with the string "cvar_risk_ni_g_", the name for CVaR Risk for Gain Normal Dependent function begins with the string "cvar_risk_nd_g_", the name for CVaR Deviation for Loss function begins with the string "cvar_dev_", the name for CVaR Deviation for Loss Normal Independent function begins with the string "cvar_ni_dev_", the name for CVaR Deviation for Loss Normal Dependent function begins with the string "cvar_nd_dev_", the name for CVaR Deviation for Gain function begins with the string "cvar_dev_g_", the name for CVaR Deviation for Gain Normal Independent function begins with the string "cvar_ni_dev_g_", the name for CVaR Deviation for Gain Normal Dependent function begins with the string "cvar_nd_dev_g_". Names of these functions may include only alphabetic characters, numbers, and the underscore sign, "_". The names of these functions are "insensitive" to the case, i.e. there is no difference between low case and upper case in these names.

1.2.3 VaR Group

Functions from this group are used for calculation of VaR-based measures of risk and deviation. VaR is a percentile of a distribution. VaR answers the question, what is the maximum outcome with the confidence level α*100% ? This means that VaR is the minimal value such that probability of outcomes which are less or equal to VaR is greater or equal to α .

The VaR group consists of the following functions:

- VaR Risk for Loss (software notation: var_risk_...) (section **Calculation of VaR Risk for Loss**)
- VaR Risk for Loss Normal Independent (software notation: var_risk_ni_...) (section **Calculation of VaR Risk for Loss Normal Independent (var_risk_ni)**)
- VaR Risk for Loss Normal Dependent (software notation: var_risk_nd_...) (section **Calculation of VaR Risk for Loss Normal Dependent (var_risk_nd)**)
- VaR Risk for Gain (software notation: var_risk_g_...) (section **Calculation of VaR Risk for Gain**)
- VaR Risk for Gain Normal Independent (software notation: var_risk_ni_g_...) (section **Calculation of VaR Risk for Gain Normal Independent (var_risk_ni_g)**)
- VaR Risk for Gain Normal Dependent (software notation: var_risk_nd_g_...) (section **Calculation of VaR Risk for Gain Normal Dependent (var_risk_nd_g)**)
- VaR Deviation for Loss (software notation: var_dev_...) (section **Calculation of VaR Deviation for Loss**)
- VaR Deviation for Loss Normal Independent (software notation: var_ni_dev_...) (section **Calculation of VaR Deviation for Loss Normal Independent (var_ni_dev)**)
- VaR Deviation for Loss Normal Dependent (software notation: var_nd_dev_...) (section **Calculation of VaR Deviation for Loss Normal Dependent (var_nd_dev)**)
- VaR Deviation for Gain (software notation: var_dev_g_...) (section **Calculation of VaR Deviation for Gain**)
- VaR Deviation for Gain Normal Independent (software notation: var_ni_dev_g_...) (section **Calculation of VaR Deviation for Gain Normal Independent (var_ni_dev_g)**)
- VaR Deviation for Gain Normal Dependent (software notation: var_nd_dev_g_...) (section **Calculation of VaR Deviation for Gain Normal Dependent (var_nd_dev_g)**)

For more details about the Properties of this Group see the section **Properties of VaR Group**.

These functions depend on the parameter α (confidence level) and are defined on some Point , $\vec{x} = (x_1, x_2, \dots, x_I)$, and use **Matrix of Scenarios** or parameters of normal distribution as described in the following subsections.

1.2.3.1 Calculation of VaR Risk for Loss (var_risk)

For continuous distributions, given $\vec{x}$ and a confidence level α in (0,1), the α-VaR Risk for Loss equals

$$\mathbf{var\_risk}_{\alpha}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \mathbf{min}\{\zeta \in R : \psi(\vec{x},\zeta) \geq \alpha\},$$

where

$\psi(\vec{x},\zeta)$ is probability distribution function of the loss $L(\vec{x},\vec{\theta})$.

For discrete distributions, considered in PSG, when models are based on scenarios and finite sampling, calculation of VaR Risk for Loss includes the following steps:

1. Calculate the values of Loss function for all scenarios

$$L(\vec{x},\vec{\theta}_j) = \theta_{j0} - \sum_{i=1}^{I} \theta_{ji} x_i \quad , \quad j = 1, \dots, J \quad .$$

2. Sort losses so that

$$L(\vec{x},\vec{\theta}_{j_1}) \leq L(\vec{x},\vec{\theta}_{j_2}) \leq \dots \leq L(\vec{x},\vec{\theta}_{j_J}) \quad .$$

3. Determine an index $l(\alpha)$ such that

$$\sum_{i=1}^{l(\alpha)} p_{ji} > \alpha \quad \text{and} \quad \sum_{i=1}^{l(\alpha)-1} p_{ji} \leq \alpha \quad .$$

4. If the index $l(\alpha) > 1$ is such that the confidence level α equals

$$\sum_{i=1}^{l(\alpha)-1} p_{j_i} = \alpha,$$

then, VaR Risk for Loss equals

$$\mathbf{var\_risk}_{\alpha}\left(L\left(\vec{x},\vec{\theta}\right)\right) = L\left(\vec{x},\vec{\theta}_{l(\alpha)-1}\right).$$

If $l(\alpha) = 1$ then

$$\mathbf{var\_risk}_{\alpha}\left(L\left(\vec{x},\vec{\theta}\right)\right) = L\left(\vec{x},\vec{\theta}_1\right).$$

If $\sum_{i=1}^{l(\alpha)-1} p_{j_i} < \alpha$, then VaR Risk for Loss equals linear interpolation between VaR Risks for Loss with confidence levels

$$\mathbf{var\_risk}_{\alpha}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \frac{\overline{\alpha} - \alpha}{\overline{\alpha} - \underline{\alpha}} \mathbf{var\_risk}_{\underline{\alpha}}\left(L\left(\vec{x},\vec{\theta}\right)\right) + \frac{\alpha - \underline{\alpha}}{\overline{\alpha} - \underline{\alpha}} \mathbf{var\_risk}_{\overline{\alpha}}\left(L\left(\vec{x},\vec{\theta}\right)\right).$$

1.2.3.2 Calculation of VaR Risk for Loss Normal Independent (var_risk_ni)

The VaR Risk for Loss Normal Independent is a special case of the **Calculation of VaR Risk for Loss Normal Dependent (var_risk_nd)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x}, \vec{\theta}) = L(\vec{x}, \theta_0, \theta_1 \ldots, \theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1, \ldots, I.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$L(\vec{x}, \vec{\theta}) \sim N(\mu_L, \sigma_L^2)$ and $F(z) = P\{L(\vec{x}, \vec{\theta}) \le z\} = \frac{1}{\sigma_L \sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i; \;\; \sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2 \sigma_I^2.$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

$VaR_\alpha^{st} = \Phi^{-1}(\alpha).$

The VaR Risk for Loss Normal Independent is calculated as follows:

$$\text{var\_risk\_ni}_\alpha \left(L\left(\vec{x}, \vec{\theta}\right) \right) = \sigma_L \text{VaR}_\alpha^{st} + \mu_L.$$

1.2.3.3 Calculation of VaR Risk for Loss Normal Dependent (var_risk_nd)

The VaR Risk for Loss Normal Dependent is a special case of the **Calculation of VaR Risk for Loss (var_risk)** for continuous distributions when random coefficients in a loss function follow the multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\dots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu}, \Sigma)$, where $\vec{\mu} = (\mu_0, \mu_1, \dots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1, \dots, I$;
Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

The parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of means and covariance Smatrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_0 & \mu_1 & \dots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i) x_i + \sum_{i=1}^{I} \sum_{k=1}^{I} cov(\theta_i,\theta_k) x_i x_k .$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

$VaR_{\alpha}^{st} = \Phi^{-1}(\alpha)$.

The VaR Risk for Loss Normal Dependent is calculated as follows:

$$\text{var\_risk\_nd}_{\alpha}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \sigma_L \text{VaR}_{\alpha}^{st} + \mu_L.$$

1.2.3.4 Calculation of VaR Risk for Gain (var_risk_g)

VaR Risk for Gain equals

$$\text{var\_risk\_g}_{\alpha}\left(G\left(\vec{x},\vec{\theta}\right)\right) = \text{var\_risk}_{\alpha}\left(L\left(\vec{x},-\vec{\theta}\right)\right).$$

1.2.3.5 Calculation of VaR Risk for Gain Normal Independent (var_risk_ni_g)

The VaR Risk for Gain Normal Independent is a special case of the **Calculation of VaR Risk for Gain Normal Dependent (var_risk_nd_g)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

Corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\ldots,I.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;\ \ \sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2\sigma_I^2$.

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

$$VaR_\alpha^{st} = \Phi^{-1}(\alpha).$$

VaR Risk for Gain Normal Independent is calculated as follows:

$$\text{var\_risk\_nd\_g}_\alpha\left(G\left(\vec{x},\vec{\theta}\right)\right) = \text{var\_risk\_nd}_\alpha\left(L\left(\vec{x},-\vec{\theta}\right)\right) = \sigma_L \text{VaR}_\alpha^{st} - \mu_L.$$

1.2.3.6 Calculation of VaR Risk for Gain Normal Dependent (var_risk_nd_g)

The VaR Risk for Gain Normal Dependent is a special case of the **Calculation of VaR Risk for Gain (var_risk_g)** for continuous distributions when coefficients in a gain function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

Corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu}, \Sigma)$, where

$\vec{\mu} = (\mu_0, \mu_1, \ldots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I;$

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

The parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of means and covariance Smatrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_0 & \mu_1 & \dots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0, \theta_0) & cov(\theta_0, \theta_1) & \dots & cov(\theta_0, \theta_I) \\ 2 & cov(\theta_1, \theta_0) & cov(\theta_1, \theta_1) & \dots & cov(\theta_1, \theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I, \theta_0) & cov(\theta_I, \theta_1) & \dots & cov(\theta_I, \theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x}, \vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x}, \vec{\theta}) \le z\} = \frac{1}{\sigma_L \sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;$$

$$\sigma_L^2 = cov(\theta_0, \theta_0) - 2\sum_{i=1}^{I} cov(\theta_0, \theta_i) x_i + \sum_{i=1}^{I} \sum_{k=1}^{I} cov(\theta_i, \theta_k) x_i x_k.$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

$$VaR_\alpha^{st} = \Phi^{-1}(\alpha).$$

VaR Risk for Gain Normal Dependent is calculated as follows:

$$\text{var\_risk\_nd\_g}_\alpha\left(G\left(\vec{x}, \vec{\theta}\right)\right) = \text{var\_risk\_nd}_\alpha\left(L\left(\vec{x}, -\vec{\theta}\right)\right) = \sigma_L \text{VaR}_\alpha^{st} - \mu_L.$$

1.2.3.7 Calculation of VaR Deviation for Loss (var_dev)

VaR Deviation for Loss equals

$$\text{var\_dev}_\alpha\left(L\left(\vec{x}, \vec{\theta}\right)\right) = \text{var\_risk}_\alpha\left(f\left(\vec{x}, \vec{\theta}\right)\right),$$

where

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] \quad .$$

1.2.3.8 Calculation of VaR Deviation for Loss Normal Independent (var_ni_dev)

The VaR Deviation for Loss Normal Independent is a special case of the **Calculation of VaR Deviation for Loss Normal Dependent (var_nd_dev)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\ldots,I.$
Consider the random function

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I} (\theta_i - E[\theta_i]) x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the matrix of variances

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$f(\vec{x},\vec{\theta}) \sim N(0,\sigma_f^2), \text{ and } F(z) = P\{f(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_f\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}} dy,$$

where

$$\sigma_f^2 = \sigma_0^2 + \sum_{i=1}^{I} x_i^2 \sigma_i^2.$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

$$VaR_\alpha^{st} = \Phi^{-1}(\alpha).$$

The VaR Deviation for Loss Normal Independent is calculated as follows:

$$\text{var\_nd\_dev}_\alpha\left(L\left(\vec{x},\vec{\theta}\right)\right) = \text{var\_risk\_nd}_\alpha\left(f\left(\vec{x},\vec{\theta}\right)\right) = \sigma_f \text{VaR}_\alpha^{st}.$$

1.2.3.9 Calculation of VaR Deviation for Loss Normal Dependent (var_nd_dev)

The VaR Deviation for Loss Normal Dependent is a special case of the **Calculation of VaR Deviation for Loss (var_dev)** for continuous distributions when coefficients in a loss function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \dots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu}, \Sigma)$, where $\vec{\mu} = (\mu_0, \mu_1, \dots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\dots,I$;
Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Consider the random function

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I} (\theta_i - E[\theta_i]) x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the covariance Smatrix, which has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$f(\vec{x},\vec{\theta}) \sim N(0,\sigma_f^2), \text{ and } F(z) = P\{f(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_f\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}} dy,$$

where

$$\sigma_f^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i) x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k) x_i x_k .$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

$VaR_\alpha^{st} = \Phi^{-1}(\alpha)$.

The VaR Deviation for Loss Normal Dependent is calculated as follows:

$$\text{var\_nd\_dev}_\alpha\left(L\left(\vec{x},\vec{\theta}\right)\right) = \text{var\_risk\_nd}_\alpha\left(f\left(\vec{x},\vec{\theta}\right)\right) = \sigma_f \text{VaR}_\alpha^{st}.$$

1.2.3.10 Calculation of VaR Deviation for Gain (var_dev_g)

VaR Deviation for Gain equals

$$\text{var\_dev\_g}_\alpha\left(G\left(\vec{x},\vec{\theta}\right)\right) = \text{var\_risk\_g}_\alpha\left(g\left(\vec{x},\vec{\theta}\right)\right),$$

where

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = L(\vec{x},-\vec{\theta}) - E[L(\vec{x},-\vec{\theta})] \quad .$$

1.2.3.11 Calculation of VaR Deviation for Gain Normal Independent (var_ni_dev_g)

The VaR Deviation for Gain Normal Independent is a special case of the **Calculation of VaR Deviation for Gain Normal Dependent (var_nd_dev_g)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

Corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N\left(\mu_i, \sigma_i^2\right), i = 0,1,\ldots,I.$
Consider the random function

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = -(\theta_0 - E[\theta_0]) + \sum_{i=1}^{I}(\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the matrix of variances

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$\text{g}\left(\vec{x},\vec{\theta}\right) \sim N\left(0,\sigma_{\text{g}}^2\right), \text{ and } F(z) = P\left\{\text{g}\left(\vec{x},\vec{\theta}\right) \le z\right\} = \frac{1}{\sigma_{\text{g}}\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_{\text{g}}^2}}dy,$$

where

$$\sigma_g^2 = \sigma_0^2 + \sum_{i=1}^{I} x_i^2 \sigma_i^2 .$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

$$VaR_\alpha^{st} = \Phi^{-1}(\alpha).$$

The VaR Deviation for Gain Normal Independent is calculated as follows:

$$\text{var\_ni\_dev\_g}_\alpha\left(G\left(\vec{x},\vec{\theta}\right)\right) = \text{var\_risk\_ni\_g}_\alpha\left(\text{g}\left(\vec{x},\vec{\theta}\right)\right) = \sigma_g \text{VaR}_\alpha^{st}.$$

1.2.3.12 Calculation of VaR Deviation for Gain Normal Dependent (var_nd_dev_g)

The VaR Deviation for Gain Normal Dependent is a special case of the **Calculation of VaR Deviation for Gain (var_dev_g)** for continuous distributions when random coefficients in a gain function follow the multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

Corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma)$, where $\vec{\mu} = \left(\mu_0, \mu_1, \ldots, \mu_I\right)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I$;

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Consider the random function

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = -(\theta_0 - E[\theta_0]) + \sum_{i=1}^{I} (\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the covariance Smatrix, which has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$\mathrm{g}\left(\vec{x},\vec{\theta}\right) \sim N(0,\sigma_{\mathrm{g}}^2), \text{ and } F(z) = P\{\mathrm{g}\left(\vec{x},\vec{\theta}\right) \le z\} = \frac{1}{\sigma_{\mathrm{g}}\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_{\mathrm{g}}^2}}dy,$$

where

$$\sigma_{\mathrm{g}}^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let

$$\emptyset(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2} \text{ be probability density function of the standard normal distribution;}$$

$$\Phi(z) = \int_{-\infty}^{z}\emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt \text{ be the standard normal distribution;}$$

$$VaR_{\alpha}^{st} = \Phi^{-1}(\alpha).$$

The VaR Deviation for Gain Normal Dependent is calculated as follows:

$$\text{var\_nd\_dev\_g}_{\alpha}\left(G\left(\vec{x},\vec{\theta}\right)\right) = \text{var\_risk\_nd\_g}_{\alpha}\left(\mathrm{g}\left(\vec{x},\vec{\theta}\right)\right) = \sigma_{\mathrm{g}}\mathrm{VaR}_{\alpha}^{st}.$$

1.2.3.13 Properties of VaR Group

Confidence level α should satisfy the following condition: $0 \le \alpha \le 1$. Functions from the VaR group are calculated with double precision. The name of any function from this group may contain up to 128 symbols. The name of the VaR Risk for Loss function should begin with the string “var_risk_”, the name for VaR Risk for Loss Normal Independent function should begin with the string “var_risk_ni_", the name for VaR Risk for Loss Normal Dependent function should begin with the string “var_risk_nd_", the name for VaR Risk for Gain function should begin with the string “var_risk_g_”, the name for VaR Risk for Gain Normal Independent function should begin with the string "var_risk_ni_g_", the name for VaR Risk for Gain Normal Dependent function should begin with the string "var_risk_nd_g_", the name for VaR Deviation for Loss function should begin with the string “var_dev_”, the name for VaR Deviation for Loss Normal Independent function should begin with the string "var_ni_dev_", the name for VaR Deviation for Loss Normal Dependent function should begin with the string "var_nd_dev_", the name for VaR Deviation for Gain function should begin with the string “var_dev_g_”, the name for VaR Deviation for Gain Normal Independent function should begin with the string "var_ni_dev_g", the name for VaR Deviation for Gain Normal Dependent function should begin with the string "var_nd_dev_g". The names of these functions should include only alphabetic characters, numbers, and the underscore sign, “_”. The names of these functions are "insensitive" to the case, i.e. there is no difference between low case and upper case in these names.

1.2.4 Maximum Group

Functions from this group are used for calculation of the Maximum-based measures of risk and deviation. For instance, the Maximum Loss of a portfolio in a specified time period is defined as the maximal value over all random loss outcomes. When the distribution of losses is continuous, this risk measure may be unbounded, unless the distribution is "truncated". For example, for normal distribution the maximum loss is infinitely large. However, for discrete loss distributions with fixed number of scenarios, for instance, for sets based on historical datasets, the Maximum Loss is a reasonable measure of risk. For discrete distributions with a fixed number of scenarios, the Maximum Loss is as a special case of CVaR when α is close to 1. The Maximum group consists of four functions:

- Maximum Risk for Loss (software notation: max_risk_...) (section **Calculation of Maximum Risk for Loss)**
- Maximum Risk for Gain (software notation: max_risk_g_...) (section **Calculation of Maximum Risk for Gain)**
- Maximum Deviation for Loss (software notation: max_dev_...) (section **Calculation of Maximum Deviation for Loss)**
- Maximum Deviation for Gain (software notation: max_dev_g_...) (section **Calculation of Maximum Deviation for Gain)**

For more details about the Properties of this Group see the section **Properties of Maximum Group**.

These functions are defined on some **Point** ,$\vec{x} = (x_1, x_2, \ldots, x_I)$, and the **Matrix of Scenarios** as described in the following subsections.

1.2.4.1 Calculation of Maximum Risk for Loss (max_risk)

Maximum Risk for Loss equals

$$\mathbf{max\_risk}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \max_{1\le j\le J} L\left(\vec{x},\vec{\theta}_j\right),$$

where

$$L(\vec{x},\vec{\theta}_j) = \theta_{j0} - \sum_{i=1}^{I}\theta_{ji}x_i \quad , \quad j = 1,\ldots,J \quad .$$

1.2.4.2 Calculation of Maximum Risk for Gain (max_risk_g)

Maximum Risk for Gain equals

$$\mathbf{max\_risk\_g}\left(G\left(\vec{x},\vec{\theta}\right)\right) = \mathbf{max\_risk}\left(L\left(\vec{x},-\vec{\theta}\right)\right).$$

1.2.4.3 Calculation of Maximum Deviation for Loss (max_dev)

Maximum Deviation for Loss equals

$$\mathbf{max\_dev}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \mathbf{max\_risk}\left(f\left(\vec{x},\vec{\theta}\right)\right),$$

where

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] \quad .$$

1.2.4.4 Calculation of Maximum Deviation for Gain (max_dev_g)

Maximum Deviation for Gain equals

$$\textbf{max\_dev\_g}\left(G\left(\vec{x},\vec{\theta}\right)\right)=\textbf{max\_dev}\left(L\left(\vec{x},-\vec{\theta}\right)\right)=\textbf{max\_risk}\left(f\left(\vec{x},-\vec{\theta}\right)\right),$$

where

$$f(\vec{x},-\vec{\theta})=L(\vec{x},-\vec{\theta})-E[L(\vec{x},-\vec{\theta})] \quad .$$

1.2.4.5 Properties of Maximum Group

Functions from the maximum group are calculated with double precision. The name of any function from this group may contain up to 128 symbols. The name of the Maximum Risk for Loss function should begin with the string “max_risk_”, the name for the Maximum Risk for Gain function should begin with the string “max_risk_g_”, the name for the Maximum Deviation for Loss function should begin with the string “max_dev_”, the name for the Maximum Deviation for Gain function should begin with the string “max_dev_g_”. The names of these functions may include only alphabetic characters, numbers, and the underscore sign, “_”. The names of these functions are "insensitive" to the case, i.e. there is no difference between low case and upper case in these names.

1.2.5 Mean Abs Group

Functions from this group are similar to the Mean-Absolute Deviation measure which is an alternative to the classical Variance measure. Konno and Shirakawa (1994) showed that Mean-Absolute Deviation optimal portfolios exhibit properties similar to those of Markowitz Mean-Variance optimal portfolios. The Mean Abs Group includes the following functions:

- Mean Absolute Penalty (software notation: meanabs_pen_...) (section **Calculation of Mean Absolute Penalty)**
- Mean Absolute Penalty Normal Independent (software notation: meanabs_pen_ni_...) (section **Calculation of Mean Absolute Penalty Normal Independent (meanabs_pen_ni)**)
- Mean Absolute Penalty Normal Dependent (software notation: meanabs_pen_nd_...) (section **Calculation of Mean Absolute Penalty Normal Dependent (meanabs_pen_nd)**)
- Mean Absolute Deviation (software notation: meanabs_dev_...) (section **Calculation of Mean Absolute Deviation)**
- Mean Absolute Deviation Normal Independent (software notation: meanabs_ni_dev_...) (section **Calculation of Mean Absolute Deviation Normal Independent (meanabs_ni_dev)**)
- Mean Absolute Deviation Normal Dependent (software notation: meanabs_nd_dev_...) (section **Calculation of Mean Absolute Deviation Normal Dependent (meanabs_nd_dev)**)
- Mean Absolute Risk for Loss (software notation: meanabs_risk_...) (section **Calculation of Mean Absolute Risk for Loss)**
- Mean Absolute Risk for Loss Normal Independent (software notation: meanabs_risk_ni_) (section **Calculation of Mean Absolute Risk for Loss Normal Independent (meanabs_risk_ni)**)
- Mean Absolute Risk for Loss Normal Dependent (software notation: meanabs_risk_nd_) (section **Calculation of Mean Absolute Risk for Loss Normal Dependent (meanabs_risk_nd)**)
- Mean Absolute Risk for Gain (software notation: meanabs_risk_g_...) (section **Calculation of Mean Absolute Risk for Gain**)
- Mean Absolute Risk for Gain Normal Independent (software notation: meanabs_risk_ni_g) (section

Calculation of Mean Absolute Risk for Gain Normal Independent (meanabs_risk_ni_g))

- Mean Absolute Risk for Gain Normal Dependent (software notation: meanabs_risk_nd_g) (section **Calculation of Mean Absolute Risk for Gain Normal Dependent (meanabs_risk_nd_g)**)

For more details about the Properties of this Group see the section **Properties of Mean Abs Group**.

These functions are defined on some **Point** , $\vec{x} = (x_1, x_2, \ldots, x_I)$, and use **Matrix of Scenarios** (in regular Matrix or in packed Pmatrix format) or **Simmetric Matrix** (Smatrix) or parameters of normal distribution.

1.2.5.1 Calculation of Mean Absolute Penalty (meanabs_pen)

Mean Absolute Penalty equals:

For continuous distributions,

$$\textbf{meanabs\_pen}(L(\vec{x},\vec{\theta})) = \int |L(\vec{x},\vec{\theta})|\, p(\vec{\theta})d\vec{\theta},$$

where $p(\vec{\theta})$ is the smooth probability density of random vector $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$.

For discrete distributions, considered in PSG

$$\textbf{meanabs\_pen}(L(\vec{x},\vec{\theta})) = \sum_{j=1}^{J} p_j\, |L(\vec{x},\vec{\theta}_j)|\,,$$

where

$$L(\vec{x},\vec{\theta}_j) = \theta_{j0} - \sum_{i=1}^{I} \theta_{ji} x_i \quad , \quad j = 1, \ldots, J \;\; .$$

1.2.5.2 Calculation of Mean Absolute Penalty Normal Independent (meanabs_pen_ni)

The Mean Absolute Penalty Normal Independent is a special case of the **Calculation of Mean Absolute Penalty Normal Dependent (meanabs_pen_nd)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\ldots,I.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \dots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;\ \ \sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2 \sigma_I^2.$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Mean Absolute Penalty Normal Independent is calculated as follows:

$$\text{meanabs\_pen\_ni}\left(L\left(\vec{x},\vec{\theta}\right)\right) = 2\sigma_L \emptyset\left(-\frac{\mu_L}{\sigma_L}\right) + \mu_L\left[1 - 2\Phi\left(-\frac{\mu_L}{\sigma_L}\right)\right].$$

1.2.5.3 Calculation of Mean Absolute Penalty Normal Dependent (meanabs_pen_nd)

The Mean Absolute Penalty Normal Dependent is a special case of the **Calculation of Mean Absolute Penalty (meanabs_pen)** for continuous distributions when coefficients in a loss function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \dots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu}, \Sigma)$, where $\vec{\mu} = (\mu_0, \mu_1, \dots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\dots,I;$

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

The parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of means and covariance Smatrix.

Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_0 & \mu_1 & \dots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0, \theta_0) & cov(\theta_0, \theta_1) & \dots & cov(\theta_0, \theta_I) \\ 2 & cov(\theta_1, \theta_0) & cov(\theta_1, \theta_1) & \dots & cov(\theta_1, \theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I, \theta_0) & cov(\theta_I, \theta_1) & \dots & cov(\theta_I, \theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x}, \vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x}, \vec{\theta}) \le z\} = \frac{1}{\sigma_L \sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;$$
$$\sigma_L^2 = cov(\theta_0, \theta_0) - 2\sum_{i=1}^{I} cov(\theta_0, \theta_i) x_i + \sum_{i=1}^{I} \sum_{k=1}^{I} cov(\theta_i, \theta_k) x_i x_k.$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Mean Absolute Penalty Normal Independent is calculated as follows:

$$\text{meanabs\_pen\_nd}\left(L\left(\vec{x}, \vec{\theta}\right)\right) = 2\sigma_L \emptyset\left(-\frac{\mu_L}{\sigma_L}\right) + \mu_L \left[1 - 2\Phi\left(-\frac{\mu_L}{\sigma_L}\right)\right].$$

1.2.5.4 Calculation of Mean Absolute Deviation (meanabs_dev)

Mean Absolute Deviation equals

$$\textbf{meanabs\_dev}(L(\vec{x}, \vec{\theta})) = \textbf{meanabs\_pen}(f(\vec{x}, \vec{\theta})),$$

where

$$f(\vec{x}, \vec{\theta}_j) = L(\vec{x}, \vec{\theta}_j) - E[L(\vec{x}, \vec{\theta})] \quad .$$

1.2.5.5 Calculation of Mean Absolute Deviation Normal Independent (meanabs_ni_dev)

The Mean Absolute Deviation Normal Independent is a special case of the **Calculation of Mean Absolute Deviation Normal Dependent (meanabs_nd_dev)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x}, \vec{\theta}) = L(\vec{x}, \theta_0, \theta_1 \ldots, \theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0, 1, \ldots, I.$
Consider the random function

$$f(\vec{x}, \vec{\theta}) = L(\vec{x}, \vec{\theta}) - E[L(\vec{x}, \vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I} (\theta_i - E[\theta_i]) x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the matrix of variances

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$f(\vec{x}, \vec{\theta}) \sim N(0, \sigma_f^2), \text{ and } F(z) = P\{f(\vec{x}, \vec{\theta}) \le z\} = \frac{1}{\sigma_f \sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}} dy,$$

where

$$\sigma_f^2 = \sigma_0^2 + \sum_{i=1}^{I} x_i^2 \sigma_i^2 .$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

The Mean Absolute Deviation Normal Independent is calculated as follows:

$$\text{meanabs\_ni\_dev}\left(L(\vec{x}, \vec{\theta})\right) = \text{meanabs\_pen\_ni}\left(f(\vec{x}, \vec{\theta})\right) = 2\sigma_f \emptyset(0) \approx 2 \cdot 0.399 \sigma_f .$$

1.2.5.6 Calculation of Mean Absolute Deviation Normal Dependent (meanabs_nd_dev)

The Mean Absolute Deviation Normal Dependent is a special case of the **Calculation of Mean Absolute Deviation (meanabs_dev)** for continuous distributions when coefficients in a gain function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \dots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma)$, where $\vec{\mu} = (\mu_0, \mu_1, \dots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\dots,I$;
Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Consider the random function

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I} (\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the covariance Smatrix, which has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$f(\vec{x},\vec{\theta}) \sim N(0,\sigma_f^2), \text{ and } F(z) = P\{f(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_f\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}} dy,$$

where

$$\sigma_f^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k .$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

The Mean Absolute Deviation Normal Dependent is calculated as follows:

$$\text{meanabs\_nd\_dev}\left(L(\vec{x},\vec{\theta})\right) = \text{meanabs\_pen\_nd}\left(f(\vec{x},\vec{\theta})\right) = 2\sigma_f \emptyset(0) \approx 2 \cdot 0.399\sigma_f .$$

1.2.5.7 Calculation of Mean Absolute Risk for Loss (meanabs_risk)

Mean Absolute Risk for Loss equals

For continuous distributions,

$$\textbf{meanabs\_risk}(L(\vec{x},\vec{\theta})) = \int L(\vec{x},\vec{\theta})\,p(\vec{\theta})d\vec{\theta} + \int |f(\vec{x},\vec{\theta})|\,p(\vec{\theta})d\vec{\theta},$$

where

$p(\vec{\theta})$ is density of the random vector $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$,

and

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] \quad .$$

For discrete distributions, considered in PSG

$$\textbf{meanabs\_risk}(L(\vec{x},\vec{\theta})) = \sum_{j=1}^{J} p_j L(\vec{x},\vec{\theta}_j) + \sum_{j=1}^{J} p_j\,|\,f(\vec{x},\vec{\theta}_j)\,|\,,$$

where

$$L(\vec{x},\vec{\theta}_j) = \theta_{j0} - \sum_{i=1}^{I} \theta_{ji} x_i \quad , \quad j = 1, \ldots, J \quad .$$

and

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] \quad .$$

1.2.5.8 Calculation of Mean Absolute Risk for Loss Normal Independent (meanabs_risk_ni)

The Mean Absolute Risk for Loss Normal Independent is a special case of the **Calculation of Mean Absolute Risk for Loss Normal Dependent (meanabs_risk_nd)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\ldots,I.$
The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \dots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i$; $\sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2 \sigma_I^2$.

Consider the random function

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I} (\theta_i - E[\theta_i]) x_i \quad .$$

In accordance with the properties of the normal distribution,

$$f(\vec{x},\vec{\theta}) \sim N(0, \sigma_f^2), \text{ and } F(z) = P\{f(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_f\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}} dy,$$

where

$$\sigma_f^2 = \sigma_0^2 + \sum_{i=1}^{I} x_i^2 \sigma_i^2.$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

The Mean Absolute Risk for Loss Normal Independent is calculated as follows:

$$\text{meanabs\_risk\_ni}\left(L(\vec{x},\vec{\theta})\right) = \mu_L + \text{ meanabs\_ni\_dev}\left(L(\vec{x},\vec{\theta})\right) = 2 \cdot 0.399\sigma_f + \mu_L.$$

1.2.5.9 Calculation of Mean Absolute Risk for Loss Normal Dependent (meanabs_risk_nd)

The Mean Absolute Risk for Loss Normal Dependent is a special case of the **Calculation of Mean Absolute Risk for Loss (meanabs_risk)** for continuous distributions when coefficients in a loss function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \dots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu}, \Sigma)$, where

$\vec{\mu} = (\mu_0, \mu_1, \dots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\dots,I$;

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

The parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of means and covariance Smatrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_0 & \mu_1 & \dots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L,\sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}}dy,$$

where

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Consider the random function

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I}(\theta_i - E[\theta_i])x_i \quad .$$

In accordance with properties of normal distribution,

$$f(\vec{x},\vec{\theta}) \sim N(0,\sigma_f^2), \text{ and } F(z) = P\{f(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_f\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}}dy,$$

where

$$\sigma_f^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

The Mean Absolute Risk for Loss Normal Dependent is calculated as follows:

$$\text{meanabs\_risk\_nd}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \mu_L + \text{ meanabs\_nd\_dev}\left(L\left(\vec{x},\vec{\theta}\right)\right) = 2 \cdot 0.399\sigma_f + \mu_L.$$

1.2.5.10 Calculation of Mean Absolute Risk for Gain (meanabs_risk_g)

Mean Absolute Risk for Gain equals

$$\mathbf{meanabs\_risk\_g}(G(\vec{x},\vec{\theta})) = \mathbf{meanabs\_risk}(\mathrm{L}(\vec{x},-\vec{\theta})).$$

1.2.5.11 Calculation of Mean Absolute Risk for Gain Normal Independent (meanabs_risk_ni_g)

The Mean Absolute Risk for Gain Normal Independent is a special case of the **Calculation of Mean Absolute Risk for Gain Normal Dependent (meanabs_risk_nd_g)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N\left(\mu_i, \sigma_i^2\right), i = 0,1,\ldots,I.$
The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$L\left(\vec{x},\vec{\theta}\right) \sim N\left(\mu_L, \sigma_L^2\right) \text{ and } F(z) = P\left\{L\left(\vec{x},\vec{\theta}\right) \le z\right\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{\left(y-\mu_L\right)^2}{2\sigma_L^2}}\,dy,$$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;\ \ \sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2\sigma_I^2.$

The corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

Consider the random function

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = -(\theta_0 - E[\theta_0]) + \sum_{i=1}^{I}(\theta_i - E[\theta_i])x_i \quad .$$

In accordance with the properties of the normal distribution,

$$\mathrm{g}\left(\vec{x},\vec{\theta}\right) \sim N(0,\sigma_{\mathrm{g}}^2), \text{ and } F(z) = P\{\mathrm{g}\left(\vec{x},\vec{\theta}\right) \le z\} = \frac{1}{\sigma_{\mathrm{g}}\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_{\mathrm{g}}^2}}dy,$$

where

$$\sigma_{\mathrm{g}}^2 = \sigma_0^2 + \sum_{i=1}^{I} x_i^2\sigma_i^2.$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

The Mean Absolute Risk for Gain Normal Independent is calculated as follows:

$$\text{meanabs\_risk\_ni\_g}\left(G\left(\vec{x},\vec{\theta}\right)\right) = -\mu_L + \text{ meanabs\_ni\_dev}\left(G\left(\vec{x},\vec{\theta}\right)\right) =$$
$$= \text{meanabs\_ni\_dev}\left(L\left(\vec{x},-\vec{\theta}\right)\right) - \mu_L = 2\cdot 0.399\sigma_{\mathrm{g}} - \mu_L.$$

1.2.5.12 Calculation of Mean Absolute Risk for Gain Normal Dependent (meanabs_risk_nd_g)

The Mean Absolute Risk for Gain Normal Dependent is a special case of the **Calculation of Mean Absolute Risk for Gain (meanabs_risk_g)** for continuous distributions when coefficients in a gain function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma)$, where $\vec{\mu} = \left(\mu_0, \mu_1, \ldots, \mu_I\right)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I$;
Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

The parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of means and covariance Smatrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_0 & \mu_1 & \dots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

The corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

Consider the random function

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = -(\theta_0 - E[\theta_0]) + \sum_{i=1}^{I}(\theta_i - E[\theta_i])x_i \quad .$$

In accordance with the properties of the normal distribution,

$$g(\vec{x},\vec{\theta}) \sim N(0,\sigma_g^2), \text{ and } F(z) = P\{g(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_g\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_g^2}} dy,$$

where

$$\sigma_g^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

The Mean Absolute Risk for Gain Normal Dependent is calculated as follows:

$$\text{meanabs\_risk\_nd\_g}\left(G\left(\vec{x},\vec{\theta}\right)\right) = -\mu_L + \text{meanabs\_nd\_dev}\left(G\left(\vec{x},\vec{\theta}\right)\right) =$$

$$= \text{meanabs\_nd\_dev}\left(L\left(\vec{x},-\vec{\theta}\right)\right) - \mu_L = 2 \cdot 0.399\sigma_{\text{g}} - \mu_L.$$

1.2.5.13 Properties of Mean Abs Group

Functions from the Mean Abs group are calculated with double precision. The name of any function from this group may contain up to 128 symbols. The name of the Mean Absolute Penalty function should begin with the string “meanabs_pen_”, the name of the Mean Absolute Penalty Normal Independent function should begin with the string “meanabs_pen_ni_", the name of the Mean Absolute Penalty Normal Dependent function should begin with the string “meanabs_pen_nd_", the name of the Mean Absolute Deviation function should begin with the string “meanabs_dev_”, the name of the Mean Absolute Deviation Normal Independent function should begin with the string "meanabs_ni_dev", the name of the Mean Absolute Deviation Normal Dependent function should begin with the string "meanabs_nd_dev_", the name of the Mean Absolute Risk for Loss function should begin with the string “meanabs_risk_”, the name of the Mean Absolute Risk for Loss Normal Independent function should begin with the string "meanabs_risk_ni_", the name of the Mean Absolute Risk for Loss Normal Dependent function should begin with the string "meanabs_risk_nd_", the name of the Mean Absolute Risk for Gain function should begin with the string “meanabs_risk_g_”, the name of the Mean Absolute Risk for Gain Normal Independent function should begin with the string "meanabs_risk_ni_g", the name of the Mean Absolute Risk for Gain Normal Dependent function should begin with the string "meanabs_risk_nd_g". The names of these functions may include only alphabetic characters, numbers, and the underscore sign, “_”. The names of these functions are "insensitive" to the case, i.e. there is no difference between low case and upper case in these names.

1.2.6 Partial Moment Group

The Partial Moment Group includes the following functions:

- Partial Moment Penalty for Loss (software notation: pm_pen_...) (section **Calculation of Partial Moment Penalty for Loss**)
- Partial Moment Penalty for Loss Normal Independent (software notation: pm_pen_ni_...) (section **Calculation of Partial Moment Penalty for Loss Normal Independent (pm_pen_ni)**)
- Partial Moment Penalty for Loss Normal Dependent (software notation: pm_pen_nd_...) (section **Calculation of Partial Moment Penalty for Loss Normal Dependent (pm_pen_nd)**)
- Partial Moment Penalty for Gain (software notation: pm_pen_g_...) (section **Calculation of Partial Moment Penalty for Gain)**
- Partial Moment Penalty for Gain Normal Independent (software notation: pm_pen_ni_g_) (section **Calculation of Partial Moment Penalty for Gain Normal Independent (pm_pen_ni_g)**)
- Partial Moment Penalty for Gain Normal Dependent (software notation: pm_pen_nd_g_) (section **Calculation of Partial Moment Penalty for Gain Normal Dependent (pm_pen_nd_g)**)
- Partial Moment Loss Deviation (software notation: pm_dev_...) (section **Calculation of Partial Moment Loss Deviation**)
- Partial Moment Loss Deviation Normal Independent (software notation: pm_ni_dev_) (section **Calculation of Partial Moment Loss Deviation Normal Independent (pm_ni_dev)**)
- Partial Moment Loss Deviation Normal Dependent (software notation: pm_nd_dev_) (section **Calculation of Partial Moment Loss Deviation Normal Dependent (pm_nd_dev)**)
- Partial Moment Gain Deviation (software notation: pm_dev_g_...) (section **Calculation of Partial Moment Gain Deviation**)

- Partial Moment Gain Deviation Normal Independent (software notation: pm_ni_dev_g) (section **Calculation of Partial Moment Gain Deviation Normal Independent (pm_ni_dev_g)**)
- Partial Moment Gain Deviation Normal Dependent (software notation: pm_ni_dev_g) (section **Calculation of Partial Moment Gain Deviation Normal Dependent (pm_nd_dev_g)**)
- Partial Moment Two Penalty for Loss (software notation: pm2_pen...) (section **Calculation of Partial Moment Two Penalty for Loss**)
- Partial Moment Two Penalty for Loss Normal Independent (software notation: pm2_pen_ni) (section **Calculation of Partial Moment Two Penalty for Loss Normal Independent (pm2_pen_ni)**)
- Partial Moment Two Penalty for Loss Normal Dependent (software notation: pm2_pen_nd) (section **Calculation of Partial Moment Two Penalty for Loss Normal Dependent (pm2_pen_nd)**)
- Partial Moment Two Penalty for Gain (software notation: pm2_pen_g...) (section **Calculation of Partial Moment Two Penalty for Gain**)
- Partial Moment Two Penalty for Gain Normal Independent (software notation: pm2_pen_ni_g) (section **Calculation of Partial Moment Two Penalty for Gain Normal Independent (pm2_pen_ni_g)**)
- Partial Moment Two Penalty for Gain Normal Dependent (software notation: pm2_pen_nd_g) (section **Calculation of Partial Moment Two Penalty for Gain Normal Dependent (pm2_pen_nd_g)**)
- Partial Moment Two Deviation for Loss (software notation: pm2_dev...) (section **Calculation of Partial Moment Two Deviation for Loss**)
- Partial Moment Two Deviation for Loss Normal Independent (software notation: pm2_ni_dev) (section **Calculation of Partial Moment Two Deviation for Loss Normal Independent (pm2_ni_dev)**)
- Partial Moment Two Deviation for Loss Normal Dependent (software notation: pm2_nd_dev) (section **Calculation of Partial Moment Two Deviation for Loss Normal Dependent (pm2_nd_dev)**)
- Partial Moment Two Deviation for Gain (software notation: pm2_dev_g...) (section **Calculation of Partial Moment Two Deviation for Gain**)
- Partial Moment Two Deviation for Gain Normal Independent (software notation: pm2_ni_dev_g) (section **Calculation of Partial Moment Two Deviation for Gain Normal Independent (pm2_ni_dev_g)**)
- Partial Moment Two Deviation for Gain Normal Dependent (software notation: pm2_nd_dev_g) (section **Calculation of Partial Moment Two Deviation for Gain Normal Dependent (pm2_nd_dev_g)**)

For more details about the Properties of this Group see the section **Properties of Partial Moment Group**.

These functions depend on the parameter w and are defined on some **Point** ,$\vec{x} = (x_1, x_2, \ldots, x_I)$, and use **Matrix of Scenarios** (in regular Matrix or in packed Pmatrix format) or **Simmetric Matrix** (Smatrix) or parameters of normal distribution.

1.2.6.1 Calculation of Partial Moment Penalty for Loss (pm_pen)

Partial Moment Penalty for Loss equals

For continuous distributions,

$$\mathbf{pm\_pen}(L(\vec{x},\vec{\theta})) = \int \max\left\{0, L(\vec{x},\vec{\theta}) - w\right\} p(\vec{\theta})d\vec{\theta},$$

where

$p(\vec{\theta})$ is the smooth probability density of the random vector $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$,

and

w is a threshold value.

For discrete distributions, considered in PSG

$$\textbf{pm\_pen}(L(\vec{x},\vec{\theta})) = \sum_{j=1}^{J} p_j \max\left\{0, L(\vec{x},\vec{\theta}_j) - w\right\},$$

where w is a threshold value.

1.2.6.2 Calculation of Partial Moment Penalty for Loss Normal Independent (pm_pen_ni)

The Partial Moment Penalty for Loss Normal Independent is a special case of the **Calculation of Partial Moment Penalty for Loss Normal Dependent (pm_pen_nd)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\ldots,I.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 \ldots & nameI \\ 1 & \mu_0 & \mu_1 \quad \ldots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 \quad \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;\ \ \sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2 \sigma_I^2.$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Partial Moment Penalty for Loss Normal Independent is calculated as follows:

$$\text{pm\_pen\_ni}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \sigma_L \emptyset\left(\frac{w-\mu_L}{\sigma_L}\right) + (\mu_L - w)\left[1 - \Phi\left(\frac{w-\mu_L}{\sigma_L}\right)\right].$$

1.2.6.3 Calculation of Partial Moment Penalty for Loss Normal Dependent (pm_pen_nd)

The Partial Moment Penalty for Loss Normal Dependent is a special case of the **Calculation of Partial Moment Penalty for Loss (pm_pen)** for continuous distributions when coefficients in a loss function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \dots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma)$, where
$\vec{\mu} = \left(\mu_0, \mu_1, \dots, \mu_I\right)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\dots,I$;
Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

The parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of means and covariance Smatrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_0 & \mu_1 & \dots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L,\sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}}\,dy,$$

where

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z}\emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

The Partial Moment Penalty for Loss Normal Dependent is calculated as follows:

$$\text{pm\_pen\_nd}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \sigma_L\emptyset\left(\frac{w-\mu_L}{\sigma_L}\right) + (\mu_L - w)\left[1 - \Phi\left(\frac{w-\mu_L}{\sigma_L}\right)\right].$$

1.2.6.4 Calculation of Average Partial Moment Penalty for Loss Normal Independent (avg_pm_pen_ni)

Let

M = number of random loss functions;

$\vec{\theta}^m = (\theta_0^m,\theta_1^m,\dots,\theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1,\dots,M.$

$L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I}\theta_i^m x_i$ = m-th loss function, $m = 1,\dots,M.$

All coefficients $\theta_0^1,\theta_1^1,\dots,\theta_I^1,\theta_0^2,\theta_1^2,\dots,\theta_I^2,\dots,\theta_0^M,\theta_1^M,\dots,\theta_I^M$ are independent and normally distributed random values:

$$\theta_i^m \sim N(\mu_{mi},\sigma_{mi}^2),\ i = 0,1,\dots,I;\ \ m = 1,\dots,M.$$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.

Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI & scenario\_probability \\ 1 & \mu_{10} & \mu_{11} & \dots & \mu_{1I} & v_1 \\ 2 & \mu_{20} & \mu_{21} & \dots & \mu_{2I} & v_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ M & \mu_{M0} & \mu_{M1} & \dots & \mu_{MI} & v_M \end{pmatrix},$$

where row with $id = m$ contains means of coefficients of m-th loss function;

$v_m \ge 0$ = weight of m-th loss function.

If scenario_probability column is absent or all $v_m = 0$ then all weights are considered as equal to 1.

$\bar{v}_m = v_m \Big/ \sum_{k=1}^{M} v_k$ is normalized weight of m-th loss function.

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots name I \\ 1 & \sigma_{10}^2 & \sigma_{11}^2 & \dots & \sigma_{1I}^2 \\ 2 & \sigma_{20}^2 & \sigma_{21}^2 & \dots & \sigma_{2I}^2 \\ \dots & \dots & \dots & \dots & \dots \\ M & \sigma_{M0}^2 & \sigma_{M1}^2 & \dots & \sigma_{MI}^2 \end{pmatrix},$$

where row with $id = m$ contains variances of coefficients of m-th loss function.

Let w = a threshold.

In accordance with the properties of the normal distribution,

$$L_m(\vec{x},\vec{\theta}^m) \sim N(\mu_{L_m}, \sigma_{L_m}^2),$$

where $\mu_{L_m} = \mu_{m0} - \sum_{i=1}^{I} \mu_{mi} x_i = E[L_m(\vec{x},\vec{\theta}^m)]$;

$$\sigma_{L_m}^2 = \sigma_{m0}^2 + \sum_{i=1}^{I} \sigma_{mi}^2\, x_i^2 = Var\left(L_m(\vec{x},\vec{\theta}^m)\right) =$$

$$= E\left[\left(L_m(\vec{x},\vec{\theta}^m) - \mu_{L_m}\right)^2\right]; \quad m = 1,\dots,M.$$

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Average Partial Moment Penalty for Loss Normal Independent is calculated as weighted mean of separate functions:

$$\text{avg\_pm\_pen\_ni}_w \left(L_1\left(\vec{x},\overrightarrow{\theta^1}\right),\dots,L_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right) =$$

$$= \sum_{m=1}^{M} \bar{v}_m \left(\sigma_{L_m} \phi\left(\frac{w-\mu_{L_m}}{\sigma_{L_m}}\right) + (\mu_{L_m} - w)\left[1 - \phi\left(\frac{w-\mu_{L_m}}{\sigma_{L_m}}\right)\right]\right).$$

1.2.6.5 Calculation of Partial Moment Penalty for Gain (pm_pen_g)

Partial Moment Penalty for Gain equals

$$\mathbf{pm\_pen\_g}\,(G(\vec{x},\vec{\theta})) = \mathbf{pm\_pen}(L(\vec{x},-\vec{\theta}))\,.$$

1.2.6.6 Calculation of Partial Moment Penalty for Gain Normal Independent (pm_pen_ni_g)

The Partial Moment Penalty for Gain Normal Independent is a special case of the **Calculation of Partial Moment Penalty for Gain Normal Dependent (pm_pen_nd_g)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

Corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\ldots,I.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with properties of normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}}\,dy,$$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;\ \sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2\sigma_I^2.$

Let

$\boldsymbol{w}$ be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

The Partial Moment Penalty for Gain Normal Independent is calculated as follows:

$$\text{pm\_pen\_ni\_g}\left(G\left(\vec{x},\vec{\theta}\right)\right) = \text{pm\_pen\_ni}\left(L\left(\vec{x},-\vec{\theta}\right)\right)$$
$$= \sigma_L \emptyset\left(\frac{w+\mu_L}{\sigma_L}\right) - (\mu_L + w)\left[1 - \Phi\left(\frac{w+\mu_L}{\sigma_L}\right)\right].$$

1.2.6.7 Calculation of Partial Moment Penalty for Gain Normal Dependent (pm_pen_nd_g)

The Partial Moment Penalty for Gain Normal Dependent is a special case of the **Calculation of Partial Moment Penalty for Gain (pm_pen_g)** for continuous distributions when coefficients in a gain function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

The corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma),$ where $\vec{\mu} = \left(\mu_0, \mu_1, \ldots, \mu_I\right)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I;$
Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

The parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of means and covariance Smatrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L,\sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}}\,dy,$$

where

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

The Partial Moment Penalty for Gain Normal Dependent is calculated as follows:

$$\text{pm\_pen\_nd\_g}\left(G\left(\vec{x},\vec{\theta}\right)\right) = \text{pm\_pen\_nd}\left(L\left(\vec{x},-\vec{\theta}\right)\right)$$
$$= \sigma_L\emptyset\left(\frac{w+\mu_L}{\sigma_L}\right) - (\mu_L + w)\left[1-\Phi\left(\frac{w+\mu_L}{\sigma_L}\right)\right].$$

1.2.6.8 Calculation of Average Partial Moment Penalty for Gain Normal Independent (avg_pm_pen_ni_g)

Let

M = number of random loss functions;

$\vec{\theta}^m = (\theta_0^m, \theta_1^m, \dots, \theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1,\dots,M$.

$L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I}\theta_i^m x_i$ = m-th loss function, $m = 1,\dots,M$.

$G_m(\vec{x},\vec{\theta}^m) = L_m(\vec{x},-\vec{\theta}^m) = -\theta_0^m + \sum_{i=1}^{I}\theta_i^m x_i$ = m-th gain function, $m = 1,\dots,M$.

All coefficients $\theta_0^1,\theta_1^1,\dots,\theta_I^1,\theta_0^2,\theta_1^2,\dots,\theta_I^2,\dots,\theta_0^M,\theta_1^M,\dots,\theta_I^M$ are independent and normally distributed random values:

$\theta_i^m \sim N(\mu_{mi}, \sigma_{mi}^2),\; i = 0,1,\dots,I;\;\; m = 1,\dots,M.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.

Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI & scenario\_probability \\ 1 & \mu_{10} & \mu_{11} & \dots & \mu_{1I} & v_1 \\ 2 & \mu_{20} & \mu_{21} & \dots & \mu_{2I} & v_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ M & \mu_{M0} & \mu_{M1} & \dots & \mu_{MI} & v_M \end{pmatrix},$$

where row with $id = m$ contains means of coefficients of m-th loss function;

$v_m \geq 0$ = weight of m-th loss function.

If scenario_probability column is absent or all $v_m = 0$ then all weights are considered as equal to 1.

$\bar{v}_m = v_m \Big/ \sum_{k=1}^{M} v_k$ is normalized weight of m-th loss function.

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_{10}^2 & \sigma_{11}^2 & \dots & \sigma_{1I}^2 \\ 2 & \sigma_{20}^2 & \sigma_{21}^2 & \dots & \sigma_{2I}^2 \\ \dots & \dots & \dots & \dots & \dots \\ M & \sigma_{M0}^2 & \sigma_{M1}^2 & \dots & \sigma_{MI}^2 \end{pmatrix},$$

where row with $id = m$ contains variances of coefficients of m-th loss function.

Let w = a threshold.

In accordance with the properties of the normal distribution,

$G_m(\vec{x}, \vec{\theta}^m) \sim N(\mu_{G_m}, \sigma_{G_m}^2),$

where

$$\mu_{G_m} = -\mu_{m0} + \sum_{i=1}^{I} \mu_{mi} x_i = -E[L_m(\vec{x}, \vec{\theta}^m)];$$

$$\sigma_{G_m}^2 = \sigma_{m0}^2 + \sum_{i=1}^{I} \sigma_{mi}^2\, x_i^2 = Var\left(G_m(\vec{x}, \vec{\theta}^m)\right) =$$

$$= E\left[\left(G_m(\vec{x}, \vec{\theta}^m) - \mu_{G_m}\right)^2\right] = Var(L_m(\vec{x}, \vec{\theta}^m));\;\; m = 1,\dots,M.$$

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Average Partial Moment Penalty for Gain Normal Independent is calculated as weighted mean of separate functions:

$$\text{avg\_pm\_pen\_ni\_g}_w\left(L_1\left(\vec{x},\overrightarrow{\theta^1}\right),\dots,L_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right)=$$

$$=\text{avg\_pm\_pen\_ni}_w\left(G_1\left(\vec{x},\overrightarrow{\theta^1}\right),\dots,G_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right)=$$

$$=\sum_{m=1}^{M}\bar{v}_m\left(\sigma_{Lm}\phi\left(\frac{w+\mu_{Lm}}{\sigma_{Lm}}\right)-\left(\mu_{Lm}+w\right)\left[1-\phi\left(\frac{w+\mu_{Lm}}{\sigma_{Lm}}\right)\right]\right).$$

1.2.6.9 Calculation of Partial Moment Loss Deviation (pm_dev)

Partial Moment Loss Deviation equals

$$\textbf{pm\_dev}(L(\vec{x},\vec{\theta}))=\textbf{pm\_pen}(f(\vec{x},\vec{\theta})),$$

where

$$f(\vec{x},\vec{\theta})=L(\vec{x},\vec{\theta})-E[L(\vec{x},\vec{\theta})] \quad .$$

1.2.6.10 Calculation of Partial Moment Loss Deviation Normal Independent (pm_ni_dev)

The Partial Moment Loss Deviation Normal Independent is a special case of the **Calculation of Partial Moment Loss Deviation Normal Dependent (pm_nd_dev)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x}=(x_1,x_2,\dots,x_I)$ be a decision vector; $\vec{\theta}=(\theta_0,\theta_1,\dots\theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta})=L(\vec{x},\theta_0,\theta_1\dots,\theta_I)=\theta_0-\sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N\left(\mu_i,\sigma_i^2\right), i=0,1,\dots,I.$
Consider the random function

$$f(\vec{x},\vec{\theta})=L(\vec{x},\vec{\theta})-E[L(\vec{x},\vec{\theta})]=(\theta_0-E[\theta_0])-\sum_{i=1}^{I}(\theta_i-E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the matrix of variances

$$V=\begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \dots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$f(\vec{x},\vec{\theta}) \sim N(0,\sigma_f^2), \text{ and } F(z) = P\{f(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_f\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}}dy,$$

where

$$\sigma_f^2 = \sigma_0^2 + \sum_{i=1}^{I} x_i^2\sigma_i^2.$$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z}\emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

The Partial Moment Loss Deviation Normal Independent is calculated as follows:

$$\text{pm\_ni\_dev}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \text{pm\_pen\_ni}\left(f\left(\vec{x},\vec{\theta}\right)\right) = \sigma_f\emptyset\left(\frac{w}{\sigma_f}\right) - w\left[1-\Phi\left(\frac{w}{\sigma_f}\right)\right].$$

1.2.6.11 Calculation of Partial Moment Loss Deviation Normal Dependent (pm_nd_dev)

The Partial Moment Loss Deviation Normal Dependent is a special case of the **Calculation of Partial Moment Loss Deviation (pm_dev)** for continuous distributions when coefficients in a loss function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\dots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma)$, where

$\vec{\mu} = \left(\mu_0, \mu_1, \dots, \mu_I\right)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\dots,I;$

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Consider the random function

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I}(\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the covariance Smatrix, which has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$f(\vec{x},\vec{\theta}) \sim N(0,\sigma_f^2), \text{ and } F(z) = P\{f(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_f\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}}dy,$$

where

$$\sigma_f^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z}\emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

The Partial Moment Loss Deviation Normal Dependent is calculated as follows:

$$\text{pm\_nd\_dev}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \text{pm\_pen\_nd}\left(f\left(\vec{x},\vec{\theta}\right)\right) = \sigma_f\emptyset\left(\frac{w}{\sigma_f}\right) - w\left[1-\Phi\left(\frac{w}{\sigma_f}\right)\right].$$

1.2.6.12 Calculation of Average Partial Moment Loss Deviation Normal Independent (avg_pm_ni_dev)

Let

M = number of random loss functions;

$\vec{\theta}^m = (\theta_0^m, \theta_1^m, \dots, \theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1,\dots,M$.

$L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I}\theta_i^m x_i$ = m-th loss function, $m = 1,\dots,M$.

All coefficients $\theta_0^1,\theta_1^1,\dots,\theta_I^1,\theta_0^2,\theta_1^2,\dots,\theta_I^2,\dots,\theta_0^M,\theta_1^M,\dots,\theta_I^M$ are independent and normally distributed random values:

$$\theta_i^m \sim N(\mu_{mi},\sigma_{mi}^2),\ i = 0,1,\dots,I;\ \ m = 1,\dots,M.$$

Consider the random functions

$$f_m(\vec{x},\vec{\theta}^m) = L_m(\vec{x},\vec{\theta}^m) - E[L_m(\vec{x},\vec{\theta}^m)] = (\theta_0^m - E[\theta_0^m]) - \sum_{i=1}^{I}(\theta_i^m - E[\theta_i^m])x_i,$$
$$m = 1,\dots,M.$$

Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI & scenario\_probability \\ 1 & \mu_{10} & \mu_{11} & \dots & \mu_{1I} & v_1 \\ 2 & \mu_{20} & \mu_{21} & \dots & \mu_{2I} & v_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ M & \mu_{M0} & \mu_{M1} & \dots & \mu_{MI} & v_M \end{pmatrix},$$

where row with $id = m$ contains means of coefficients of m-th loss function;

$v_m \geq 0$ = weight of m-th loss function.

If scenario_probability column is absent or all $v_m = 0$ then all weights are considered as equal to 1.

$\bar{v}_m = v_m \Big/ \sum_{k=1}^{M} v_k$ is normalized weight of m-th loss function.

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_{10}^2 & \sigma_{11}^2 & \dots & \sigma_{1I}^2 \\ 2 & \sigma_{20}^2 & \sigma_{21}^2 & \dots & \sigma_{2I}^2 \\ \dots & \dots & \dots & \dots & \dots \\ M & \sigma_{M0}^2 & \sigma_{M1}^2 & \dots & \sigma_{MI}^2 \end{pmatrix},$$

where row with $id = m$ contains variances of coefficients of m-th loss function.

Let w = a threshold.

In accordance with the properties of the normal distribution,

$$f_m(\vec{x},\vec{\theta}^m) \sim N(0,\sigma_{fm}^2),$$

where

$$\sigma_{fm}^2 = \sigma_{m0}^2 + \sum_{i=1}^{I} \sigma_{mi}^2 x_i^2 = Var\left(f_m(\vec{x},\vec{\theta}^m)\right) =$$
$$= E\left[\left(f_m(\vec{x},\vec{\theta}^m)\right)^2\right] = Var(L_m(\vec{x},\vec{\theta}^m)); \quad m = 1,\dots,M.$$

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Average Partial Moment Loss Deviation Normal Independent is calculated as weighted mean of separate functions:

$$\text{avg\_pm\_ni\_dev}_w\left(L_1\left(\vec{x},\overrightarrow{\theta^1}\right),\dots,L_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right)=$$

$$=\text{avg\_pm\_pen\_ni}_w\left(f_1\left(\vec{x},\overrightarrow{\theta^1}\right),\dots,f_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right)=$$

$$=\sum_{m=1}^{M}\bar{v}_m\left(\sigma_{f\,m}\phi\left(\frac{w}{\sigma_{f\,m}}\right)-w\left[1-\phi\left(\frac{w}{\sigma_{f\,m}}\right)\right]\right).$$

1.2.6.13 Calculation of Partial Moment Gain Deviation (pm_dev_g)

Partial Moment Gain Deviation equals

$$\mathbf{pm\_dev\_g}(G(\vec{x},\vec{\theta}))=\mathbf{pm\_dev}(\mathrm{L}(\vec{x},-\vec{\theta})).$$

1.2.6.14 Calculation of Partial Moment Gain Deviation Normal Independent (pm_ni_dev_g)

The Partial Moment Gain Deviation Normal Independent is a special case of the **Calculation of Partial Moment Gain Deviation Normal Dependent (pm_nd_dev_g)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x}=(x_1,x_2,\dots,x_I)$ be a decision vector; $\vec{\theta}=(\theta_0,\theta_1,\dots\theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta})=L(\vec{x},\theta_0,\theta_1\dots,\theta_I)=\theta_0-\sum_{i=1}^{I}\theta_i x_i\quad.$$

The corresponding Gain Function is

$$G(\vec{x},\vec{\theta})=L(\vec{x},-\vec{\theta})=-L(\vec{x},\vec{\theta})=-\theta_0+\sum_{i=1}^{I}\theta_i x_i\quad.$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i,\sigma_i^2), i=0,1,\dots,I.$
Consider the random function

$$g(\vec{x},\vec{\theta})=G(\vec{x},\vec{\theta})-E[G(\vec{x},\vec{\theta})]=-(\theta_0-E[\theta_0])+\sum_{i=1}^{I}(\theta_i-E[\theta_i])x_i\quad.$$

Since the mean of this function is zero, it is sufficient to consider only the matrix of variances

$$V=\begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \dots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$g(\vec{x},\vec{\theta}) \sim N(0,\sigma_g^2), \text{ and } F(z) = P\{g(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_g\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_g^2}} dy,$$

where

$$\sigma_g^2 = \sigma_0^2 + \sum_{i=1}^{I} x_i^2 \sigma_i^2.$$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Partial Moment Gain Deviation Normal Independent is calculated as follows:

$$\text{pm\_ni\_dev}\left(G\left(\vec{x},\vec{\theta}\right)\right) = \text{pm\_pen\_ni}\left(g\left(\vec{x},\vec{\theta}\right)\right) = \sigma_g \emptyset\left(\frac{w}{\sigma_g}\right) - w\left[1 - \Phi\left(\frac{w}{\sigma_g}\right)\right].$$

1.2.6.15 Calculation of Partial Moment Gain Deviation Normal Dependent (pm_nd_dev_g)

The Partial Moment Gain Deviation Normal Dependent is a special case of the **Calculation of Partial Moment Gain Deviation (pm_dev_g)** for continuous distributions when coefficients in a gain function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

Corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma)$, where $\vec{\mu} = (\mu_0, \mu_1, \ldots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I$;

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Consider the random function

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = -(\theta_0 - E[\theta_0]) + \sum_{i=1}^{I}(\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the covariance Smatrix, which has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$\mathrm{g}(\vec{x},\vec{\theta}) \sim N(0,\sigma_{\mathrm{g}}^2), \text{ and } F(z) = P\{\mathrm{g}(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_{\mathrm{g}}\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_{\mathrm{g}}^2}}dy,$$

where

$$\sigma_{\mathrm{g}}^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k .$$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z}\emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

The Partial Moment Gain Deviation Normal Dependent is calculated as follows:

$$\mathrm{pm\_nd\_dev}\left(G\left(\vec{x},\vec{\theta}\right)\right) = \mathrm{pm\_pen\_nd}\left(\mathrm{g}\left(\vec{x},\vec{\theta}\right)\right) = \sigma_{\mathrm{g}}\emptyset\left(\frac{w}{\sigma_{\mathrm{g}}}\right) - w\left[1 - \Phi\left(\frac{w}{\sigma_{\mathrm{g}}}\right)\right].$$

1.2.6.16 Calculation of Average Partial Moment Gain Deviation Normal Independent (avg_pm_ni_dev_g)

Let

M = number of random loss functions;

$\vec{\theta}^m = (\theta_0^m, \theta_1^m, \dots, \theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1, \dots, M.$

$L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I}\theta_i^m x_i$ = m-th loss function, $m = 1, \dots, M.$

$G_m(\vec{x},\vec{\theta}^m) = L_m(\vec{x},-\vec{\theta}^m) = -\theta_0^m + \sum_{i=1}^{I}\theta_i^m x_i$ = m-th gain function, $m = 1, \dots, M.$

All coefficients $\theta_0^1, \theta_1^1, \ldots, \theta_I^1, \theta_0^2, \theta_1^2, \ldots, \theta_I^2, \ldots, \theta_0^M, \theta_1^M, \ldots, \theta_I^M$ are independent and normally distributed random values:

$\theta_i^m \sim N(\mu_{mi}, \sigma_{mi}^2),\ i = 0,1,\ldots,I;\ \ m = 1,\ldots,M.$

Consider the random functions

$$g_m(\vec{x}, \vec{\theta}^m) = G_m(\vec{x}, \vec{\theta}^m) - E[G_m(\vec{x}, \vec{\theta}^m)] = -(\theta_0^m - E[\theta_0^m]) + \sum_{i=1}^I (\theta_i^m - E[\theta_i^m]) x_i,$$

$m = 1, \ldots, M.$

Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI & scenario\_probability \\ 1 & \mu_{10} & \mu_{11} & \ldots & \mu_{1I} & v_1 \\ 2 & \mu_{20} & \mu_{21} & \ldots & \mu_{2I} & v_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ M & \mu_{M0} & \mu_{M1} & \ldots & \mu_{MI} & v_M \end{pmatrix},$$

where row with $id = m$ contains means of coefficients of m-th loss function;

$v_m \geq 0$ = weight of m-th loss function.

If scenario_probability column is absent or all $v_m = 0$ then all weights are considered as equal to 1.

$\bar{v}_m = v_m \Big/ \sum_{k=1}^{M} v_k$ is normalized weight of m-th loss function.

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_{10}^2 & \sigma_{11}^2 & \ldots & \sigma_{1I}^2 \\ 2 & \sigma_{20}^2 & \sigma_{21}^2 & \ldots & \sigma_{2I}^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ M & \sigma_{M0}^2 & \sigma_{M1}^2 & \ldots & \sigma_{MI}^2 \end{pmatrix},$$

where row with $id = m$ contains variances of coefficients of m-th loss function.

Let w = a threshold.

In accordance with the properties of the normal distribution,

$$g_m(\vec{x}, \vec{\theta}^m) \sim N(0, \sigma_{g_m}^2),$$

where

$$\sigma_{g_m}^2 = \sigma_{m0}^2 + \sum_{i=1}^{I} \sigma_{mi}^2 x_i^2 = Var\left(f_g(\vec{x}, \vec{\theta}^m)\right) =$$

$$= E\left[\left(g_m(\vec{x}, \vec{\theta}^m)\right)^2\right] = Var(L_m(\vec{x}, \vec{\theta}^m));\quad m = 1,\ldots,M.$$

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Average Partial Moment Gain Deviation Normal Independent is calculated as weighted mean of separate functions:

$$\text{avg\_pm\_ni\_dev\_g}_w\left(L_1\left(\vec{x},\overrightarrow{\theta^1}\right),\ldots,L_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right) =$$

$$=\text{avg\_pm\_pen\_ni}_w\left(g_1\left(\vec{x},\overrightarrow{\theta^1}\right),\ldots,g_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right) =$$

$$= \sum_{m=1}^{M} \bar{v}_m \left(\sigma_{g_m} \phi\left(\frac{w}{\sigma_{g_m}}\right) - w\left[1-\phi\left(\frac{w}{\sigma_{g_m}}\right)\right]\right).$$

1.2.6.17 Calculation of Partial Moment Two Penalty for Loss (pm2_pen)

Partial Moment Two Penalty for Loss equals

$$\mathbf{pm2\_pen}(L(\vec{x},\vec{\theta})) = \sum_{j=1}^{J} p_j\left(\max\left\{0, L(\vec{x},\vec{\theta}_j) - w\right\}\right)^2,$$

where w is a threshold value.

1.2.6.18 Calculation of Partial Moment Two Penalty for Loss Normal Independent (pm2_pen_ni)

The Partial Moment Two Penalty for Loss Normal Independent is a special case of the **Calculation of Partial Moment Two Penalty for Loss Normal Dependent (pm2_pen_nd)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented in the shape one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\ldots,I.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \dots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;\ \ \sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2 \sigma_I^2.$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Partial Moment Two Penalty for Loss Normal Independent is calculated as follows:

$$\text{pm2\_pen\_ni}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \sigma_L(\mu_L - w)\emptyset\left(\frac{w-\mu_L}{\sigma_L}\right) + \left(\sigma_L^2 + (\mu_L - w)^2\right)\left[1 - \Phi\left(\frac{w-\mu_L}{\sigma_L}\right)\right].$$

1.2.6.19 Calculation of Partial Moment Two Penalty for Loss Normal Dependent (pm2_pen_nd)

The Partial Moment Two Penalty for Loss Normal Dependent is a special case of the **Calculation of Partial Moment Two Penalty for Loss (pm2_pen)** for continuous distributions when random coefficients in a loss function follow the multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \dots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu}, \Sigma)$, where

$\vec{\mu} = (\mu_0, \mu_1, \dots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\dots,I;$

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

The parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of

means and covariance Smatrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_0 & \mu_1 & \dots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L,\sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}}\,dy,$$

where

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

The Partial Moment Two Penalty for Loss Normal Dependent is calculated as follows:

$$\text{pm2\_pen\_nd}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \sigma_L(\mu_L - w)\emptyset\left(\frac{w-\mu_L}{\sigma_L}\right) + \left(\sigma_L^2 + (\mu_L - w)^2\right)\left[1 - \Phi\left(\frac{w-\mu_L}{\sigma_L}\right)\right].$$

1.2.6.20 Calculation of Partial Moment Two Penalty for Gain (pm2_pen_g)

Partial Moment Two Penalty for Gain equals

$$\text{pm2\_pen\_g}(G(\vec{x},\vec{\theta})) = \text{pm2\_pen}(L(\vec{x},-\vec{\theta})).$$

1.2.6.21 Calculation of Partial Moment Two Penalty for Gain Normal Independent (pm2_pen_ni_g)

The Partial Moment Two Penalty for Gain Normal Independent is a special case of the **Calculation of Partial Moment Two Penalty for Gain Normal Dependent (pm2_pen_nd_g)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x}, \vec{\theta}) = L(\vec{x}, \theta_0, \theta_1 \ldots, \theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

The corresponding Gain Function is

$$G(\vec{x}, \vec{\theta}) = L(\vec{x}, -\vec{\theta}) = -L(\vec{x}, \vec{\theta}) = -\theta_0 + \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0, 1, \ldots, I.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$L(\vec{x}, \vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x}, \vec{\theta}) \le z\} = \frac{1}{\sigma_L \sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;\ \sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2 \sigma_I^2.$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Partial Moment Two Penalty for Gain Normal Independent is calculated as follows:

$$\text{pm2\_pen\_ni\_g}\left(G\left(\vec{x},\vec{\theta}\right)\right) = \text{pm2\_pen\_ni}\left(L\left(\vec{x},-\vec{\theta}\right)\right)$$
$$= -\sigma_L(\mu_L + w)\emptyset\left(\frac{w+\mu_L}{\sigma_L}\right) + \left(\sigma_L^2 + (\mu_L + w)^2\right)\left[1 - \Phi\left(\frac{w+\mu_L}{\sigma_L}\right)\right].$$

1.2.6.22 Calculation of Partial Moment Two Penalty for Gain Normal Dependent (pm2_pen_nd_g)

The Partial Moment Two Penalty for Gain Normal Dependent is a special case of the **Calculation of Partial Moment Two Penalty for Gain (pm2_pen_g)** for continuous distributions when coefficients in a gain function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

The corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma)$, where

$\vec{\mu} = \left(\mu_0, \mu_1, \ldots, \mu_I\right)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I;$

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

The parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of means and covariance Smatrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0, \theta_0) & cov(\theta_0, \theta_1) & \dots & cov(\theta_0, \theta_I) \\ 2 & cov(\theta_1, \theta_0) & cov(\theta_1, \theta_1) & \dots & cov(\theta_1, \theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I, \theta_0) & cov(\theta_I, \theta_1) & \dots & cov(\theta_I, \theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x}, \vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x}, \vec{\theta}) \le z\} = \frac{1}{\sigma_L \sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;$$

$$\sigma_L^2 = cov(\theta_0, \theta_0) - 2\sum_{i=1}^{I} cov(\theta_0, \theta_i) x_i + \sum_{i=1}^{I} \sum_{k=1}^{I} cov(\theta_i, \theta_k) x_i x_k.$$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Partial Moment Two Penalty for Gain Normal Dependent is calculated as follows:

$$\text{pm2\_pen\_nd\_g}\left(G\left(\vec{x}, \vec{\theta}\right)\right) = \text{pm2\_pen\_nd}\left(L\left(\vec{x}, -\vec{\theta}\right)\right)$$
$$= -\sigma_L(\mu_L + w)\emptyset\left(\frac{w + \mu_L}{\sigma_L}\right) + \left(\sigma_L^2 + (\mu_L + w)^2\right)\left[1 - \Phi\left(\frac{w + \mu_L}{\sigma_L}\right)\right].$$

1.2.6.23 Calculation of Partial Moment Two Deviation for Loss (pm2_dev)

Partial Moment Two Deviation for Loss equals

$$\mathbf{pm2\_dev}(L(\vec{x}, \vec{\theta})) = \mathbf{pm2\_pen}(f(\vec{x}, \vec{\theta})) = \sum_{j=1}^{J} p_j \left(max\left\{0, f(\vec{x}, \vec{\theta}_j) - w\right\}\right)^2,$$

where w threshold value;

$$f(\vec{x}, \vec{\theta}) = L(\vec{x}, \vec{\theta}) - E[L(\vec{x}, \vec{\theta})] \quad .$$

1.2.6.24 Calculation of Partial Moment Two Deviation for Loss Normal Independent (pm2_ni_dev)

The Partial Moment Two Deviation for Loss Normal Independent is a special case of the **Calculation of Partial Moment Two Deviation for Loss Normal Dependent (pm2_nd_dev)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x}, \vec{\theta}) = L(\vec{x}, \theta_0, \theta_1 \dots, \theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1, \dots, I.$
Consider the random function

$$f(\vec{x}, \vec{\theta}) = L(\vec{x}, \vec{\theta}) - E[L(\vec{x}, \vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I} (\theta_i - E[\theta_i]) x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the matrix of variances

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \dots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$f(\vec{x}, \vec{\theta}) \sim N(0, \sigma_f^2), \text{ and } F(z) = P\{f(\vec{x}, \vec{\theta}) \le z\} = \frac{1}{\sigma_f \sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}} dy,$$

where

$$\sigma_f^2 = \sigma_0^2 + \sum_{i=1}^{I} x_i^2 \sigma_i^2 .$$

Let
w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2} t^2} dt$ be the standard normal distribution;

The Partial Moment Two Deviation for Loss Normal Independent is calculated as follows:

$$\text{pm2\_ni\_dev}\left(L\left(\vec{x}, \vec{\theta}\right)\right) = \text{pm2\_pen\_ni}\left(f\left(\vec{x}, \vec{\theta}\right)\right) = -\sigma_f w \emptyset\left(\frac{w}{\sigma_f}\right) + \left(\sigma_f^2 + w^2\right)\left[1 - \Phi\left(\frac{w}{\sigma_f}\right)\right].$$

1.2.6.25 Calculation of Partial Moment Two Deviation for Loss Normal Dependent (pm2_nd_dev)

The Partial Moment Two Deviation for Loss Normal Dependent is a special case of the **Calculation of Partial Moment Two Deviation for Loss (pm2_dev)** for continuous distributions when coefficients in a loss function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu}, \Sigma)$, where
$\vec{\mu} = (\mu_0, \mu_1, \ldots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I;$
Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Consider the random function

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I} (\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the covariance Smatrix, which has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$f(\vec{x},\vec{\theta}) \sim N(0,\sigma_f^2), \text{ and } F(z) = P\{f(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_f\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}} dy,$$

where

$$\sigma_f^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k .$$

Let
w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Partial Moment Two Deviation for Loss Normal Dependent is calculated as follows:

$$\text{pm2\_nd\_dev}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \text{pm2\_pen\_nd}\left(f\left(\vec{x},\vec{\theta}\right)\right) = -\sigma_f w \emptyset\left(\frac{w}{\sigma_f}\right) + \left(\sigma_f^2 + w^2\right)\left[1 - \Phi\left(\frac{w}{\sigma_f}\right)\right].$$

1.2.6.26 Calculation of Partial Moment Two Deviation for Gain (pm2_dev_g)

Partial Moment Two Deviation for Gain equals

$$\mathbf{pm2\_dev\_g}(G(\vec{x},\vec{\theta})) = \mathbf{pm2\_dev}(L(\vec{x},-\vec{\theta})).$$

1.2.6.27 Calculation of Partial Moment Two Deviation for Gain Normal Independent (pm2_ni_dev_g)

The Partial Moment Two Deviation for Gain Normal Independent is a special case of the **Calculation of Partial Moment Two Deviation for Gain Normal Dependent (pm2_nd_dev_g)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

The corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N\left(\mu_i, \sigma_i^2\right), i = 0,1,\ldots,I.$
Consider the random function

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = -(\theta_0 - E[\theta_0]) + \sum_{i=1}^{I} (\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the matrix of variances

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$\mathrm{g}\left(\vec{x},\vec{\theta}\right) \sim N\left(0,\sigma_{\mathrm{g}}^2\right), \text{ and } F(z) = P\left\{\mathrm{g}\left(\vec{x},\vec{\theta}\right) \le z\right\} = \frac{1}{\sigma_{\mathrm{g}}\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_{\mathrm{g}}^2}} dy,$$

where

$$\sigma_g^2 = \sigma_0^2 + \sum_{i=1}^{I} x_i^2 \sigma_i^2 .$$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Partial Moment Two Deviation for Gain Normal Independent is calculated as follows:

$$\text{pm2\_ni\_dev\_g}\left(G\left(\vec{x},\vec{\theta}\right)\right) = \text{pm2\_pen\_ni}\left(g\left(\vec{x},\vec{\theta}\right)\right) = -\sigma_g w \emptyset\left(\frac{w}{\sigma_g}\right) + \left(\sigma_g^2 + w^2\right)\left[1 - \Phi\left(\frac{w}{\sigma_g}\right)\right].$$

1.2.6.28 Calculation of Partial Moment Two Deviation for Gain Normal Dependent (pm2_nd_dev_g)

The Partial Moment Two Deviation for Gain Normal Dependent is a special case of the **Calculation of Partial Moment Two Deviation for Gain (pm2_dev_g)** for continuous distributions when coefficients in a gain function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \dots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

The corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma)$, where $\vec{\mu} = \left(\mu_0, \mu_1, \dots, \mu_I\right)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\dots,I$;

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Consider the random function

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = -(\theta_0 - E[\theta_0]) + \sum_{i=1}^{I} (\theta_i - E[\theta_i]) x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the covariance Smatrix, which has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$\mathrm{g}\left(\vec{x},\vec{\theta}\right) \sim N(0,\sigma_{\mathrm{g}}^2), \text{ and } F(z) = P\{\mathrm{g}\left(\vec{x},\vec{\theta}\right) \le z\} = \frac{1}{\sigma_{\mathrm{g}}\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_{\mathrm{g}}^2}}dy,$$

where

$$\sigma_{\mathrm{g}}^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let

w be a threshold;

$\varnothing(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z}\varnothing(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

The Partial Moment Two Deviation for Gain Normal Dependent is calculated as follows:

$$\mathrm{pm2\_nd\_dev\_g}\left(G\left(\vec{x},\vec{\theta}\right)\right) = \mathrm{pm2\_pen\_nd}\left(\mathrm{g}\left(\vec{x},\vec{\theta}\right)\right) = -\sigma_{\mathrm{g}} w \varnothing\left(\frac{w}{\sigma_{\mathrm{g}}}\right) + (\sigma_{\mathrm{g}}^2 + w^2)\left[1 - \Phi\left(\frac{w}{\sigma_{\mathrm{g}}}\right)\right].$$

1.2.6.29 Properties of Partial Moment Group

The threshold value, ***w***, may be any real number. Functions from the Partial Moment group are calculated with double precision. The name of any function from this group may contain up to 128 symbols. The name of the Partial Moment Penalty for Loss function should begin with the string "pm_pen_", the name of the Partial Moment Penalty for Loss Normal Independent function should begin with the string "pm_pen_ni_", the name of the Partial Moment Penalty for Loss Normal Dependent function should begin with the string "pm_pen_nd_", the name of the Partial Moment Penalty for Gain function should begin with the string "pm_pen_g_", the name of the Partial Moment Penalty for Gain Normal Independent function should begin with the string "pm_pen_ni_g_", the name of the Partial Moment Penalty for Gain Normal Dependent function should begin with the string "pm_pen_nd_g_", the name of the Partial Moment Loss Deviation function should begin with the string "pm_dev_", the name of the Partial Moment Loss Deviation Normal Independent function should begin with the string "pm_ni_dev_", the name of the Partial Moment Loss Deviation Normal Dependent function should begin with the string "pm_nd_dev_", the name of the Partial Moment Gain Deviation function should begin with the string "pm_dev_g_", the name of the Partial Moment Gain Deviation Normal Independent function should begin with the string "pm_ni_dev_g_", the name of the Partial Moment Gain Deviation Normal Dependent function should begin with the string "pm_nd_dev_g_", the name of the Partial Moment Two Penalty for Loss function should begin with the string "pm2_pen", the name of the Partial Moment Two Penalty for Loss Normal Independent function should begin with the string "pm2_pen_ni", the name of the Partial

Moment Two Penalty for Loss Normal Dependent function should begin with the string "pm2_pen_nd", the name of the Partial Moment Two Penalty for Gain function should begin with the string "pm2_pen_g", the name of the Partial Moment Two Penalty for Gain Normal Independent function should begin with the string "pm2_pen_ni_g", the name of the Partial Moment Two Penalty for Gain Normal Dependent function should begin with the string "pm2_pen_nd_g", the name of the Partial Moment Two Deviation for Loss function should begin with the string "pm2_dev", the name of the Partial Moment Two Deviation for Loss Normal Independent function should begin with the string "pm2_ni_dev", the name of the Partial Moment Two Deviation for Loss Normal Dependent function should begin with the string "pm2_nd_dev", the name of the Partial Moment Two Deviation for Gain function should begin with the string "pm2_dev_g", the name of the Partial Moment Two Deviation for Gain Normal Independent function should begin with the string "pm2_ni_dev_g", the name of the Partial Moment Two Deviation for Gain Normal Independent function should begin with the string "pm2_ni_dev_g". The names of these functions may include only alphabetic characters, numbers, and the underscore sign, "_". The names of these functions are "insensitive" to the case, i.e. there is no difference between low case and upper case in these names.

1.2.7 Probability Group

The Probability Group includes the following functions:

- Probability Exceeding Penalty for Loss (software notation: pr_pen_...) (section **Calculation of Probability Exceeding Penalty for Loss**)
- Probability Exceeding Penalty for Loss Normal Independent (software notation: pr_pen_ni_...) (section **Calculation of Probability Exceeding Penalty for Loss Normal Independent (pr_pen_ni)**)
- Probability Exceeding Penalty for Loss Normal Dependent (software notation: pr_pen_nd_...) (section **Calculation of Probability Exceeding Penalty for Loss Normal Dependent (pr_pen_nd)**)
- Probability Exceeding Penalty for Gain (software notation: pr_pen_g_...) (section **Calculation of Probability Exceeding Penalty for Gain**)
- Probability Exceeding Penalty for Gain Normal Independent (software notation: pr_pen_ni_g_...) (section **Calculation of Probability Exceeding Penalty for Gain Normal Independent (pr_pen_ni_g)**)
- Probability Exceeding Penalty for Gain Normal Dependent (software notation: pr_pen_nd_g_...) (section **Calculation of Probability Exceeding Penalty for Gain Normal Dependent (pr_pen_nd_g)**)
- Probability Exceeding Deviation for Loss (software notation: pr_dev_...) (section **Calculation of Probability Exceeding Deviation for Loss**)
- Probability Exceeding Deviation for Loss Normal Independent (software notation: pr_ni_dev_...) (section **Calculation of Probability Exceeding Deviation for Loss Normal Independent (pr_ni_dev)**)
- Probability Exceeding Deviation for Loss Normal Dependent (software notation: pr_nd_dev_...) (section **Calculation of Probability Exceeding Deviation for Loss Normal Dependent (pr_nd_dev)**)
- Probability Exceeding Deviation for Gain (software notation: pr_dev_g_...) (section **Calculation of Probability Exceeding Deviation for Gain**)
- Probability Exceeding Deviation for Gain Normal Independent (software notation: pr_ni_dev_g_...) (section **Calculation of Probability Exceeding Deviation for Gain Normal Independent (pr_ni_dev_g)**)
- Probability Exceeding Deviation for Gain Normal Dependent (software notation: pr_nd_dev_g_...) (section **Calculation of Probability Exceeding Deviation for Gain Normal Dependent (pr_nd_dev_g)**)
- Probability Exceeding Penalty for Loss Multiple (software notation: prmulti_pen_...) (section **Calculation of Probability Exceeding Penalty for Loss Multiple**)
- Probability Exceeding Penalty for Loss Multiple Normal Independent (software notation: prmulti_pen_ni_...) (section **Calculation of Probability Exceeding Penalty for Loss Multiple Normal Independent (prmulti_pen_ni)**)
- Average Probability Exceeding Penalty for Loss Normal Independent (software notation: avg_pr_pen_ni)

(section **Calculation of Average Probability Exceeding Penalty for Loss Normal Independent (avg_pr_pen_ni)**)

- Probability Exceeding Penalty for Loss Multiple Normal Dependent (software notation: prmulti_pen_nd_...) (section **Calculation of Probability Exceeding Penalty for Loss Multiple Normal Dependent (prmulti_pen_nd)**)
- Probability Exceeding Penalty for Gain Multiple (software notation: prmulti_pen_g_...) (section **Calculation of Probability Exceeding Penalty for Gain Multiple**)
- Probability Exceeding Penalty for Gain Multiple Normal Independent (software notation: prmulti_pen_ni_g) (section **Calculation of Probability Exceeding Penalty for Gain Multiple Normal Independent (prmulti_pen_ni_g)**)
- Average Probability Exceeding Penalty for Gain Normal Independent (software notation: avg_pr_pen_ni_g) (section **Calculation of Average Probability Exceeding Penalty for Gain Normal Independent (avg_pr_pen_ni_g)**)
- Probability Exceeding Penalty for Gain Multiple Normal Dependent (software notation: prmulti_pen_nd_g) (section **Calculation of Probability Exceeding Penalty for Gain Multiple Normal Dependent (prmulti_pen_nd_g)**)
- Probability Exceeding Deviation for Loss Multiple (software notation: prmulti_dev_...) (section **Calculation of Probability Exceeding Deviation for Loss Multiple**)
- Probability Exceeding Deviation for Loss Multiple Normal Independent (software notation: prmulti_ni_dev) (section **Calculation of Probability Exceeding Deviation for Loss Multiple Normal Independent (prmulti_ni_dev)**)
- Average Probability Exceeding Deviation for Loss Normal Independent (software notation: avg_pr_ni_dev) (section **Calculation of Average Probability Exceeding Deviation for Loss Normal Independent (avg_pr_ni_dev)**)
- Probability Exceeding Deviation for Loss Multiple Normal Dependent (software notation: prmulti_nd_dev) (section **Calculation of Probability Exceeding Deviation for Loss Multiple Normal Dependent (prmulti_nd_dev)**)
- Probability Exceeding Deviation for Gain Multiple (software notation: prmulti_dev_g_...) (section **Calculation of Probability Exceeding Deviation for Gain Multiple**)
- Probability Exceeding Deviation for Gain Multiple Normal Independent (software notation: prmulti_ni_dev_g) (section **Calculation of Probability Exceeding Deviation for Gain Multiple Normal Independent (prmulti_ni_dev_g)**)
- Average Probability Exceeding Deviation for Gain Normal Independent (software notation: avg_pr_ni_dev_g) (section **Calculation of Average Probability Exceeding Deviation for Gain Normal Independent (avg_pr_ni_dev_g)**)
- Probability Exceeding Deviation for Gain Multiple Normal Dependent (software notation: prmulti_nd_dev_g) (section **Calculation of Probability Exceeding Deviation for Gain Multiple Normal Dependent (prmulti_nd_dev_g)**)

For more details about the Properties of this Group see the section **Properties of Probability Group**.

These functions depend on the threshold, ***w***, and are defined on some **Point** , $\vec{x} = (x_1, x_2, \ldots, x_I)$, and the **Matrix of Scenarios** (in regular Matrix or in packed format) or **Simmetric Matrix** (Smatrix).

1.2.7.1 Calculation of Probability Exceeding Penalty for Loss (pr_pen)

For some threshold ***w,*** the Probability Exceeding Penalty for Loss is the probability $\Pr\left\{L(\vec{x},\vec{\theta}) \geq w\right\}$.

The natural way of calculation of the Probability Exceeding Penalty for Loss is

$$\mathbf{pr\_pen}(L(\vec{x},\vec{\theta})) = \sum_{j=1}^{J} p_j h(L(\vec{x},\vec{\theta}_j),w),$$

where

$$h(y,w) = \begin{cases} 1, & \text{if } y \geq w \\ 0, & \text{otherwise} \end{cases};$$

$$L(\vec{x},\vec{\theta}_j) = \theta_{j0} - \sum_{i=1}^{I} \theta_{ji} x_i \;, \quad j = 1,\dots,J \;,$$

and ***w*** is the threshold.

However, to maintain stability of optimization algorithms, PSG uses the following interpolated formula consistent with var_risk.

$$\mathbf{pr\_pen}(L(\vec{x},\vec{\theta})) = \alpha,$$

where α is calculated as follows.

If there is no $VaR = w$ or there is one discrete $VaR = w$, then

$$\alpha = 1 - \frac{w(\bar{\alpha} - \underline{\alpha}) + \underline{\alpha} \cdot VaR_{\bar{\alpha}}(L(x,\theta)) - \bar{\alpha} \cdot VaR_{\underline{\alpha}}(L(x,\theta))}{VaR_{\bar{\alpha}}(L(x,\theta)) - VaR_{\underline{\alpha}}(L(x,\theta))},$$

where $VaR_{\bar{\alpha}}(L(x,\theta)) \geq w$ is the nearest to ***w*** discrete VaR, $VaR_{\underline{\alpha}}(L(x,\theta)) \leq w$ is the nearest to ***w*** discrete VaR.

If there are more than one discrete $VaR = w$, then define $VaR_{\bar{\alpha}}(L(x,\theta)) = w$ with the biggest $\bar{\alpha}$ and $VaR_{\underline{\alpha}}(L(x,\theta)) = w$ with the smallest $\underline{\alpha}$ and set $\alpha = 1 - \dfrac{\bar{\alpha} + \underline{\alpha}}{2}$.

If $VaR_{\bar{\alpha}}(L(x,\theta))$ doesn't exist, then $\alpha = 0$. If $VaR_{\underline{\alpha}}(L(x,\theta))$ doesn't exist then $\alpha = 1$.

1.2.7.2 Calculation of Probability Exceeding Penalty for Loss Normal Independent (pr_pen_ni)

The Probability Exceeding Penalty for Loss Normal Independent is a special case of **Calculation of Probability Exceeding Penalty for Loss Normal Dependent (pr_pen_nd)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\dots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\dots,I.$

The parameters of normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_0 & \mu_1 & \dots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \dots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$L(\vec{x},\vec{\theta}) \sim N(\mu_L, \sigma_L^2)$ and $F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;\ \sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2\sigma_I^2.$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Probability Exceeding Penalty for Loss Normal Independent is calculated as follows:

$$\text{pr\_pen\_ni}\left(L(\vec{x},\vec{\theta})\right) = 1 - \Phi\left(\frac{w-\mu_L}{\sigma_L}\right).$$

1.2.7.3 Calculation of Probability Exceeding Penalty for Loss Normal Dependent (pr_pen_nd)

The Probability Exceeding Penalty for Loss Normal Dependent is a special case of **Calculation of Probability Exceeding Penalty for Loss (pr_pen)** for continuous distributions when coefficients in a loss function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\dots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma)$, where $\vec{\mu} = (\mu_0,\mu_1,\dots,\mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\dots,I$;
Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

The parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of means and covariance Smatrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_0 & \mu_1 & \dots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L,\sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where

$$\mu_L = \mu_0 - \textstyle\sum_{i=1}^{I} x_i\mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\textstyle\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let
w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Probability Exceeding Penalty for Loss Normal Dependent is calculated as follows:

$$\text{pr\_pen\_nd}\left(L(\vec{x},\vec{\theta})\right) = 1 - \Phi\left(\frac{w-\mu_L}{\sigma_L}\right).$$

1.2.7.4 Calculation of Probability Exceeding Penalty for Gain (pr_pen_g)

For some threshold ***w***, the Probability Exceeding Penalty for Gain is the probability $\Pr\left\{G(\vec{x},\vec{\theta}) \geq w\right\}$.

The natural way to calculate the Probability Exceeding Penalty for Gain is:

$$\text{pr\_pen\_g}(G(\vec{x},\vec{\theta})) = \sum_{j=1}^{J} p_j h(G(\vec{x},\vec{\theta}_j), w),$$

where

$$h(y,w) = \begin{cases} 1, & \text{if } y \geq w \\ 0, & \text{otherwise} \end{cases},$$

$$G(\vec{x},\vec{\theta}_j) = -L(\vec{x},\vec{\theta}_j) = -\theta_{j0} + \sum_{i=1}^{I} \theta_{ji} x_i \quad , \quad j = 1,\ldots,J \quad ,$$

and ***w*** is the threshold.

However, to maintain stability of optimization algorithms PSG uses the interpolated formula consistent with the var_risk_g (see section **Calculation of Probability Exceeding Penalty for Loss)** and replace functions $L(\vec{x},\vec{\theta}_j)$ by functions $G(\vec{x},\vec{\theta}_j)$.

1.2.7.5 Calculation of Probability Exceeding Penalty for Gain Normal Independent (pr_pen_ni_g)

The Probability Exceeding Penalty for Gain Normal Independent is a special case of **Calculation of Probability Exceeding Penalty for Gain Normal Dependent (pr_pen_nd_g)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

The corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\ldots,I.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}}dy,$$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i$; $\sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2\sigma_I^2$.

Let

w be a threshold;

$\varnothing(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z}\varnothing(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

The Probability Exceeding Penalty for Gain Normal Independent is calculated as follows:

$$\text{pr\_pen\_ni\_g}\left(G(\vec{x},\vec{\theta})\right) = \text{pr\_pen\_ni}\left(L(\vec{x},-\vec{\theta})\right) = 1 - \Phi\left(\frac{w+\mu_L}{\sigma_L}\right).$$

1.2.7.6 Calculation of Probability Exceeding Penalty for Gain Normal Dependent (pr_pen_nd_g)

The Probability Exceeding Penalty for Gain Normal Dependent is a special case of **Calculation of Probability Exceeding Penalty for Gain (pr_pen_g)** for continuous distributions when coefficients in a gain function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

The corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma)$, where

$\vec{\mu} = (\mu_0, \mu_1, \ldots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I$;

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

The parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of means and covariance Smatrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_0 & \mu_1 & \dots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L, \sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}} dy,$$

where

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Probability Exceeding Penalty for Gain Normal Dependent is calculated as follows:

$$\text{pr\_pen\_nd\_g}\left(G(\vec{x},\vec{\theta})\right) = \text{pr\_pen\_nd}\left(L(\vec{x},-\vec{\theta})\right) = 1 - \Phi\left(\frac{w+\mu_L}{\sigma_L}\right).$$

1.2.7.7 Calculation of Probability Exceeding Deviation for Loss (pr_dev)

For some threshold w, the Probability Exceeding Deviation for Loss is the probability $\Pr\{f(\vec{x},\vec{\theta}) \geq w\}$.

The natural way to calculate the Probability Exceeding Deviation for Loss is

$$\text{pr\_dev}(L(\vec{x},\vec{\theta})) = \sum_{j=1}^{J} p_j h(f(\vec{x},\vec{\theta}_j), w),$$

where

$$h(y,w) = \begin{cases} 1, & \text{if } y \geq w \\ 0, & \text{otherwise} \end{cases},$$

$$f(\vec{x},\vec{\theta}_j) = L(\vec{x},\vec{\theta}_j) - E[L(\vec{x},\vec{\theta})] = (\theta_{j0} - E[\theta_0]) - \sum_{i=1}^{I}(\theta_{ji} - E[\theta_i])x_i \;\; , \quad j = 1,\dots,J \;,$$

and w is the threshold.

However, to maintain stability of optimization algorithms PSG uses the interpolated formula consistent with var_risk (see section **Calculation of Probability Exceeding Penalty for Loss**) and replace functions $L(\vec{x},\vec{\theta}_j)$ with functions $f(\vec{x},\vec{\theta}_j)$.

1.2.7.8 Calculation of Probability Exceeding Deviation for Loss Normal Independent (pr_ni_dev)

The Probability Exceeding Deviation for Loss Normal Independent is a special case of **Calculation of Probability Exceeding Deviation for Loss Normal Dependent (pr_nd_dev)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\dots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\dots,I.$

Consider the random function

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I}(\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the matrix of variances

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \dots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$f(\vec{x},\vec{\theta}) \sim N(0,\sigma_f^2), \text{ and } F(z) = P\{f(\vec{x},\vec{\theta}) \leq z\} = \frac{1}{\sigma_f\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}}dy,$$

where

$$\sigma_f^2 = \sigma_0^2 + \sum_{i=1}^{I} x_i^2 \sigma_i^2 .$$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Probability Exceeding Deviation for Loss Normal Independent is calculated as follows:

$$\text{pr\_ni\_dev}\left(L(\vec{x},\vec{\theta})\right) = \text{pr\_pen\_ni}\left(f(\vec{x},\vec{\theta})\right) = 1 - \Phi\left(\frac{w}{\sigma_f}\right).$$

1.2.7.9 Calculation of Probability Exceeding Deviation for Loss Normal Dependent (pr_nd_dev)

The Probability Exceeding Deviation for Loss Normal Dependent is a special case of **Calculation of Probability Exceeding Deviation for Loss (pr_dev)** for continuous distributions when coefficients in a loss function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \dots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu}, \Sigma)$, where $\vec{\mu} = (\mu_0, \mu_1, \dots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\dots,I$;

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Consider the random function

$$f(\vec{x},\vec{\theta}) = L(\vec{x},\vec{\theta}) - E[L(\vec{x},\vec{\theta})] = (\theta_0 - E[\theta_0]) - \sum_{i=1}^{I} (\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the covariance Smatrix, which has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$f(\vec{x},\vec{\theta}) \sim N(0,\sigma_f^2), \text{ and } F(z) = P\{f(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_f\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_f^2}}dy,$$

where

$$\sigma_f^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let w be a threshold;

$\varnothing(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z}\varnothing(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

The Probability Exceeding Deviation for Loss Normal Dependent is calculated as follows:

$$\text{pr\_nd\_dev}\left(L(\vec{x},\vec{\theta})\right) = \text{pr\_pen\_nd}\left(f(\vec{x},\vec{\theta})\right) = 1 - \Phi\left(\frac{w}{\sigma_f}\right).$$

1.2.7.10 Calculation of Probability Exceeding Deviation for Gain (pr_dev_g)

For some threshold w, the Probability Exceeding Deviation for Gain is the probability $\Pr\{g(\vec{x},\vec{\theta}) \ge w\}$.

The simplest approach (which is NOT used in PSG) for calculating the Probability Exceeding Deviation for Gain is:

$$\text{pr\_dev\_g}(G(\vec{x},\vec{\theta})) = \sum_{j=1}^{J} p_j h(g(\vec{x},\vec{\theta}_j), w),$$

where

$$h(y,w) = \begin{cases} 1, & \text{if } y \ge w \\ 0, & \text{otherwise} \end{cases},$$

$$g(\vec{x},\vec{\theta}_j) = G(\vec{x},\vec{\theta}_j) - E[G(\vec{x},\vec{\theta})] = -(\theta_{j0} - E[\theta_0]) + \sum_{i=1}^{I}(\theta_{ji} - E[\theta_i])x_i \quad , \quad j = 1,\ldots,J \quad ,$$

and w is the threshold.

However, to maintain stability of optimization algorithms PSG uses the interpolated formula consistent with var_risk_g (see the section **Calculation of Probability Exceeding Penalty for Loss)** and replace functions

$L(\vec{x},\vec{\theta}_j)$ by functions $g(\vec{x},\vec{\theta}_j)$.

1.2.7.11 Calculation of Probability Exceeding Deviation for Gain Normal Independent (pr_ni_dev_g)

The Probability Exceeding Deviation for Gain Normal Independent is a special case of **Calculation of Probability Exceeding Deviation for Gain Normal Dependent (pr_nd_dev_g)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

The corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\ldots,I.$
Consider the random function

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = -(\theta_0 - E[\theta_0]) + \sum_{i=1}^{I} (\theta_i - E[\theta_i]) x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the matrix of variances

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$g(\vec{x},\vec{\theta}) \sim N(0,\sigma_g^2), \text{ and } F(z) = P\{g(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_g\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_g^2}} dy,$$

where

$$\sigma_g^2 = \sigma_0^2 + \sum_{i=1}^{I} x_i^2 \sigma_i^2.$$

Let
w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z} \emptyset(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}t^2} dt$ be the standard normal distribution;

The Probability Exceeding Deviation for Gain Normal Independent is calculated as follows:

$$\text{pr\_ni\_dev\_g}\left(G\left(\vec{x},\vec{\theta}\right)\right) = \text{pr\_pen\_ni\_g}\left(\text{g}\left(\vec{x},\vec{\theta}\right)\right) = 1 - \Phi\left(\frac{w}{\sigma_{\text{g}}}\right).$$

1.2.7.12 Calculation of Probability Exceeding Deviation for Gain Normal Dependent (pr_nd_dev_g)

The Probability Exceeding Deviation for Gain Normal Dependent is a special case of the **Calculation of Probability Exceeding Deviation for Gain (pr_dev_g)** for continuous distributions when coefficients in a gain function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

The corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma),$ where $\vec{\mu} = \left(\mu_0, \mu_1, \ldots, \mu_I\right)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I;$
Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Consider the random function

$$g(\vec{x},\vec{\theta}) = G(\vec{x},\vec{\theta}) - E[G(\vec{x},\vec{\theta})] = -(\theta_0 - E[\theta_0]) + \sum_{i=1}^{I}(\theta_i - E[\theta_i])x_i \quad .$$

Since the mean of this function is zero, it is sufficient to consider only the covariance Smatrix, which has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$g(\vec{x},\vec{\theta}) \sim N(0,\sigma_g^2), \text{ and } F(z) = P\{g(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_g\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{y^2}{2\sigma_g^2}}dy,$$

where

$$\sigma_g^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Let

w be a threshold;

$\emptyset(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ be probability density function of the standard normal distribution;

$\Phi(z) = \int_{-\infty}^{z}\emptyset(t)dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}dt$ be the standard normal distribution;

The Probability Exceeding Deviation for Gain Normal Dependent is calculated as follows:

$$\text{pr\_nd\_dev\_g}\left(G(\vec{x},\vec{\theta})\right) = \text{pr\_pen\_nd\_g}\left(g(\vec{x},\vec{\theta})\right) = 1 - \Phi\left(\frac{w}{\sigma_g}\right).$$

1.2.7.13 Calculation of Probability Exceeding Penalty for Loss Multiple (prmulti_pen)

For some threshold w, the Probability Exceeding Penalty for Loss Multiple is calculated as follows:

$$\text{prmulti\_pen}(L_1(\vec{x},\vec{\theta}^1),\dots,L_M(\vec{x},\vec{\theta}^M)) =$$
$$= (1 - P\{L_1(\vec{x},\vec{\theta}^1) \le w;\dots;L_M(\vec{x},\vec{\theta}^M) \le w\}) = (1 - \sum_{\substack{j: L_1(\vec{x},\vec{\theta}^1)\le w \\ \text{-----------------} \\ L_M(\vec{x},\vec{\theta}^M)\le w}} p_j),$$

where

M = number of random loss functions;

$\vec{\theta}^m = (\theta_0^m,\theta_1^m,\dots,\theta_I^m) =$ vector of random coefficients for m-th Loss Function, $m = 1,2,\dots,M$;

$L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I}\theta_i^m x_i$ = m-th loss function, $m = 1,2,\dots,M$;

$\vec{\theta}_j^m = (\theta_{j0}^m,\theta_{j1}^m,\dots,\theta_{jI}^m)$ = j-th scenario of the random vector $\vec{\theta}^m$ for m-th loss function, $j = 1,2,\dots,J$, $m = 1,2,\dots M$.

1.2.7.14 Calculation of Probability Exceeding Penalty for Loss Multiple Normal Independent (prmulti_pen_ni)

The Probability Exceeding Penalty for Loss Multiple Normal Independent is a special case of **Calculation of Probability Exceeding Penalty for Loss Multiple (prmulti_pen)** when all coefficients in all loss functions are independent and normally distributed random values.

Let

M = number of random loss functions;
w = a threshold;
$\vec{\theta}^m = (\theta_0^m, \theta_1^m, \ldots, \theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1, \ldots, M.$

$L_m(\vec{x}, \vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I} \theta_i^m x_i$ = m-th loss function, $m = 1, \ldots, M.$

All coefficients $\theta_0^1, \theta_1^1, \ldots, \theta_I^1, \theta_0^2, \theta_1^2, \ldots, \theta_I^2, \ldots, \theta_0^M, \theta_1^M, \ldots, \theta_I^M$ are independent and normally distributed random values:

$\theta_i^m \sim N(\mu_{mi}, \sigma_{mi}^2),\ i = 0,1, \ldots, I;\ \ m = 1, \ldots, M.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots name I \\ 1 & \mu_{10} & \mu_{11} & \ldots & \mu_{1I} \\ 2 & \mu_{20} & \mu_{21} & \ldots & \mu_{2I} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ M & \mu_{M0} & \mu_{M1} & \ldots & \mu_{MI} \end{pmatrix},$$

where row with $id = m$ contains means of coefficients of m-th loss function.
Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots name I \\ 1 & \sigma_{10}^2 & \sigma_{11}^2 & \ldots & \sigma_{1I}^2 \\ 2 & \sigma_{20}^2 & \sigma_{21}^2 & \ldots & \sigma_{2I}^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ M & \sigma_{M0}^2 & \sigma_{M1}^2 & \ldots & \sigma_{MI}^2 \end{pmatrix},$$

where row with $id = m$ contains variances of coefficients of m-th loss function.
In accordance with the properties of the normal distribution,

$L_m(\vec{x}, \vec{\theta}^m) \sim N(\mu_{L_m}, \sigma_{L_m}^2)$, and $p_m = P\{L_m(\vec{x}, \vec{\theta}^m) \le w\} = \dfrac{1}{\sigma_{L_m}\sqrt{2\pi}} \displaystyle\int_{-\infty}^{w} e^{-\frac{(y-\mu_{L_m})^2}{2\sigma_{L_m}^2}} dy;$

where $\mu_{L_m} = \mu_{m0} - \displaystyle\sum_{i=1}^{I} \mu_{mi} x_i;\quad \sigma_{L_m}^2 = \sigma_{m0}^2 + \displaystyle\sum_{i=1}^{I} x_i^2 \sigma_{mi}^2;\quad m = 1, \ldots, M.$

The Probability Exceeding Penalty for Loss Multiple Normal Independent is calculated as follows:

$$\text{prmulti\_pen\_ni}\left(L_1\left(\vec{x},\overrightarrow{\theta^1}\right),\dots,L_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right) =$$

$$= \left(1 - P\{L_1(\vec{x},\vec{\theta}^1) \le w; \dots; L_M(\vec{x},\vec{\theta}^M) \le w\}\right) = 1 - \prod_{m=1}^{M} p_m .$$

1.2.7.15 Calculation of Average Probability Exceeding Penalty for Loss Normal Independent (avg_pr_pen_ni)

Let

M = number of random loss functions;

$\vec{\theta}^m = (\theta_0^m, \theta_1^m, \dots, \theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1, \dots, M$.

$L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I} \theta_i^m x_i$ = m-th loss function, $m = 1, \dots, M$.

All coefficients $\theta_0^1, \theta_1^1, \dots, \theta_I^1, \theta_0^2, \theta_1^2, \dots, \theta_I^2, \dots, \theta_0^M, \theta_1^M, \dots, \theta_I^M$ are independent and normally distributed random values:

$\theta_i^m \sim N(\mu_{mi}, \sigma_{mi}^2),\ i = 0,1,\dots,I;\ \ m = 1,\dots,M.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots nameI & & scenario\_probability \\ 1 & \mu_{10} & \mu_{11} & \dots & \mu_{1I} & v_1 \\ 2 & \mu_{20} & \mu_{21} & \dots & \mu_{2I} & v_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ M & \mu_{M0} & \mu_{M1} & \dots & \mu_{MI} & v_M \end{pmatrix},$$

where row with $id = m$ contains means of coefficients of m-th loss function;
$v_m \ge 0$ = weight of m-th loss function.
If scenario_probability column is absent or all $v_m = 0$ then all weights are considered as equal to 1.

$\bar{v}_m = v_m \Big/ \sum_{k=1}^{M} v_k$ is normalized weight of m-th loss function.

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots nameI & \\ 1 & \sigma_{10}^2 & \sigma_{11}^2 & \dots & \sigma_{1I}^2 \\ 2 & \sigma_{20}^2 & \sigma_{21}^2 & \dots & \sigma_{2I}^2 \\ \dots & \dots & \dots & \dots & \dots \\ M & \sigma_{M0}^2 & \sigma_{M1}^2 & \dots & \sigma_{MI}^2 \end{pmatrix},$$

where row with $id = m$ contains variances of coefficients of m-th loss function.
Let w = a threshold.
In accordance with the properties of the normal distribution,

$$L_m(\vec{x},\vec{\theta}^m) \sim N(\mu_{L_m},\sigma^2_{L_m}), \text{ and } \quad p_m = P\{L_m(\vec{x},\vec{\theta}^m) \le w\} = \frac{1}{\sigma_{L_m}\sqrt{2\pi}} \int_{-\infty}^{w} e^{-\frac{(y-\mu_{L_m})^2}{2\sigma^2_{L_m}}} dy;$$

where $\mu_{L_m} = \mu_{m0} - \sum_{i=1}^{I} \mu_{mi} x_i = E[L_m(\vec{x},\vec{\theta}^m)];$

$$\sigma^2_{L_m} = \sigma^2_{m0} + \sum_{i=1}^{I} \sigma^2_{mi}\, x_i^2 = Var\left(L_m(\vec{x},\vec{\theta}^m)\right) =$$

$$= E\left[\left(L_m(\vec{x},\vec{\theta}^m) - \mu_{L_m}\right)^2\right]; \quad m = 1,\dots,M.$$

The Average Probability Exceeding Penalty for Loss Normal Independent is calculated as weighted mean of separate functions:

$$\text{avg\_pr\_pen\_ni}_{\text{w}}\left(L_1\left(\vec{x},\overrightarrow{\theta^1}\right),\dots,L_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right) =$$

$$= 1 - \sum_{m=1}^{M} \bar{v}_m\, p_m$$

1.2.7.16 Calculation of Probability Exceeding Penalty for Loss Multiple Normal Dependent (prmulti_pen_nd)

The Probability Exceeding Penalty for Loss Multiple Normal Dependent is a special case of **Calculation of Probability Exceeding Penalty for Loss Multiple (prmulti_pen)** when all coefficients in each loss function are mutually dependent normally distributed random values.

Let

M = number of random loss functions;
w = a threshold;
$\vec{\theta}^m = (\theta_0^m,\theta_1^m,\dots,\theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1,\dots,M.$

$L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I} \theta_i^m x_i$ = m-th loss function, $m = 1,\dots,M.$

For fixed m, the vector is normally distributed, where
$\vec{\mu}^m = (\mu_0^m,\mu_1^m,\dots,\mu_I^m)$ is the vector of means: $\mu_i^m = E\theta_i^m, i = 0,1,\dots,I;$
Σ_m is the covariance matrix:

$$\Sigma_m = \begin{pmatrix} cov(\theta_0^m,\theta_0^m) & cov(\theta_0^m,\theta_1^m) \ldots & cov(\theta_0^m,\theta_I^m) \\ cov(\theta_1^m,\theta_0^m) & cov(\theta_1^m,\theta_1^m) \ldots & cov(\theta_1^m,\theta_I^m) \\ \ldots & \ldots & \ldots \\ cov(\theta_I^m,\theta_0^m) & cov(\theta_I^m,\theta_1^m) \ldots & cov(\theta_I^m,\theta_I^m) \end{pmatrix}.$$

The parameters of the normal distributions of M vectors of random coefficients, $\vec{\theta}^m = (\theta_0^m, \theta_1^m, \ldots, \theta_I^m)$, should be presented in form of M+1 matrices: one matrix of means and M covariance Smatrices.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_{10} & \mu_{11} & \ldots & \mu_{1I} \\ 2 & \mu_{20} & \mu_{21} & \ldots & \mu_{2I} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ M & \mu_{M0} & \mu_{M1} & \ldots & \mu_{MI} \end{pmatrix},$$

where row with *id* = *m* contains means of coefficients of *m*-th loss function.
The *m*-th covariance Smatrix has the following form:

$$V_m = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & cov(\theta_0^m,\theta_0^m) & cov(\theta_0^m,\theta_1^m) & \ldots & cov(\theta_0^m,\theta_I^m) \\ 2 & cov(\theta_1^m,\theta_0^m) & cov(\theta_1^m,\theta_1^m) & \ldots & cov(\theta_1^m,\theta_I^m) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ I+1 & cov(\theta_I^m,\theta_0^m) & cov(\theta_I^m,\theta_1^m) & \ldots & cov(\theta_I^m,\theta_I^m) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$L_m(\vec{x},\vec{\theta}^m) \sim N(\mu_{L_m}, \sigma_{L_m}^2), \text{ and } p_m = P\{L_m(\vec{x},\vec{\theta}^m) \le w\} = \frac{1}{\sigma_{L_m}\sqrt{2\pi}} \int_{-\infty}^{w} e^{-\frac{(y-\mu_{L_m})^2}{2\sigma_{L_m}^2}} dy;$$

where

$$\mu_{L_m} = \mu_{m0} - \sum_{i=1}^{I} \mu_{mi} x_i;$$

$$\sigma_{L_m}^2 = cov(\theta_0^m,\theta_0^m) - 2\sum_{i=1}^{I} cov(\theta_0^m,\theta_i^m) x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i^m,\theta_k^m) x_i x_k.$$

The Probability Exceeding Penalty for Loss Multiple Normal Dependent is calculated as follows:

$$\text{prmulti\_pen\_nd}\left(L_1(\vec{x},\vec{\theta}^1), \ldots, L_M(\vec{x},\vec{\theta}^M)\right) =$$

$$= \left(1 - P\{L_1(\vec{x},\vec{\theta}^1) \le w; \ldots; L_M(\vec{x},\vec{\theta}^M) \le w\}\right) = 1 - \prod_{m=1}^{M} p_m.$$

1.2.7.17 Calculation of Probability Exceeding Penalty for Gain Multiple (prmulti_pen_g)

For some threshold ***w***, the Probability Exceeding Penalty for Gain Multiple is calculated as follows:

$$\text{prmulti\_pen\_g}\left(G_1(\vec{x},\vec{\theta}^1),\dots,G_M(\vec{x},\vec{\theta}^M)\right)=$$

$$=\left(\mathbf{1}-P\{G_1(\vec{x},\vec{\theta}^1)\le w;\dots;G_M(\vec{x},\vec{\theta}^M)\le w\}\right)=(\mathbf{1}-\sum_{\substack{j:G_1(\vec{x},\vec{\theta}^1)\le w\\ \dots\dots\dots\\ G_M(\vec{x},\vec{\theta}^M)\le w}} p_j)$$

where

M = number of random loss functions;

$\vec{\theta}^m=(\theta_0^m,\theta_1^m,\dots,\theta_I^m)=$ vector of random coefficients for m-th loss function $m=1,2,\dots,M$;

$G_m(\vec{x},\vec{\theta}^m)=-L_m(\vec{x},\vec{\theta}^m)=-\theta_0^m+\sum_{i=1}^{I}\theta_i^m x_i$ is m-th gain function, $m=1,2,\dots,M$;

$\vec{\theta}_j^m=(\theta_{j0}^m,\theta_{j1}^m,\dots,\theta_{jI}^m)$ = j-th scenario of the random vector $\vec{\theta}^m$, for m-th loss function, $j=1,2,\dots,J$, $m=1,2,\dots M$.

1.2.7.18 Calculation of Probability Exceeding Penalty for Gain Multiple Normal Independent (prmulti_pen_ni_g)

The Probability Exceeding Penalty for Gain Multiple Normal Independent is a special case of **Calculation of Probability Exceeding Penalty for Gain Multiple (prmulti_pen_g)** when all coefficients in a gain functions are independent and normally distributed random values.

Let

M = number of random loss functions;
w = threshold;
$\vec{\theta}^m=(\theta_0^m,\theta_1^m,\dots,\theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m=1,\dots,M.$

$L_m(\vec{x},\vec{\theta}^m)=\theta_0^m-\sum_{i=1}^I\theta_i^m x_i$ = m-th loss function, $m=1,\dots,M.$

$G_m(\vec{x},\vec{\theta}^m)=L_m(\vec{x},-\vec{\theta}^m)=-\theta_0^m+\sum_{i=1}^I\theta_i^m x_i$ = m-th gain function, $m=1,\dots,M.$

All coefficients $\theta_0^1,\theta_1^1,\dots,\theta_I^1,\theta_0^2,\theta_1^2,\dots,\theta_I^2,\dots,\theta_0^M,\theta_1^M,\dots,\theta_I^M$ are independent and normally distributed random values:

$\theta_i^m\sim N(\mu_{mi},\sigma_{mi}^2),\ i=0,1,\dots,I;\ \ m=1,\dots,M.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_{10} & \mu_{11} & \dots & \mu_{1I} \\ 2 & \mu_{20} & \mu_{21} & \dots & \mu_{2I} \\ \dots & \dots & \dots & \dots & \dots \\ M & \mu_{M0} & \mu_{M1} & \dots & \mu_{MI} \end{pmatrix},$$

where row with $id = m$ contains means of coefficients of m-th loss function.
Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_{10}^2 & \sigma_{11}^2 & \dots & \sigma_{1I}^2 \\ 2 & \sigma_{20}^2 & \sigma_{21}^2 & \dots & \sigma_{2I}^2 \\ \dots & \dots & \dots & \dots & \dots \\ M & \sigma_{M0}^2 & \sigma_{M1}^2 & \dots & \sigma_{MI}^2 \end{pmatrix},$$

where row with $id = m$ contains variances of coefficients of m-th loss function.
In accordance with the properties of the normal distribution,

$G_m(\vec{x},\vec{\theta}^m) \sim N(\mu_{G_m},\sigma_{G_m}^2)$, and $p_m = P\{G_m(\vec{x},\vec{\theta}^m) \le w\} = \frac{1}{\sigma_{G_m}\sqrt{2\pi}} \int_{-\infty}^{w} e^{-\frac{(y-\mu_{G_m})^2}{2\sigma_{G_m}^2}} dy,$

where

$$\mu_{G_m} = -\mu_{m0} + \sum_{i=1}^{I} \mu_{mi} x_i = -\mu_{L_m};\ \sigma_{G_m}^2 = \sigma_{m0}^2 + \sum_{i=1}^{I} x_i^2 \sigma_{mi}^2 = \sigma_{L_m}^2;\ m = 1,\dots,M.$$

The Probability Exceeding Penalty for Gain Multiple Normal Independent is calculated as follows:

$$\text{prmulti\_pen\_ni\_g}\left(G_1(\vec{x},\vec{\theta}^1),\dots,G_M(\vec{x},\vec{\theta}^M)\right) =$$

$$= \left(1 - P\left\{G_1\left(\vec{x},\vec{\theta}^1\right) \le w;\dots;G_M\left(\vec{x},\vec{\theta}^M\right) \le w\right\}\right) = 1 - \prod_{m=1}^{M} p_m.$$

1.2.7.19 Calculation of Average Probability Exceeding Penalty for Gain Normal Independent (avg_pr_pen_ni_g)

Let

M = number of random loss functions;

$\vec{\theta}^m = (\theta_0^m, \theta_1^m, \dots, \theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1,\dots,M.$

$L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I} \theta_i^m x_i$ = m-th loss function, $m = 1,\dots,M.$

$G_m(\vec{x},\vec{\theta}^m) = L_m(\vec{x},-\vec{\theta}^m) = -\theta_0^m + \sum_{i=1}^{I} \theta_i^m x_i$ = m-th gain function, $m = 1,\dots,M.$

All coefficients $\theta_0^1,\theta_1^1,\dots,\theta_I^1,\theta_0^2,\theta_1^2,\dots,\theta_I^2,\dots,\theta_0^M,\theta_1^M,\dots,\theta_I^M$ are independent and normally distributed random values:

$\theta_i^m \sim N(\mu_{mi}, \sigma_{mi}^2), \; i = 0,1,\dots,I; \;\; m = 1,\dots,M.$

The parameters of the normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.

Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI & scenario\_probability \\ 1 & \mu_{10} & \mu_{11} & \dots & \mu_{1I} & v_1 \\ 2 & \mu_{20} & \mu_{21} & \dots & \mu_{2I} & v_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ M & \mu_{M0} & \mu_{M1} & \dots & \mu_{MI} & v_M \end{pmatrix},$$

where row with $id = m$ contains means of coefficients of m-th loss function;

$v_m \geq 0$ = weight of m-th loss function.

If scenario_probability column is absent or all $v_m = 0$ then all weights are considered as equal to 1.

$\bar{v}_m = v_m \Big/ \sum_{k=1}^{M} v_k$ is normalized weight of m-th loss function.

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_{10}^2 & \sigma_{11}^2 & \dots & \sigma_{1I}^2 \\ 2 & \sigma_{20}^2 & \sigma_{21}^2 & \dots & \sigma_{2I}^2 \\ \dots & \dots & \dots & \dots & \dots \\ M & \sigma_{M0}^2 & \sigma_{M1}^2 & \dots & \sigma_{MI}^2 \end{pmatrix},$$

where row with $id = m$ contains variances of coefficients of m-th loss function.

Let w = a threshold.

In accordance with the properties of the normal distribution,

$G_m(\vec{x}, \vec{\theta}^m) \sim N(\mu_{G_m}, \sigma_{G_m}^2)$, and $p_m = P\{G_m(\vec{x}, \vec{\theta}^m) \leq w\} = \frac{1}{\sigma_{G_m}\sqrt{2\pi}} \int_{-\infty}^{w} e^{-\frac{(y-\mu_{G_m})^2}{2\sigma_{G_m}^2}} dy,$

where

$$\mu_{G_m} = -\mu_{m0} + \sum_{i=1}^{I} \mu_{mi} x_i = -E[L_m(\vec{x}, \vec{\theta}^m)];$$

$$\sigma_{G_m}^2 = \sigma_{m0}^2 + \sum_{i=1}^{I} \sigma_{mi}^2 \, x_i^2 = Var\left(G_m(\vec{x}, \vec{\theta}^m)\right) =$$

$$= E\left[\left(G_m(\vec{x}, \vec{\theta}^m) - \mu_{G_m}\right)^2\right] = Var(L_m(\vec{x}, \vec{\theta}^m)); \;\; m = 1, \dots, M.$$

The Average Probability Exceeding Penalty for Gain Normal Independent is calculated as weighted mean of separate functions:

$$\text{avg\_pr\_pen\_ni\_g}_w\left(L_1\left(\vec{x},\overrightarrow{\theta^1}\right),\ldots,L_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right)=$$

$$=\text{avg\_pr\_pen\_ni}_w\left(G_1\left(\vec{x},\overrightarrow{\theta^1}\right),\ldots,G_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right)=$$

$$=1-\sum_{m=1}^{M}\bar{v}_m\,p_m$$

1.2.7.20 Calculation of Probability Exceeding Penalty for Gain Multiple Normal Dependent (prmulti_pen_nd_g)

The Probability Exceeding Penalty for Gain Multiple Normal Dependent is a special case of the **Calculation of Probability Exceeding Penalty for Gain Multiple (prmulti_pen_g)** when all coefficients in each loss function are mutually dependent normally distributed random values.

Let

M = number of random loss functions;
w = a threshold;
$\vec{\theta}^m=(\theta_0^m,\theta_1^m,\ldots,\theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m=1,\ldots,M.$

$L_m\left(\vec{x},\vec{\theta}^m\right)=\theta_0^m-\sum_{i=1}^{I}\theta_i^m x_i$ = m-th loss function, $m=1,\ldots,M.$

$G_m\left(\vec{x},\vec{\theta}^m\right)=L_m\left(\vec{x},-\vec{\theta}^m\right)=-\theta_0^m+\sum_{i=1}^{I}\theta_i^m x_i$ = m-th gain function, $m=1,\ldots,M.$

For fixed m, the vector of random coefficients $\vec{\theta}^m\sim N(\vec{\mu}^m,\Sigma_m)$, where

$\vec{\mu}^m=(\mu_0^m,\mu_1^m,\ldots,\mu_I^m)$ is the vector of means: $\mu_i^m=E\theta_i^m, i=0,1,\ldots,I;$

Σ_m is the covariance matrix:

$$\Sigma_m=\begin{pmatrix} cov(\theta_0^m,\theta_0^m) & cov(\theta_0^m,\theta_1^m) \ldots & cov(\theta_0^m,\theta_I^m)\\ cov(\theta_1^m,\theta_0^m) & cov(\theta_1^m,\theta_1^m) \ldots & cov(\theta_1^m,\theta_I^m)\\ \ldots & \ldots & \ldots\\ cov(\theta_I^m,\theta_0^m) & cov(\theta_I^m,\theta_1^m) \ldots & cov(\theta_I^m,\theta_I^m)\end{pmatrix}.$$

The parameters of the normal distributions of M vectors of random coefficients, $\vec{\theta}^m=(\theta_0^m,\theta_1^m,\ldots,\theta_I^m)$, should be presented in form of M+1 matrices: one matrix of means and M covariance Smatrices.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_{10} & \mu_{11} & \ldots & \mu_{1I} \\ 2 & \mu_{20} & \mu_{21} & \ldots & \mu_{2I} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ M & \mu_{M0} & \mu_{M1} & \ldots & \mu_{MI} \end{pmatrix},$$

where row with $id = m$ contains means of coefficients of m-th loss function.
The m-th covariance Smatrix has the following form:

$$V_m = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & cov(\theta_0^m, \theta_0^m) & cov(\theta_0^m, \theta_1^m) & \ldots & cov(\theta_0^m, \theta_I^m) \\ 2 & cov(\theta_1^m, \theta_0^m) & cov(\theta_1^m, \theta_1^m) & \ldots & cov(\theta_1^m, \theta_I^m) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ I+1 & cov(\theta_I^m, \theta_0^m) & cov(\theta_I^m, \theta_1^m) & \ldots & cov(\theta_I^m, \theta_I^m) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$G_m(\vec{x}, \vec{\theta}^m) \sim N(\mu_{G_m}, \sigma_{G_m}^2)$, and $p_m = P\{G_m(\vec{x}, \vec{\theta}^m) \le w\} = \frac{1}{\sigma_{G_m}\sqrt{2\pi}} \int_{-\infty}^{w} e^{-\frac{(y-\mu_{G_m})^2}{2\sigma_{G_m}^2}} dy,$

where

$$\mu_{G_m} = -\mu_{m0} + \sum_{i=1}^{I} \mu_{mi} x_i = -\mu_{L_m};$$

$$\sigma_{G_m}^2 = cov(\theta_0^m, \theta_0^m) - 2\sum_{i=1}^{I} cov(\theta_0^m, \theta_i^m) x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i^m, \theta_k^m) x_i x_k\,; \;\; m = 1, \ldots, M.$$

The probability Exceeding Penalty for Gain Multiple Normal Dependent is calculated as follows:

$$\text{prmulti\_pen\_nd\_g}\left(G_1(\vec{x}, \vec{\theta}^1), \ldots, G_M(\vec{x}, \vec{\theta}^M)\right) =$$
$$= \left(1 - P\left\{G_1\left(\vec{x}, \vec{\theta}^1\right) \le w; \ldots; G_M\left(\vec{x}, \vec{\theta}^M\right) \le w\right\}\right) = 1 - \prod_{m=1}^{M} p_m.$$

1.2.7.21 Calculation of Probability Exceeding Deviation for Loss Multiple (prmulti_dev)

For some threshold w**,** the Probability Exceeding Deviation for Loss Multiple is calculated as follows:

$$\text{prmulti\_dev}\left(L_1(\vec{x}, \vec{\theta}^1), \ldots, L_M(\vec{x}, \vec{\theta}^M)\right) =$$

$$= (\mathbf{1} - P\{f_1(\vec{x}, \vec{\theta}^1) \le w; \ldots; f_M(\vec{x}, \vec{\theta}^M) \le w\}) = (\mathbf{1} - \sum_{\substack{j: f_1(\vec{x}, \vec{\theta}^1) \le w \\ \ldots\ldots \\ f_M(\vec{x}, \vec{\theta}^M) \le w}} p_j)$$

where
M = number of random loss functions;

$\vec{\theta}^m = (\theta_0^m, \theta_1^m, \dots, \theta_I^m) =$ vector of random coefficients for m-th loss function, $m = 1,2,\dots,M$;

$f_m(\vec{x},\vec{\theta}^m) = (\theta_0^m - E\theta_0^m) - \sum_{i=1}^{I}(\theta_i^m - E\theta_i^m)x_i \quad , \quad m = 1,2,\dots M$;

$\vec{\theta}_j^m = (\theta_{j0}^m, \theta_{j1}^m, \dots, \theta_{jI}^m)$ = j-th scenario of the random vector $\vec{\theta}^m$, for m-th loss function, $j = 1,2,\dots,J \;,\; m = 1,2,\dots M$.

1.2.7.22 Calculation of Probability Exceeding Deviation for Loss Multiple Normal Independent (prmulti_ni_dev)

The Probability Exceeding Deviation for Loss Multiple Normal Independent is a special case of Calculation of Probability Exceeding Deviation for Loss Multiple (prmulti_dev) when all coefficients in all loss functions are independent and normally distributed random values.

Let

M = number of random loss functions;
w = a threshold;
$\vec{\theta}^m = (\theta_0^m, \theta_1^m, \dots, \theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1, \dots, M$.

$L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I}\theta_i^m x_i$ = m-th loss function, $m = 1, \dots, M$.

All coefficients $\theta_0^1, \theta_1^1, \dots, \theta_I^1, \theta_0^2, \theta_1^2, \dots, \theta_I^2, \dots, \theta_0^M, \theta_1^M, \dots, \theta_I^M$ are independent and normally distributed random values:

$\theta_i^m \sim N(\mu_{mi}, \sigma_{mi}^2), \; i = 0,1,\dots,I; \;\; m = 1,\dots,M.$
Consider the random functions

$$f_m(\vec{x},\vec{\theta}^m) = L_m(\vec{x},\vec{\theta}^m) - E\left[L_m(\vec{x},\vec{\theta}^m)\right] = (\theta_0^m - E[\theta_0^m]) - \sum_{i=1}^{I}(\theta_i^m - E[\theta_i^m])x_i,$$
$m = 1, \dots, M.$

Since the means of these functions are zero, it is sufficient to consider only the matrix of variances, which has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_{10}^2 & \sigma_{11}^2 & \dots & \sigma_{1I}^2 \\ 2 & \sigma_{20}^2 & \sigma_{21}^2 & \dots & \sigma_{2I}^2 \\ \dots & \dots & \dots & \dots & \dots \\ M & \sigma_{M0}^2 & \sigma_{M1}^2 & \dots & \sigma_{MI}^2 \end{pmatrix},$$

where row with $id = m$ contains variances of coefficients of $L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I}\theta_i^m x_i$ loss function.
In accordance with the properties of the normal distribution,

$f_m(\vec{x},\vec{\theta}^m) \sim N(0,\sigma_{f_m}^2)$, and $p_m = P\{f_m(\vec{x},\vec{\theta}^m) \le w\} = \frac{1}{\sigma_{f_m}\sqrt{2\pi}}\int_{-\infty}^{w} e^{-\frac{y^2}{2\sigma_{f_m}^2}} dy,$

where

$\sigma_{f_m}^2 = \sigma_{m0}^2 + \sum_{i=1}^{I} x_i^2 \sigma_{mi}^2 = \sigma_{L_m}^2;\ \ m = 1,\dots,M.$

The Probability Exceeding Deviation for Loss Multiple Normal Independent is calculated as follows:

$$\text{prmulti\_ni\_dev}\left(L_1(\vec{x},\vec{\theta}^1),\dots,L_M(\vec{x},\vec{\theta}^M)\right) =$$
$$= \left(1 - P\left\{f_1\left(\vec{x},\vec{\theta}^1\right) \le w;\dots;f_M\left(\vec{x},\vec{\theta}^M\right) \le w\right\}\right) = 1 - \prod_{m=1}^{M} p_m.$$

1.2.7.23 Calculation of Average Probability Exceeding Deviation for Loss Normal Independent (avg_pr_ni_dev)

Let

M = number of random loss functions;

$\vec{\theta}^m = (\theta_0^m,\theta_1^m,\dots,\theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1,\dots,M.$

$L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I}\theta_i^m x_i$ = m-th loss function, $m = 1,\dots,M.$

All coefficients $\theta_0^1,\theta_1^1,\dots,\theta_I^1,\theta_0^2,\theta_1^2,\dots,\theta_I^2,\dots,\theta_0^M,\theta_1^M,\dots,\theta_I^M$ are independent and normally distributed random values:

$\theta_i^m \sim N(\mu_{mi},\sigma_{mi}^2),\ i = 0,1,\dots,I;\ \ m = 1,\dots,M.$

Consider the random functions

$f_m(\vec{x},\vec{\theta}^m) = L_m(\vec{x},\vec{\theta}^m) - E[L_m(\vec{x},\vec{\theta}^m)] = (\theta_0^m - E[\theta_0^m]) - \sum_{i=1}^{I}(\theta_i^m - E[\theta_i^m])x_i,$

$m = 1,\dots,M.$

Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI & scenario\_probability \\ 1 & \mu_{10} & \mu_{11} & \dots & \mu_{1I} & v_1 \\ 2 & \mu_{20} & \mu_{21} & \dots & \mu_{2I} & v_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ M & \mu_{M0} & \mu_{M1} & \dots & \mu_{MI} & v_M \end{pmatrix},$$

where row with $id = m$ contains means of coefficients of $L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I}\theta_i^m x_i$ -th loss function;

$v_m \ge 0$ = weight of m-th loss function.

If scenario_probability column is absent or all $v_m = 0$ then all weights are considered as equal to 1.

$\bar{v}_m = v_m \Big/ \sum_{k=1}^{M} v_k$ is normalized weight of m-th loss function.

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_{10}^2 & \sigma_{11}^2 & \dots & \sigma_{1I}^2 \\ 2 & \sigma_{20}^2 & \sigma_{21}^2 & \dots & \sigma_{2I}^2 \\ \dots & \dots & \dots & \dots & \dots \\ M & \sigma_{M0}^2 & \sigma_{M1}^2 & \dots & \sigma_{MI}^2 \end{pmatrix},$$

where row with $id = m$ contains variances of coefficients of $L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I} \theta_i^m x_i$ -th loss function.

Let w = a threshold.

In accordance with the properties of the normal distribution,

$$f_m(\vec{x},\vec{\theta}^m) \sim N(0,\sigma_{fm}^2), \text{ and } p_m = P\{f_m(\vec{x},\vec{\theta}^m) \le w\} = \frac{1}{\sigma_{f_m}\sqrt{2\pi}} \int_{-\infty}^{w} e^{-\frac{y^2}{2\sigma_{f_m}^2}} dy,$$

where

$$\sigma_{fm}^2 = \sigma_{m0}^2 + \sum_{i=1}^{I} \sigma_{mi}^2 x_i^2 = Var\left(f_m(\vec{x},\vec{\theta}^m)\right) =$$

$$= E\left[\left(f_m(\vec{x},\vec{\theta}^m)\right)^2\right] = Var(L_m(\vec{x},\vec{\theta}^m)); \quad m = 1,\dots,M.$$

The Average Probability Exceeding Deviation for Loss Normal Independent is calculated as weighted mean of separate functions:

$$\text{avg\_pr\_pen\_ni\_dev}_w\left(L_1\left(\vec{x},\overrightarrow{\theta^1}\right),\dots,L_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right) =$$

$$= \text{avg\_pr\_pen\_ni}_w\left(f_1\left(\vec{x},\overrightarrow{\theta^1}\right),\dots,f_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right) =$$

$$= 1 - \sum_{m=1}^{M} \bar{v}_m \, p_m$$

1.2.7.24 Calculation of Probability Exceeding Deviation for Loss Multiple Normal Dependent (prmulti_nd_dev)

The Probability Exceeding Deviation for Loss Multiple Normal Dependent is a special case of the **Calculation of Probability Exceeding Deviation for Loss Multiple (prmulti_dev)** when all coefficients in each loss function are mutually dependent normally distributed random values.

Let

M = number of random loss functions;
w = a threshold;
$\vec{\theta}^m = (\theta_0^m, \theta_1^m, \dots, \theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1, \dots, M.$

$L_m(\vec{x}, \vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I} \theta_i^m x_i$ = m-th loss function, $m = 1, \dots, M.$

For fixed m, the vector of random coefficients $\vec{\theta}^m \sim N(\vec{\mu}^m, \Sigma_m)$, where
$\vec{\mu}^m = (\mu_0^m, \mu_1^m, \dots, \mu_I^m)$ is the vector of means: $\mu_i^m = E\theta_i^m, i = 0,1, \dots, I;$
Σ_m is the covariance matrix:

$$\Sigma_m = \begin{pmatrix} cov(\theta_0^m, \theta_0^m) & cov(\theta_0^m, \theta_1^m) \dots & cov(\theta_0^m, \theta_I^m) \\ cov(\theta_1^m, \theta_0^m) & cov(\theta_1^m, \theta_1^m) \dots & cov(\theta_1^m, \theta_I^m) \\ \dots \dots \dots & \dots \dots \dots & \dots \dots \dots \\ cov(\theta_I^m, \theta_0^m) & cov(\theta_I^m, \theta_1^m) \dots & cov(\theta_I^m, \theta_I^m) \end{pmatrix}.$$

Consider the random functions

$$f_m(\vec{x}, \vec{\theta}^m) = L_m(\vec{x}, \vec{\theta}^m) - E[L_m(\vec{x}, \vec{\theta}^m)] = (\theta_0^m - E[\theta_0^m]) - \sum_{i=1}^{I} (\theta_i^m - E[\theta_i^m]) x_i,$$
$$m = 1, \dots, M.$$

Since the means of these functions are zero, it is sufficient to consider only M covariance Smatrices.
The m-th Smatrix has the following form:

$$V_m = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0^m, \theta_0^m) & cov(\theta_0^m, \theta_1^m) & \dots & cov(\theta_0^m, \theta_I^m) \\ 2 & cov(\theta_1^m, \theta_0^m) & cov(\theta_1^m, \theta_1^m) & \dots & cov(\theta_1^m, \theta_I^m) \\ \dots & \dots \dots & \dots \dots & \dots & \dots \dots \\ I+1 & cov(\theta_I^m, \theta_0^m) & cov(\theta_I^m, \theta_1^m) & \dots & cov(\theta_I^m, \theta_I^m) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$$f_m(\vec{x}, \vec{\theta}^m) \sim N(0, \sigma_{f_m}^2), \text{ and } p_m = P\{f_m(\vec{x}, \vec{\theta}^m) \le w\} = \frac{1}{\sigma_{f_m}\sqrt{2\pi}} \int_{-\infty}^{w} e^{-\frac{y^2}{2\sigma_{f_m}^2}} dy,$$

where

$$\sigma_{f_m}^2 = cov(\theta_0^m, \theta_0^m) - 2\sum_{i=1}^{I} cov(\theta_0^m, \theta_i^m)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i^m, \theta_k^m)x_i x_k\,; \quad m = 1, \dots, M.$$

The Probability Exceeding Deviation for Loss Multiple Normal Dependent is calculated as follows:

$$\text{prmulti\_nd\_dev}\left(L_1\left(\vec{x}, \vec{\theta}^1\right), \dots, L_M\left(\vec{x}, \vec{\theta}^M\right)\right) =$$

$$= \left(1 - P\left\{f_1\left(\vec{x}, \vec{\theta}^1\right) \le w; \dots; f_M\left(\vec{x}, \vec{\theta}^M\right) \le w\right\}\right) = 1 - \prod_{m=1}^{M} p_m.$$

1.2.7.25 Calculation of Probability Exceeding Deviation for Gain Multiple (prmulti_dev_g)

For some threshold ***w*,** the Probability Exceeding Deviation for Gain Multiple is calculated as follows:

$$\textbf{prmulti\_dev\_g}(G_1(\vec{x}, \vec{\theta}^1), \dots, G_M(\vec{x}, \vec{\theta}^M)) =$$

$$= (1 - P\{g_1(\vec{x}, \vec{\theta}^1) \le w; \dots; g_M(\vec{x}, \vec{\theta}^M) \le w\}) = (1 - \sum_{\substack{j: g_1(\vec{x}, \vec{\theta}^1) \le w \\ \text{-----------} \\ g_M(\vec{x}, \vec{\theta}^M) \le w}} p_j) \quad ,$$

where

M = number of random loss functions;

$\vec{\theta}^m = (\theta_0^m, \theta_1^m, \dots, \theta_I^m)$ is vector of random coefficients for m-th loss function, $m = 1, 2, \dots, M$;

$$g_m(\vec{x}, \vec{\theta}^m) = -(\theta_0^m - E\theta_0^m) + \sum_{i=1}^{I}(\theta_i^m - E\theta_i^m)x_i \quad , \quad m = 1, 2, \dots M \quad ;$$

$\vec{\theta}_j^m = (\theta_{j0}^m, \theta_{j1}^m, \dots, \theta_{jI}^m)$ is j-th scenario of the random vector $\vec{\theta}^m$, for m-th loss function, $j = 1, 2, \dots, J$, $m = 1, 2, \dots M$.

1.2.7.26 Calculation of Probability Exceeding Deviation for Gain Multiple Normal Independent (prmulti_ni_dev_g)

The Probability Exceeding Deviation for Gain Multiple Normal Independent is a special case of the **Calculation of Probability Exceeding Deviation for Gain Multiple (prmulti_dev_g)** when all coefficients in all gain functions are independent and normally distributed random values.

Let

M = number of random loss functions;
w = a threshold;
$\vec{\theta}^m = (\theta_0^m, \theta_1^m, \dots, \theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1, \dots, M.$

$L_m\left(\vec{x}, \vec{\theta}^m\right) = \theta_0^m - \sum_{i=1}^{I} \theta_i^m x_i$ = m-th loss function, $m = 1, \dots, M.$

$G_m(\vec{x},\vec{\theta}^m) = L_m(\vec{x},-\vec{\theta}^m) = -\theta_0^m + \sum_{i=1}^{I} \theta_i^m x_i$ = m-th gain function, $m = 1,\dots,M$.

All coefficients $\theta_0^1,\theta_1^1,\dots,\theta_I^1,\theta_0^2,\theta_1^2,\dots,\theta_I^2,\dots,\theta_0^M,\theta_1^M,\dots,\theta_I^M$ are independent and normally distributed random values:

$\theta_i^m \sim N(\mu_{mi},\sigma_{mi}^2),\ i = 0,1,\dots,I;\ \ m = 1,\dots,M.$

Consider the random functions

$g_m(\vec{x},\vec{\theta}^m) = G_m(\vec{x},\vec{\theta}^m) - E[G_m(\vec{x},\vec{\theta}^m)] = -(\theta_0^m - E[\theta_0^m]) + \sum_{i=1}^{I}(\theta_i^m - E[\theta_i^m])x_i,$
$m = 1,\dots,M.$

Since the means of these functions are zero, it is sufficient to consider only the matrix of variances, which has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_{10}^2 & \sigma_{11}^2 & \dots & \sigma_{1I}^2 \\ 2 & \sigma_{20}^2 & \sigma_{21}^2 & \dots & \sigma_{2I}^2 \\ \dots & \dots & \dots & \dots & \dots \\ M & \sigma_{M0}^2 & \sigma_{M1}^2 & \dots & \sigma_{MI}^2 \end{pmatrix},$$

where row with $id = m$ contains variances of coefficients of $L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I}\theta_i^m x_i$ -th loss function.

In accordance with the properties of the normal distribution,

$g_m(\vec{x},\vec{\theta}^m) \sim N\left(0,\sigma_{g_m}^2\right)$, and $p_m = P\{g_m(\vec{x},\vec{\theta}^m) \le w\} = \frac{1}{\sigma_{g_m}\sqrt{2\pi}}\int_{-\infty}^{w} e^{-\frac{y^2}{2\sigma_{g_m}^2}}dy,$

where

$$\sigma_{g_m}^2 = \sigma_{m0}^2 + \sum_{i=1}^{I} x_i^2\sigma_{mi}^2 = \sigma_{L_m}^2;\ m = 1,\dots,M.$$

The Probability Exceeding Deviation for Gain Multiple Normal Independent is calculated as follows:

$$\text{prmulti\_ni\_dev\_g}\left(G_1(\vec{x},\vec{\theta}^1),\dots,G_M(\vec{x},\vec{\theta}^M)\right) =$$
$$= \left(1 - P\left\{g_1\left(\vec{x},\vec{\theta}^1\right) \le w;\dots;g_M\left(\vec{x},\vec{\theta}^M\right) \le w\right\}\right) = 1 - \prod_{m=1}^{M} p_m.$$

1.2.7.27 Calculation of Average Probability Exceeding Deviation for Gain Normal Independent (avg_pr_ni_dev_g)

Let

M = number of random loss functions;

$\vec{\theta}^m = (\theta_0^m,\theta_1^m,\dots,\theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1,\dots,M.$

$L_m(\vec{x},\vec{\theta}^m) = \theta_0^m - \sum_{i=1}^{I} \theta_i^m x_i$ = *m*-th loss function, $m = 1, \dots, M.$

$G_m(\vec{x},\vec{\theta}^m) = L_m(\vec{x},-\vec{\theta}^m) = -\theta_0^m + \sum_{i=1}^{I} \theta_i^m x_i$ = *m*-th gain function, $m = 1, \dots, M.$

All coefficients $\theta_0^1, \theta_1^1, \dots, \theta_I^1, \theta_0^2, \theta_1^2, \dots, \theta_I^2, \dots, \theta_0^M, \theta_1^M, \dots, \theta_I^M$ are independent and normally distributed random values:

$\theta_i^m \sim N(\mu_{mi}, \sigma_{mi}^2),\ i = 0,1,\dots,I;\ \ m = 1,\dots,M.$

Consider the random functions

$g_m(\vec{x},\vec{\theta}^m) = G_m(\vec{x},\vec{\theta}^m) - E[G_m(\vec{x},\vec{\theta}^m)] = -(\theta_0^m - E[\theta_0^m]) + \sum_{i=1}^{I}(\theta_i^m - E[\theta_i^m])x_i,$
$m = 1,\dots,M.$

Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI & scenario\_probability \\ 1 & \mu_{10} & \mu_{11} & \dots & \mu_{1I} & v_1 \\ 2 & \mu_{20} & \mu_{21} & \dots & \mu_{2I} & v_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ M & \mu_{M0} & \mu_{M1} & \dots & \mu_{MI} & v_M \end{pmatrix},$$

where row with *id* = *m* contains means of coefficients of *m*-th loss function;

$v_m \geq 0$ = weight of m-th loss function.

If scenario_probability column is absent or all $v_m = 0$ then all weights are considered as equal to 1.

$\bar{v}_m = v_m \Big/ \sum_{k=1}^{M} v_k$ is normalized weight of m-th loss function.

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_{10}^2 & \sigma_{11}^2 & \dots & \sigma_{1I}^2 \\ 2 & \sigma_{20}^2 & \sigma_{21}^2 & \dots & \sigma_{2I}^2 \\ \dots & \dots & \dots & \dots & \dots \\ M & \sigma_{M0}^2 & \sigma_{M1}^2 & \dots & \sigma_{MI}^2 \end{pmatrix},$$

where row with *id* = *m* contains variances of coefficients of *m*-th loss function.

Let w = a threshold.

In accordance with the properties of the normal distribution,

$g_m(\vec{x},\vec{\theta}^m) \sim N\left(0, \sigma_{g_m}^2\right)$, and $p_m = P\{g_m(\vec{x},\vec{\theta}^m) \leq w\} = \frac{1}{\sigma_{g_m}\sqrt{2\pi}} \int_{-\infty}^{w} e^{-\frac{y^2}{2\sigma_{g_m}^2}} dy,$

where

$$\sigma_{g_m}^2 = \sigma_{m0}^2 + \sum_{i=1}^{I} \sigma_{mi}^2\, x_i^2 = Var\left(f_g\left(\vec{x},\vec{\theta}^m\right)\right) =$$

$$= E\left[\left(g_m\left(\vec{x},\vec{\theta}^m\right)\right)^2\right] = Var\left(L_m\left(\vec{x},\vec{\theta}^m\right)\right); \quad m = 1,\ldots,M.$$

The Average Probability Exceeding Deviation for Gain Normal Independent is calculated as weighted mean of separate functions:

$$\text{avg\_pr\_\_ni\_dev\_g}_{\text{w}}\left(L_1\left(\vec{x},\overrightarrow{\theta^1}\right),\ldots,L_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right) =$$

$$=\text{avg\_pr\_pen\_ni}_{\text{w}}\left(g_1\left(\vec{x},\overrightarrow{\theta^1}\right),\ldots,g_M\left(\vec{x},\overrightarrow{\theta^M}\right)\right) =$$

$$= 1 - \sum_{m=1}^{M} \bar{v}_m\, p_m$$

1.2.7.28 Calculation of Probability Exceeding Deviation for Gain Multiple Normal Dependent (prmulti_nd_dev_g)

The Probability Exceeding Deviation for Gain Multiple Normal Dependent is a special case of the Calculation of Probability Exceeding Deviation for Gain Multiple (prmulti_dev_g) when all coefficients in each gain function are mutually dependent normally distributed random values.

Let

M = number of random loss functions;
w = a threshold;
$\vec{\theta}^m = (\theta_0^m, \theta_1^m, \ldots, \theta_I^m)$ = vector of random coefficients for m-th Loss Function, $m = 1, \ldots, M.$

$L_m\left(\vec{x},\vec{\theta}^m\right) = \theta_0^m - \sum_{i=1}^{I} \theta_i^m x_i$ = m-th loss function, $m = 1, \ldots, M.$

$G_m\left(\vec{x},\vec{\theta}^m\right) = L_m\left(\vec{x},-\vec{\theta}^m\right) = -\theta_0^m + \sum_{i=1}^{I} \theta_i^m x_i$ = m-th gain function, $m = 1, \ldots, M.$

For fixed m, the vector of random coefficients $\vec{\theta}^m \sim N(\vec{\mu}^m, \Sigma_m)$, where
$\vec{\mu}^m = (\mu_0^m, \mu_1^m, \ldots, \mu_I^m)$ is the vector of means: $\mu_i^m = E\theta_i^m, i = 0,1,\ldots,I;$
Σ_m is the covariance matrix:

$$\Sigma_m = \begin{pmatrix} cov(\theta_0^m,\theta_0^m) & cov(\theta_0^m,\theta_1^m) \ldots & cov(\theta_0^m,\theta_I^m) \\ cov(\theta_1^m,\theta_0^m) & cov(\theta_1^m,\theta_1^m) \ldots & cov(\theta_1^m,\theta_I^m) \\ \ldots & \ldots & \ldots \\ cov(\theta_I^m,\theta_0^m) & cov(\theta_I^m,\theta_1^m) \ldots & cov(\theta_I^m,\theta_I^m) \end{pmatrix}.$$

Consider the random functions

$$g_m(\vec{x},\vec{\theta}^m) = G_m(\vec{x},\vec{\theta}^m) - E\left[G_m(\vec{x},\vec{\theta}^m)\right] = -(\theta_0^m - E[\theta_0^m]) + \sum_{i=1}^{I}(\theta_i^m - E[\theta_i^m])x_i,$$
$$m = 1,\dots,M.$$

Since the means of these functions are zero, it is sufficient to consider only M covariance Smatrices.
The *m*-th Smatrix has the following form:

$$V_m = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0^m,\theta_0^m) & cov(\theta_0^m,\theta_1^m) & \dots & cov(\theta_0^m,\theta_I^m) \\ 2 & cov(\theta_1^m,\theta_0^m) & cov(\theta_1^m,\theta_1^m) & \dots & cov(\theta_1^m,\theta_I^m) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I^m,\theta_0^m) & cov(\theta_I^m,\theta_1^m) & \dots & cov(\theta_I^m,\theta_I^m) \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$g_m(\vec{x},\vec{\theta}^m) \sim N\left(0,\sigma_{g_m}^2\right)$, and $p_m = P\{g_m(\vec{x},\vec{\theta}^m) \le w\} = \frac{1}{\sigma_{g_m}\sqrt{2\pi}}\int_{-\infty}^{w} e^{-\frac{y^2}{2\sigma_{g_m}^2}}dy,$

where

$$\sigma_{g_m}^2 = cov(\theta_0^m,\theta_0^m) - 2\sum_{i=1}^{I} cov(\theta_0^m,\theta_i^m)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i^m,\theta_k^m)x_i x_k\,; \quad m = 1,\dots,M.$$

The Probability Exceeding Deviation for Gain Multiple Normal Dependent is calculated as follows:

$$\text{prmulti\_nd\_dev\_g}\left(G_1(\vec{x},\vec{\theta}^1),\dots,G_M(\vec{x},\vec{\theta}^M)\right) =$$
$$= \left(1 - P\left\{g_1\left(\vec{x},\vec{\theta}^1\right) \le w;\dots;g_M\left(\vec{x},\vec{\theta}^M\right) \le w\right\}\right) = 1 - \prod_{m=1}^{M} p_m.$$

1.2.7.29 Properties of Probability Group

Threshold ***w*** may be any real number. Functions from the Probability group are calculated with double precision. The name of any function from this group may contain up to 128 symbols. The name of the Probability Exceeding Penalty for Loss function should begin with the string "pr_pen_", the name of the Probability Exceeding Penalty for Loss Normal Independent should begin with the string "pr_pen_ni", the name of the Probability Exceeding Penalty for Loss Normal Dependent should begin with the string "pr_pen_nd", the name of the Probability Exceeding Penalty for Gain function should begin with the string "pr_pen_g_", the name of the Probability Exceeding Penalty for Gain Normal Independent should begin with the string "pr_pen_ni_g", the name of the Probability Exceeding Penalty for Gain Normal Dependent should begin with the string "pr_pen_nd_g", the name of the Probability Exceeding Deviation for Loss function should begin with the string "pr_dev_", the name of the Probability Exceeding Deviation for Loss Normal Independent should begin with the string "pr_ni_dev", the name of the Probability Exceeding Deviation for Loss Normal Dependent should begin with the string "pr_nd_dev", the name of the Probability Exceeding Deviation for Gain function should begin with the string "pr_dev_g_", the name of the Probability Exceeding Deviation for Gain Normal Independent should begin with the string "pr_ni_dev_g", the name of the Probability Exceeding Deviation for Gain Normal Dependent should begin with the string "pr_nd_dev_g", the name of the Probability Exceeding Penalty for Loss Multiple function should begin with the string "prmulti_pen_", the name of the Probability Exceeding Penalty for Loss Multiple Normal Independent should begin with the string "prmulti_pen_ni", the name of the Probability Exceeding Penalty for Loss Multiple Normal Dependent should begin with the string "prmulti_pen_nd", the name of the Probability Exceeding Penalty for Gain Multiple function should begin with the string "prmulti_pen_g_", the name of the Probability Exceeding Penalty for Gain Multiple Normal Independent should begin with the string "prmulti_pen_ni_g", the name of the Probability Exceeding Penalty for Gain Multiple Normal Dependent should begin with the string "prmulti_pen_nd_g", the name of the Probability Exceeding Deviation for Loss Multiple should begin with the string "prmulti_dev_", the name of the Probability Exceeding Deviation for Loss Multiple Normal Independent should begin with the string "prmulti_ni_dev", the name of the Probability Exceeding Deviation for Loss Multiple Normal Dependent should begin with the string "prmulti_nd_dev", the name of the Probability Exceeding Deviation for Gain Multiple should begin with the string "prmulti_dev_g_", the name of the Probability Exceeding Deviation for Gain Multiple Normal Independent should begin with the string "prmulti_ni_dev_g", the name of the Probability Exceeding Deviation for Gain Multiple Normal Dependent should begin with the string "prmulti_nd_dev_g". The names of these functions may include only alphabetic characters, numbers, and the underscore sign, "_". The names of these functions are "insensitive" to the case, i. e. there is no difference between low case and upper case in these names.

1.2.8 CDaR Group

Functions considered in this group can be used to draw curves similar to drawdown (underwater) curves considered in active portfolio management.
For some value of the tolerance parameter, α, the Conditional Drawdown-at-Risk (CDaR) deviation is defined as the mean of the worst $(1-\alpha) * 100\%$ drawdowns.
The CDaR risk function contains the maximal drawdown and average drawdown as its limiting cases. The CDaR Group includes twelve functions:

- CDaR Deviation (software notation: cdar_dev _...) (section **Calculation of CDaR Deviation)**
- CDaR Deviation for Gain (software notation: cdar_dev_g...) (section **Calculation of CDaR Deviation for Gain)**
- CDaR Deviation Multiple (software notation: cdarmulti_dev...) (section **Calculation of CDaR Deviation Multiple)**
- CDaR Deviation for Gain Multiple (software notation: cdarmulti_dev_g...) (section **Calculation of CDaR Deviation for Gain Multiple)**
- Drawdown Deviation Maximum (software notation: drawdown_dev_max…) (section **Calculation of Drawdown Deviation Maximum)**
- Drawdown Deviation Maximum for Gain (software notation: drawdown_dev_max_g…) (section **Calculation of Drawdown Deviation Maximum for Gain)**
- Drawdown Deviation Maximum Multiple (software notation: drawdownmulti_dev_max…) (section **Calculation of Drawdown Deviation Maximum Multiple)**
- Drawdown Deviation Maximum for Gain Multiple (software notation: drawdownmulti_dev_max_g…) (see section **Calculation of Drawdown Deviation Maximum for Gain Multiple)**
- Drawdown Deviation Average (software notation: drawdown_dev_avg…) (section **Calculation of Drawdown Deviation Average)**
- Drawdown Deviation Average for Gain (software notation: drawdown_dev_avg_g…) (section **Calculation of Drawdown Deviation Average for Gain)**
- Drawdown Deviation Average Multiple (software notation: drawdownmulti_dev_avg…) (section **Calculation of Drawdown Deviation Average Multiple)**
- Drawdown Deviation Average for Gain Multiple (software notation: drawdownmulti_dev_avg_g…) (section **Calculation of Drawdown Deviation Average for Gain Multiple)**

For more details about the Properties of this Group see the section **Properties of CDaR Group**.

1.2.8.1 Calculation of CDaR Deviation (cdar_dev)

Suppose a portfolio return sample-path is defined by a single matrix of scenarios. The sample-path has J equally probable scenarios $(\, p_j = \frac{1}{J},\, j = 1,\dots,J\,)$. We consider that the scenarios are sorted according to time and the j-th scenario corresponds to time moment $j,\;\; j = 1,\dots,J$. For the matrix of scenarios, calculate the scenarios $G(x,j)$ for the gain function $G(x,\theta)$

$$G(x,j) = -\theta_{0j} + \sum_{i=1}^{I} \theta_{ij} x_i\,,\;\; j = 1,\dots,J$$

and drawdown scenarios $d(x,j)$ for the drawdown function $d(x,\theta)$

$$d(x,j) = \max_{0 \le n \le j}\{\sum_{l=1}^{n} G(x,l)\} - \sum_{l=1}^{j} G(x,l),\;\; j = 1,\dots,J\,.$$

By definition, the CDaR Deviation with confidence level $\alpha\,(0 < \alpha < 1)$ equals

$$\mathbf{cdar\_dev}_{\alpha}(G(x,\theta)) = \mathbf{cvar\_risk}_{\alpha}(d\,(x,\theta))\,.$$

1.2.8.2 Calculation of CDaR Deviation for Gain (cdar_dev_g)

Suppose a portfolio return sample-path is defined by a single matrix of scenarios. The sample-path has J equally probable scenarios $(\, p_j = \frac{1}{J},\, j = 1,\dots,J\,)$. We consider that the scenarios are sorted according to time and the j-th scenario corresponds to time moment $j,\;\; j = 1,\dots,J$. For the matrix of scenarios calculate scenarios $L(x,j)$ for the loss function $L(x,\theta)$

$$L(x,j) = \theta_{0j} - \sum_{i=1}^{I} \theta_{ij} x_i\,,\;\; j = 1,\dots,J$$

and scenarios $d(x,j)$ for the drawdown function $d(x,\theta)$

$$d(x,j) = \max_{0 \le n \le j}\{\sum_{l=1}^{n} L(x,l)\} - \sum_{l=1}^{j} L(x,l),\;\; j = 1,\dots,J\,.$$

By definition, the CDaR Deviation for Gain with confidence level $\alpha\,(0 < \alpha < 1)$ equals

$$\mathbf{cdar\_dev\_g}_{\alpha}(L(x,\theta)) = \mathbf{cvar\_risk}_{\alpha}(d\,(x,\theta))\,.$$

1.2.8.3 Calculation of CDaR Deviation Multiple (cdarmulti_dev)

Suppose we have ***K*** portfolio return sample-paths, $A_1, A_2, \dots, A_K$, defined by ***K*** matrices of scenarios. Each of the sample-paths has the same probability and has ***J*** equally probable scenarios $p_{kj} = \frac{1}{KJ},\, k = 1,\dots,K;\; j = 1,\dots,J$. We consider that the scenarios are sorted according to time and the j-th scenario corresponds to the time moment j, $j = 1,\ldots,J$. For the k-th path(matrix of scenarios), calculate the scenarios $G_k(x,j)$ for the k-th gain function $G_k(x,\theta)$

$$G_k(x,j) = -\theta_{0j}^k + \sum_{i=1}^{I} \theta_{ij}^k x_i, \; j = 1,\ldots,J; k = 1,\ldots K$$

and the scenarios $d_k(x,j)$ for the k-th drawdown function $d_k(x,\theta)$

$$d_k(x,j) = \max_{0 \le n \le j}\{\sum_{l=1}^{n} G_k(x,l)\} - \sum_{l=1}^{j} G_k(x,l), \; j = 1,\ldots,J; k = 1,\ldots,K.$$

Let the function $d(x,\theta)$ have scenarios, $d_k(x,j), \; j = 1,\ldots,J; k = 1,\ldots,K$.

By definition, the CDaR Deviation Multiple with confidence level $\alpha\,(0 < \alpha < 1)$ equals

$$\mathbf{cdarmulti\_dev}_{\alpha}(G_1(x,\theta),\ldots,G_K(x,\theta)) = \mathbf{cvar\_risk}_{\alpha}(d(x,\theta)).$$

1.2.8.4 Calculation of CDaR Deviation for Gain Multiple (cdarmulti_dev_g)

We have K portfolio return sample-paths, $A_1, A_2, \ldots, A_K$, defined by K matrices of scenarios. Each of the sample-paths has the same probability and has J equally probable scenarios, $p_{kj} = \dfrac{1}{KJ}, k = 1,\ldots,K; \; j = 1,\ldots,J$. We consider that scenarios are sorted according to time and the j-th scenario corresponds to time moment j, j=1,...J. For the k-th path (matrix of scenarios), calculate the scenarios $L_k(x,j)$ for the k-th loss function $L_k(x,\theta)$,

$$L_k(x,j) = \theta_{0j}^k - \sum_{i=1}^{I} \theta_{ij}^k x_i, \; j = 1,\ldots,J; k = 1,\ldots K$$

and the scenarios $d_k(x,j)$ for the k-th drawdown function $d_k(x,\theta)$,

$$d_k(x,j) = \max_{0 \le n \le j}\{\sum_{l=1}^{n} L_k(x,l)\} - \sum_{l=1}^{j} L_k(x,l), \; j = 1,\ldots,J; k = 1,\ldots,K.$$

Let the function $d(x,\theta)$ have scenarios, $d_k(x,j), \; j = 1,\ldots,J; \; k = 1,\ldots,K$.

By definition, the CDaR Deviation for Gain Multiple with confidence level $\alpha\,(0 < \alpha < 1)$ equals

$$\mathbf{cdarmulti\_dev\_g}_{\alpha}(L_1(x,\theta),\ldots,L_K(x,\theta)) = \mathbf{cvar\_risk}_{\alpha}(d(x,\theta)).$$

1.2.8.5 Calculation of Drawdown Deviation Maximum (drawdown_dev_max)

Suppose we have a portfolio return sample-path defined by the single matrix of scenarios. The sample-path has J equally probable scenarios, $(p_j = \dfrac{1}{J}, j = 1,\ldots,J)$. We consider that scenarios are sorted according to time and the j-th scenario corresponds to time moment j, j=1,...J. For the matrix of scenarios calculate the scenarios $G(x,j)$ for the gain function $G(x,\theta)$

$$G(x,j) = -\theta_{0j} + \sum_{i=1}^{I} \theta_{ij} x_i\,, \;\; j = 1,\dots,J$$

and the scenarios $d(x,j)$ for the drawdown function $d(x,\theta)$

$$d(x,j) = \max_{0 \le n \le j}\{\sum_{l=1}^{n} G(x,l)\} - \sum_{l=1}^{j} G(x,l), \;\; j = 1,\dots,J\,.$$

The Drawdown Deviation Maximum equals:

$$\text{drawdown\_dev\_max}(G(x,\theta)) = \max_{1 \le j \le J} d(x,j)\,.$$

1.2.8.6 Calculation of Drawdown Deviation Maximum for Gain (drawdown_dev_max_g)

Suppose we have a portfolio return sample-path defined by the single matrix of scenarios. The sample-path has J equally probable scenarios $(\, p_j = \frac{1}{J}, \; j = 1,\dots,J \,)$. We consider that scenarios are sorted according to time and the *j*-th scenario corresponds to time moment ***j, j*=1,...*J*.** For the matrix of scenarios calculate scenarios $L(x,j)$ for the loss function $L(x,\theta)$

$$L(x,j) = \theta_{0j} - \sum_{i=1}^{I} \theta_{ij} x_i\,, \;\; j = 1,\dots,J$$

and scenarios $d(x,j)$ for the drawdown function $d(x,\theta)$

$$d(x,j) = \max_{0 \le n \le j}\{\sum_{l=1}^{n} L(x,l)\} - \sum_{l=1}^{j} L(x,l), \;\; j = 1,\dots,J\,.$$

The Drawdown Deviation Maximum for Gain equals:

$$\text{drawdown\_dev\_max\_g}(L(x,\theta)) = \max_{1 \le j \le J} d(x,j)$$

1.2.8.7 Calculation of Drawdown Deviation Maximum Multiple (drawdownmulti_dev_max)

Suppose we have ***K*** portfolio return sample-paths,, $A_1, A_2, \dots, A_K$, defined by ***K*** matrices of scenarios. Each of the sample-paths has the same probability and has ***J*** equally probable scenarios, $p_{kj} = \frac{1}{KJ}, k = 1,\dots,K; \; j = 1,\dots,J$. We consider that scenarios are sorted according to time and the *j*-th scenario corresponds to time moment ***j, j*=1,...*J*.** For the ***k***-th path (matrix of scenarios), calculate scenarios $G_k(x,j)$ for the ***k***-th gain function $G_k(x,\theta)$

$$G_k(x,j) = -\theta_{0j}^k + \sum_{i=1}^{I} \theta_{ij}^k x_i, \; j = 1,\ldots,J; k = 1,\ldots K$$

and scenarios $d_k(x,j)$ for the k-th drawdown function $d_k(x,\theta)$

$$d_k(x,j) = \max_{0 \le n \le j}\{\sum_{l=1}^{n} G_k(x,l)\} - \sum_{l=1}^{j} G_k(x,l), \; j = 1,\ldots,J; k = 1,\ldots,K.$$

The Drawdown Deviation Maximum Multiple equals:

$$\textbf{drawdownmulti\_dev\_max}(G_1(x,\theta),\ldots,G_K(x,\theta)) = \max_{\substack{1 \le j \le J \\ 1 \le k \le K}} d_k(x,j).$$

1.2.8.8 Calculation of Drawdown Deviation Maximum for Gain Multiple (drawdownmulti_dev_max_g)

Suppose we have K portfolio return sample-paths, $A_1, A_2, \ldots, A_K$, defined by K matrices of scenarios. Each of the sample-paths has the same probability and has J equally probable scenarios, $p_{kj} = \dfrac{1}{KJ}, k = 1,\ldots,K; \; j = 1,\ldots,J$. We consider that scenarios are sorted according to time and the j-th scenario corresponds to time moment j, j=1,...J. For the k-th path (matrix of scenarios), calculate scenarios $L_k(x,j)$ for the k-th loss function $L_k(x,\theta)$

$$L_k(x,j) = \theta_{0j}^k - \sum_{i=1}^{I} \theta_{ij}^k x_i, \; j = 1,\ldots,J; k = 1,\ldots K$$

and scenarios $d_k(x,j)$ for the k-th drawdown function $d_k(x,\theta)$

$$d_k(x,j) = \max_{0 \le n \le j}\{\sum_{l=1}^{n} L_k(x,l)\} - \sum_{l=1}^{j} L_k(x,l), \; j = 1,\ldots,J; k = 1,\ldots,K.$$

Drawdown Deviation Maximum for Gain Multiple equals:

$$\textbf{drawdownmulti\_dev\_max\_g}(L_1(x,\theta),\ldots,L_K(x,\theta)) = \max_{\substack{1 \le j \le J \\ 1 \le k \le K}} d_k(x,j).$$

1.2.8.9 Calculation of Drawdown Deviation Average (drawdown_dev_avg)

Suppose we have a portfolio return sample-path defined by the single matrix of scenarios. The sample-path has J equally probable scenarios, $(\, p_j = \dfrac{1}{J}, j = 1,\ldots,J \,)$. We consider that the scenarios are sorted according to time and the j-th scenario corresponds to time moment j, j=1,...J. For the matrix of scenarios calculate scenarios $G(x,j)$ for the gain function $G(x,\theta)$

$$G(x,j) = -\theta_{0j} + \sum_{i=1}^{I} \theta_{ij} x_i, \; j = 1,\ldots,J$$

and scenarios $d(x,j)$ for the drawdown function $d(x,\theta)$

$$d(x,j)=\max_{0\le n\le j}\{\sum_{l=1}^{n}G(x,l)\}-\sum_{l=1}^{j}G(x,l),\ \ j=1,\ldots,J.$$

The Drawdown Deviation Average equals:

$$\mathbf{drawdown\_dev\_avg}(G(x,\theta))=\frac{1}{J}\sum_{j=1}^{J}d(x,j).$$

1.2.8.10 Calculation of Drawdown Deviation Average for Gain (drawdown_dev_avg_g)

Suppose we have a portfolio return sample-path defined by the single matrix of scenarios. The sample-path has J equally probable scenarios, $(p_j=\frac{1}{J},\ j=1,\ldots,J)$. We consider that the scenarios are sorted according to time and the j-th scenario corresponds to time moment j, $j=1,\ldots,J$. For the matrix of scenarios calculate scenarios $L(x,j)$ for the loss function $L(x,\theta)$

$$L(x,j)=\theta_{0j}-\sum_{i=1}^{I}\theta_{ij}x_i,\ \ j=1,\ldots,J$$

and scenarios $d(x,j)$ for the drawdown function $d(x,\theta)$

$$d(x,j)=\max_{0\le n\le j}\{\sum_{l=1}^{n}L(x,l)\}-\sum_{l=1}^{j}L(x,l),\ \ j=1,\ldots,J.$$

The Drawdown Deviation Average for Gain equals:

$$\mathbf{drawdown\_dev\_avg\_g}(L(x,\theta))=\frac{1}{J}\sum_{j=1}^{J}d(x,j).$$

1.2.8.11 Calculation of Drawdown Deviation Average Multiple (drawdownmulti_dev_avg)

Suppose we have K portfolio return sample-paths, $A_1, A_2, \ldots, A_K$,defined by K matrices of scenarios. Each of the sample-paths has the same probability and has J equally probable scenarios, $p_{kj}=\frac{1}{KJ},\ k=1,\ldots,K;\ j=1,\ldots,J$. We consider that scenarios are sorted according to time and the j-th scenario corresponds to time moment j, $j=1,\ldots,J$. For the k-th path (matrix of scenarios), calculate scenarios $G_k(x,j)$ for the k-th loss function $G_k(x,\theta)$

$$G_k(x,j)=-\theta_{0j}^{k}+\sum_{i=1}^{I}\theta_{ij}^{k}x_i,\ \ j=1,\ldots,J;k=1,\ldots K$$

and scenarios $d_k(x,j)$ for the k-th drawdown function $d_k(x,\theta)$

$$d_k(x,j)=\max_{0\le n\le j}\{\sum_{l=1}^{n}G_k(x,l)\}-\sum_{l=1}^{j}G_k(x,l),\ \ j=1,\ldots,J;k=1,\ldots,K.$$

The Drawdown Deviation Average Multiple equals:

$$\mathbf{drawdownmulti\_dev\_avg}(G_1(x,\theta),\ldots,G_K(x,\theta)) = \frac{1}{JK}\sum_{j=1}^{J}\sum_{k=1}^{K} d_k(x,j).$$

1.2.8.12 Calculation of Drawdown Deviation Average for Gain Multiple (drawdownmulti_dev_avg_g)

Suppose we have ***K*** portfolio return sample-paths, $A_1, A_2, \ldots, A_K$, defined by ***K*** matrices of scenarios. Each of the sample-paths has the same probability and has ***J*** equally probable scenarios, $p_{kj} = \dfrac{1}{KJ}, k = 1,\ldots,K;\ j = 1,\ldots,J$, . We consider that scenarios are sorted according to time and the ***j***-th scenario corresponds to time moment ***j, j*=1,...*J*.** For the ***k***-th path (matrix of scenarios), calculate scenarios $L_k(x,j)$ for the ***k***-th gain function $L_k(x,\theta)$

$$L_k(x,j) = \theta_{0j}^k - \sum_{i=1}^{I}\theta_{ij}^k x_i,\ \ j = 1,\ldots,J; k = 1,\ldots K$$

and scenarios $d_k(x,j)$ for the ***k***-th drawdown function $d_k(x,\theta)$

$$d_k(x,j) = \max_{0\le n\le j}\{\sum_{l=1}^{n} L_k(x,l)\} - \sum_{l=1}^{j} L_k(x,l),\ \ j = 1,\ldots,J; k = 1,\ldots,K.$$

The Drawdown Deviation Average for Gain Multiple equals:

$$\mathbf{drawdownmulti\_dev\_avg\_g}(L_1(x,\theta),\ldots,L_K(x,\theta)) = \frac{1}{JK}\sum_{j=1}^{J}\sum_{k=1}^{K} d_k(x,j).$$

1.2.8.13 Properties of CDaR Group

Functions from CDaR group are calculated with double precision. The name of any function from this group may contain up to 128 symbols. The name of the CDaR Deviation function should begin with the string “cdar_dev_”, the name of the CDaR Deviation for Gain function should begin with the string “cdar_dev_g_”, the name of the CDaR Deviation Multiple function should begin with the string “cdar_dev_mult_”, the name of the CDaR Deviation for Gain Multiple function should begin with the string “cdar_dev_mult_g_”, the name of the Drawdown Deviation Maximum function should begin with the string “drawdown_dev_max_”, the name of the Drawdown Deviation Maximum for Gain function should begin with the string “drawdown_dev_max_g_”, the name of the Drawdown Deviation Maximum Multiple function should begin with the string “drawdown_dev_max_mult_”, the name of the Drawdown Deviation Maximum for Gain Multiple function should begin with the string “drawdown_dev_max_mult_g_”, the name of the Drawdown Deviation Average function should begin with the string “drawdown_dev_avg_”, the name of the Drawdown Deviation Average for Gain function should begin with the string “drawdown_dev_avg_g_”, the name of the Drawdown Deviation Average Multiple function should begin with the string “drawdown_dev_avg_mult_”, the name of the Drawdown Deviation Average for Gain Multiple function should begin with the string “drawdown_dev_avg_mult_g_”. The names of these functions are "insensitive" to the case, i.e. there is no difference between low case and upper case in these names.

1.2.9 Standard Group

Functions from this group are used for calculating the Standard Deviation-based measures of risk and deviation. This Group of functions may be defined by using both a matrix of scenarios and a symmetric matrix. The Standard Group consists of six functions:

- Standard Penalty (software notation: st_pen_...) (section **Calculation of Standard Penalty**)
- Standard Deviation (software notation: st_dev_...) (section **Calculation of Standard Deviation**)
- Standard Risk (software notation: st_risk_...) (section **Calculation of Standard Risk**)
- Standard Gain (software notation: st_risk_g_...) (section **Calculation of Standard Gain**)
- Mean Square Penalty (software notation: meansquare_...) (section **Calculation of Mean Square Penalty**)
- Variance (software notation: variance_...) (section **Calculation of Variance**)

For more details about the Properties of this Group see the section **Properties of Standard Group**.

These functions are defined on some **Point** , $\vec{x} = (x_1, x_2, \ldots, x_I)$,and **Matrix of Scenarios**.

1.2.9.1 Calculation of Standard Penalty (st_pen)

Standard Penalty is calculated as follows:

$$\mathbf{st\_pen}(L(\vec{x},\vec{\theta})) = \left[\sum_{j=1}^{J} p_j \{L(\vec{x},\vec{\theta}_j)\}^2\right]^{\frac{1}{2}} \quad ,$$

where

$$L(\vec{x},\vec{\theta}_j) = \theta_{j0} - \sum_{i=1}^{I} \theta_{ji} x_i \quad ,$$

$$j = 1,\ldots,J \quad .$$

1.2.9.2 Calculation of Standard Deviation (st_dev)

Standard Deviation is calculated as follows:

$$\mathbf{st\_dev}(L(\vec{x},\vec{\theta})) = \left[\sum_{j=1}^{J} p_j \left\{f(\vec{x},\vec{\theta}_j)\right\}^2\right]^{\frac{1}{2}} \quad ,$$

where

$$f(\vec{x},\vec{\theta}_j) = L(\vec{x},\vec{\theta}_j) - E[L(\vec{x},\vec{\theta})] = (\theta_{j0} - E[\theta_0]) - \sum_{i=1}^{I} (\theta_{ji} - E[\theta_i]) x_i \, ,$$

$$j = 1,\ldots,J \, .$$

1.2.9.3 Calculation of Standard Risk (st_risk)

Standard Risk is calculated as follows:

$$\textbf{st\_risk}(L(\vec{x},\vec{\theta})) = \sum_{j=1}^{J} p_j L(\vec{x},\vec{\theta}_j) + \left[\sum_{j=1}^{J} p_j \left\{f(\vec{x},\vec{\theta}_j)\right\}^2\right]^{\frac{1}{2}} ,$$

where

$$L(\vec{x},\vec{\theta}_j) = \theta_{j0} - \sum_{i=1}^{I} \theta_{ji} x_i ,$$
$$j = 1,\ldots,J .$$

1.2.9.4 Calculation of Standard Gain (st_risk_g)

Standard Gain is calculated as follows:

$$\textbf{st\_risk\_g}(L(\vec{x},\vec{\theta})) = -\sum_{j=1}^{J} p_j L(\vec{x},\vec{\theta}_j) + \left[\sum_{j=1}^{J} p_j \left\{f(\vec{x},\vec{\theta}_j)\right\}^2\right]^{\frac{1}{2}} ,$$

where

$$f(\vec{x},\vec{\theta}_j) = L(\vec{x},\vec{\theta}_j) - E[L(\vec{x},\vec{\theta})] = (\theta_{j0} - E[\theta_0]) - \sum_{i=1}^{I} (\theta_{ji} - E[\theta_i]) x_i ,$$
$$L(\vec{x},\vec{\theta}_j) = \theta_{j0} - \sum_{i=1}^{I} \theta_{ji} x_i ,$$
$$j = 1,\ldots,J .$$

1.2.9.5 Calculation of Mean Square Penalty (meansquare)

Mean Square Penalty is calculated as follows:

$$\textbf{meansquare}(L(\vec{x},\vec{\theta})) = \sum_{j=1}^{J} p_j \{L(\vec{x},\vec{\theta}_j)\}^2 ,$$

where

$$L(\vec{x},\vec{\theta}_j) = \theta_{j0} - \sum_{i=1}^{I} \theta_{ji} x_i , \quad j = 1,\ldots,J.$$

1.2.9.6 Calculation of Variance (variance)

Variance is calculated as follows:

$$\textbf{variance}(L(\vec{x},\vec{\theta})) = \sum_{j=1}^{J} p_j \left\{f(\vec{x},\vec{\theta}_j)\right\}^2 ,$$

where

$$f(\vec{x},\vec{\theta}_j) = L(\vec{x},\vec{\theta}_j) - E[L(\vec{x},\vec{\theta})] = (\theta_{j0} - E[\theta_0]) - \sum_{i=1}^{I}(\theta_{ji} - E[\theta_i])x_i \ ,$$
$$j = 1,\ldots,J \ .$$

1.2.9.7 Properties of Standard Group

Functions from the Standard group are calculated with double precision. The name of any function from this group may contain up to 128 symbols. The name of the Standard Penalty function should begin with the string “st_pen_”, the name of the Standard Deviation function should begin with the string “st_dev_”, the name of the Standard Risk function should begin with the string “st_risk_”, the name of the Standard Gain function should begin with the string “st_risk_g_”, the name of the Mean Square Penalty function should begin with the string "meansquare_", the name of the Variance function should begin with the string "variance_". The names of these functions may include only alphabetic characters, numbers, and the underscore sign, “_”. The names of these functions are "insensitive" to the case, i.e. there is no difference between low case and upper case in these names.

1.2.10 Utilities Group

The Utilities Group includes the following functions:

- Exponential Utility (software notation: exp_eut_...) (section **Exponential Utility)**
- Exponential Utility Normal Independent (software notation: exp_eut_ni_...) (section **Exponential Utility Normal Independent (exp_eut_ni)**)
- Exponential Utility Normal Dependent (software notation: exp_eut_nd_...) (section **Exponential Utility Normal Dependent (exp_eut_nd)**)
- Logarithmic Utility (software notation: log_eut_...) (section **Logarithmic Utility)**
- Power Utility (software notation: pow_eut_...) (section **Power Utility)**

For more details about the Properties of this Group see the section **Properties of Utilities Group**.

Functions depend on the parameter w and are defined on some **Point** , $\vec{x} = (x_1, x_2, \ldots, x_I)$, and the **Matrix of Scenarios** (in regular Matrix or packed in Pmatrix format) or Simmetric Matrix (Smatrix).

1.2.10.1 Exponential Utility (exp_eut)

Matrix of scenarios is defined as follows:

$$\begin{pmatrix} id & scenario\_probability & scenario\_benchmark & name1 & name2 & \ldots nameI \\ 1 & p_1 & \theta_{10} & \theta_{11} & \theta_{12} & \cdots \ \theta_{1I} \\ 2 & p_2 & \theta_{20} & \theta_{21} & \theta_{22} & \cdots \ \theta_{2I} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ J & p_J & \theta_{J0} & \theta_{J1} & \theta_{J2} & \cdots \ \theta_{JI} \end{pmatrix} .$$

We consider that a random vector, $\vec{\theta} = (\theta_0, \theta_1, \ldots, \theta_I)$, has I+1 components, and the random vector, $\vec{\theta}$, has J discrete scenarios with probabilities, p_j , $j = 1,\ldots,J$ presented in this matrix.

The Gain Function is calculated as follows:

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

Calculate values of the Gain function for all scenarios:

$$G(\vec{x},\vec{\theta}_j) = -\theta_{j0} + \sum_{i=1}^{I}\theta_{ji} x_i \quad , \quad j = 1,\ldots,J \quad .$$

The **Exponential Utility** is calculated as follows:

$$\mathbf{exp\_eut}\left(G\left(\vec{x},\vec{\theta}\right)\right) = E\left[-e^{-aG\left(\vec{x},\vec{\theta}\right)}\right] = \sum_{j=1}^{J} p_j e^{-a\left(-\theta_{j0}+\sum_{i=1}^{I}\theta_{ji}x_i\right)},$$

where a >0.

1.2.10.2 Exponential Utility Normal Independent (exp_eut_ni)

The Exponential Utility Normal Independent is a special case of the **Exponential Utility Normal Dependent (exp_eut_nd)** when all elements of Smatrix are zeros except diagonal elements. In this case the diagonal elements of Smatrix are presented with one raw.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

Corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are independent and normally distributed random values: $\theta_i \sim N(\mu_i, \sigma_i^2), i = 0,1,\ldots,I.$

Parameters of normal distributions of all random coefficients should be presented in form of two matrices of scenarios: matrix of means and matrix of variances.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Matrix of variances has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

In accordance with the properties of the normal distribution,

$L(\vec{x},\vec{\theta}) \sim N(\mu_L, \sigma_L^2)$ and $F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}}\,dy,$

where $\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;\;\; \sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2\sigma_I^2.$

The Exponential Utility Normal Independent is calculated as follows:

$$\text{exp\_eut\_ni}\left(G(\vec{x},\vec{\theta})\right) = E\left[-e^{-aG(\vec{x},\vec{\theta})}\right] = -\frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ay}\, e^{-\frac{(y+\mu_L)^2}{2\sigma_L^2}}\,dy = -e^{\frac{1}{2}a^2\sigma_L^2 + a\mu_L},$$

where a >0.

1.2.10.3 Exponential Utility Normal Dependent (exp_eut_nd)

The Exponential Utility Normal Dependent is a special case of the **Exponential Utility (exp_eut)** for continuous distributions when coefficients in a gain function have multivariate normal distribution.

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

Corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are mutually dependent normally distributed random values: $\vec{\theta} \sim N(\vec{\mu},\Sigma),$ where $\vec{\mu} = (\mu_0, \mu_1, \ldots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I;$

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Parameters of the multivariate normal distribution should be presented in form of two matrices: matrix of means and covariance matrix.

Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Covariance matrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

In accordance with the properties of the multivariate normal distribution,

$$L(\vec{x},\vec{\theta}) \sim N(\mu_L,\sigma_L^2) \text{ and } F(z) = P\{L(\vec{x},\vec{\theta}) \le z\} = \frac{1}{\sigma_L\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{(y-\mu_L)^2}{2\sigma_L^2}}\,dy,$$

where

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

The Exponential Utility Normal Dependent is calculated as follows:

$$\text{exp\_eut\_nd}\left(G(\vec{x},\vec{\theta})\right) = E\left[-e^{-aG(\vec{x},\vec{\theta})}\right] = -\frac{1}{\sigma_I\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ay}\, e^{-\frac{(y+\mu_L)^2}{2\sigma_L^2}}\,dy = -e^{\frac{1}{2}a^2\sigma_L^2 + a\mu_L},$$

where a >0.

1.2.10.4 Logarithmic Utility (log_eut)

Matrix of Scenarios is defined as follows:

$$\begin{pmatrix} id & scenario\_probability & scenario\_benchmark & name1 & name2 & \ldots nameI \\ 1 & p_1 & \theta_{10} & \theta_{11} & \theta_{12} & \cdots\ \theta_{1I} \\ 2 & p_2 & \theta_{20} & \theta_{21} & \theta_{22} & \cdots\ \theta_{2I} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ J & p_J & \theta_{J0} & \theta_{J1} & \theta_{J2} & \cdots\ \theta_{JI} \end{pmatrix}.$$

We consider that a random vector, $\vec{\theta} = (\theta_0, \theta_1, \ldots, \theta_I)$, has I+1 components, and the random vector, $\vec{\theta}$, has J discrete scenarios with probabilities, p_j , $j = 1,\ldots,J$ presented in this matrix.

The Gain Function is calculated as follows:

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I}\theta_i x_i \quad .$$

Calculate values of the Gain function for all scenarios:

$$G(\vec{x},\vec{\theta}_j) = -\theta_{j0} + \sum_{i=1}^{I}\theta_{ji}x_i \quad , \quad j = 1,\ldots,J \quad .$$

The **Logarithmic Utility** is calculated as follows:

$$\mathbf{log\_eut}\left(G\left(\vec{x},\vec{\theta}\right)\right)=E\left[ln\left(G\left[\vec{x},\vec{\theta}\right]\right)\right]=\sum_{j=1}^{J}p_j\, ln\left[-\theta_{j0}+\sum_{i=1}^{I}\theta_{ji}x_i\right],$$

where

$$-\theta_{j0}+\sum_{i=1}^{I}\theta_{ji}x_i>0,\ j=1,\ldots,J.$$

1.2.10.5 Power Utility (pow_eut)

Matrix of Scenarios is defined as follows:

$$\begin{pmatrix} id & scenario\_probability & scenario\_benchmark & name1 & name2 & \ldots nameI \\ 1 & p_1 & \theta_{10} & \theta_{11} & \theta_{12} & \cdots\ \theta_{1I} \\ 2 & p_2 & \theta_{20} & \theta_{21} & \theta_{22} & \cdots\ \theta_{2I} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ J & p_J & \theta_{J0} & \theta_{J1} & \theta_{J2} & \cdots\ \theta_{JI} \end{pmatrix}.$$

We consider that a random vector, $\vec{\theta}=(\theta_0,\theta_1,\ldots,\theta_I)$, has $I+1$ components, and the random vector, $\vec{\theta}$, has J discrete scenarios with probabilities, p_j , $j=1,\ldots,J$ presented in this matrix.

The Gain Function is calculated as follows:

$$G(\vec{x},\vec{\theta})=L(\vec{x},-\vec{\theta})=-L(\vec{x},\vec{\theta})=-\theta_0+\sum_{i=1}^{I}\theta_i x_i\ .$$

Calculate values of the Gain function for all scenarios:

$$G(\vec{x},\vec{\theta}_j)=-\theta_{j0}+\sum_{i=1}^{I}\theta_{ji}x_i\ ,\ j=1,\ldots,J\ .$$

The **Power Utility** is calculated as follows:

$$\mathbf{pow\_eut}\left(G\left(\vec{x},\vec{\theta}\right)\right)=b\cdot E\left[\left(G\left[\vec{x},\vec{\theta}\right]\right)^b\right]=b\cdot\sum_{j=1}^{J}p_j\left(-\theta_{j0}+\sum_{i=1}^{I}\theta_{ji}x_i\right)^b,$$

where $b\le 1,\ b\ne 0,$ and

$$-\theta_{j0}+\sum_{i=1}^{I}\theta_{ji}x_i>0,\ j=1,\ldots,J.$$

1.2.10.6 Properties of Utilities Group

Functions from the Utility group are calculated with double precision. The name of the Exponential Utility function should begin with the string "exp_eut_". The name of the Exponential Utility Normal Independent function should begin with the string "exp_eut_ni_". The name of the Exponential Utility Normal Dependent function should begin with the string "exp_eut_nd_". The name of the Logarithmic Utility function should begin

with the string "log_eut_". The name of the Power Utility function should begin with the string "pow_eut_". The name of these functions may include only alphabetic characters, numbers, and the underscore sign, "_".
The names of these functions are "insensitive" to the case, i.e. there is no difference between low case and upper case in these names.

1.3 Risk Functions Defined on Smatrix

Some types of Risk and Deviation Functions (section **Risk Function)** are defined on a **Symmetric Matrix** denoted by Smatrix. For instance, quadratic risk and deviation functions from the Standard Group are defined on the Smatrix as well on the **Matrix of Scenarios**.

1.3.1 Definition of Standard Group Using Smatrix

Smatrix has the following general form (section **Symmetric Matrix):**

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & a_{00} & a_{01} & \cdots & a_{0I} \\ 2 & a_{10} & a_{11} & \cdots & a_{1I} \\ \hdashline I+1 & a_{I0} & a_{I,1} & \cdots & a_{II} \end{pmatrix}$$

Some of the functions in the Standard Group calculate the square root of the quadratic form:

$$\sqrt{a_{00} - 2\sum_{j=1}^{I} a_{0j} x_j + \sum_{i=1}^{I}\sum_{j=1}^{I} a_{ij} x_i x_j} \qquad (1)$$

where

$a_{ij},\ i = 0,1,\dots,I;\ j = 0,1,\dots,I,$ are coefficients of the Smatrix;

$x_i,\ i = 1,\dots,I$, are components of the Point $\vec{x}$.

To make sure that the expression under the square root is nonnegative we require positive definiteness of the Smatrix.

If the Smatrix is a Covariance Matrix (see the section **Symmetric Matrix**), the formula (1) calculates the Standard Deviation (section **Calculation of Standard Deviation using Smatrix**) of the following loss function (section **Risk Functions Defined by Matrix of Scenarios**):

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \dots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \qquad (2)$$

In this case, square of (1) gives the Variance (section **Calculation of Variance Using Smatrix**).

If the Smatrix is a Matrix of Expectations of Products (see the section **Symmetric Matrix**), the formula (1) calculates the Standard Penalty (section **Calculation of Standard Penalty Using Products Smatrix (st_pen)**) of the loss function (2). In this case, square of (1) gives the Mean Square Penalty (section **Calculation of Mean Square Penalty Using Products Smatrix (meansquare)**).

Standard Penalty, and Mean Square Penalty functions can be also calculated using mean matrix, and covariance Smatrix (section **Calculation of Standard Penalty Using Mean Matrix and Covariance Smatrix (st_pen_d)** , and section **Calculation of Mean Square Penalty Using Mean Matrix and Covariance**

Smatrix (meansquare_d)). If all the coefficients of the loss function (2) are independent random variables, then Standard Penalty, and Mean Square Penalty functions are calculated using mean matrix, and variance matrix (section **Calculation of Standard Penalty Using Mean Matrix and Variance Matrix (st_pen_i)** , and section **Calculation of Mean Square Penalty Using Mean Matrix and Variance Matrix (meansquare_i)**).

Similarly, Standard Risk, and Standard Gain functions are calculated using mean matrix, and covariance Smatrix (section **Calculation of Standard Risk Using Mean Matrix and Covariance Smatrix (st_risk_d)** , and section **Calculation of Standard Gain Using Mean Matrix and Covariance Smatrix (st_risk_d_g)**).If all the coefficients of the loss function (2) are independent random variables, then Standard Risk, and Standard Gain functions are calculated using mean matrix, and variance matrix (section **Calculation of Standard Risk Using Mean Matrix and Variance Matrix (st_risk_i)** , and section **Calculation of Standard Gain Using Mean Matrix and Variance Matrix (st_risk_i_g)**).

For more details about the Properties of this Group see the section **Properties of Standard Group**.

1.3.1.1 Calculation of Standard Penalty Using Products Smatrix (st_pen)

The Standard Penalty is calculated as follows:

$$\mathrm{st\_pen}(\vec{x},A) = \left[a_{00} - 2\sum_{j=1}^{I} a_{0j} x_j + \sum_{i=1}^{I}\sum_{j=1}^{I} a_{ij} x_i x_j \right]^{\frac{1}{2}},$$

where $x_i,\ i = 1,\ldots,I$, are components of the Point $\vec{x}$,

and $a_{ij},\ i = 0,1,\ldots,I;\ j = 0,1,\ldots,I,$ are coefficients of the following Smatrix:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \cdots & nameI \\ 1 & a_{00} & a_{01} & \cdots & a_{0I} \\ 2 & a_{10} & a_{11} & \cdots & a_{1I} \\ \hline I+1 & a_{I0} & a_{I,1} & \cdots & a_{II} \end{pmatrix}$$

Smatrix can be imported to PSG or generated inside of PSG Shell Environment from a Matrix of scenarios (see the PSG Help section "Symmetric Matrix", subsections "**Conversion of Matrix of Scenarios to Covariance Matrix**", and "**Conversion of Matrix of Scenarios to Matrix of Expectations of Products**").

Particularly, if elements of the Smatrix are expectations of products

$$a_{ik} = E[\theta_i \cdot \theta_k] = \sum_{j=1}^{J} p_j \theta_{ji} \theta_{jk} \quad , \quad i,k = 0,1,\ldots,I,$$

then the Standard Penalty is calculated as follows:

$$\mathrm{st\_pen}(\vec{x},\vec{\theta}) = \left[E\left[\theta_0 \cdot \theta_0\right] - 2\sum_{i=1}^{I} E\left[\theta_0 \cdot \theta_i\right] x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} E\left[\theta_i \cdot \theta_k\right] x_i x_k \right]^{\frac{1}{2}}.$$

1.3.1.2 Calculation of Standard Penalty Using Mean Matrix and Covariance Smatrix (st_pen_d)

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x}, \vec{\theta}) = L(\vec{x}, \theta_0, \theta_1 \ldots, \theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent random values.

$\vec{\mu} = (\mu_0, \mu_1, \ldots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I;$

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

These parameters should be presented in form of two matrices: matrix of means and covariance Smatrix. Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Mean and variance of the loss function are calculated using elements of Mean Matrix, ***A***, and Covariance Smatrix, ***V***, as follows:

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i) x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k) x_i x_k.$$

Given Mean Matrix, and Covariance Smatrix, the Standard Penalty is calculated as follows:

$$\text{st\_pen\_d}\left(L(\vec{x},\vec{\theta})\right) = \sqrt{\sigma_L^2 + \mu_L^2}.$$

1.3.1.3 Calculation of Standard Penalty Using Mean Matrix and Variance Matrix (st_pen_i)

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are independent random values with parameters:

$$\mu_i = E[\theta_i],\ \sigma_i^2 = E[(\theta_i - \mu_i)^2],\ i = 0,1,\ldots,I.$$

These parameters should be presented in form of two matrices: matrix of means and variance matrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Variance matrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

Mean and variance of the loss function are calculated using elements of Mean Matrix, ***A*****,** and variance matrix, ***V*** , as follows:

$$\mu_L = \mu_0 - \textstyle\sum_{i=1}^{I} x_i\mu_i;$$
$$\sigma_L^2 = \sigma_0^2 + \textstyle\sum_{i=1}^{I} x_I^2\sigma_I^2.$$

Given Mean Matrix, and Variance Matrix, the Standard Penalty is calculated as follows:

$$\text{st\_pen\_i}\left(L\left(\vec{x},\vec{\theta}\right)\right) = \sqrt{\sigma_L^2 + \mu_L^2}.$$

1.3.1.4 Calculation of Standard Deviation using Smatrix (st_dev)

The Standard Deviation is calculated as follows:

$$\textbf{st\_dev}\,(\vec{x}, A) = \left[a_{00} - 2\sum_{j=1}^{I} a_{0j}x_j + \sum_{i=1}^{I}\sum_{j=1}^{I} a_{ij}x_i x_j \right]^{\frac{1}{2}},$$

where $x_i, i = 1,\ldots,I$, are components of the Point $\vec{x}$**, and**
$a_{ij}, i = 0,1,\ldots,I;\ j = 0,1,\ldots,I,$ are coefficients of the following Smatrix:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & a_{00} & a_{01} & \cdots & a_{0I} \\ 2 & a_{10} & a_{11} & \cdots & a_{1I} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ I+1 & a_{I0} & a_{I,1} & \cdots & a_{II} \end{pmatrix}$$

Smatrix can be imported to PSG or generated inside of PSG Shell Environment from a Matrix of Scenarios (see the PSG Help section "Symmetric Matrix", subsections "**Conversion of Matrix of Scenarios to Covariance Matrix**", and "**Conversion of Matrix of Scenarios to Matrix of Expectations of Products**").

Particularly, if elements of the Smatrix are covariances:

$$a_{ik} = \mathbf{cov}(\theta_i, \theta_k) = \sum_{j=1}^{J} p_j(\theta_{ji} - E[\theta_i])(\theta_{jk} - E[\theta_k]) = \sum_{j=1}^{J} p_j \theta_{ji}\theta_{jk} - E[\theta_i]E[\theta_k],$$

$$i, k = 0, 1, \ldots, I,$$

$$E[\theta_i] = \sum_{j=1}^{J} p_j \theta_{ji}\ ,\ i = 1, 2, \ldots, I,$$

then the Standard Deviation is calculated as follows:

$$\mathbf{st\_dev}(\vec{x}, \vec{\theta}) = \left[\mathbf{cov}(\theta_0, \theta_0) - 2\sum_{i=1}^{I} \mathbf{cov}(\theta_0, \theta_i) x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} \mathbf{cov}(\theta_i, \theta_k) x_i x_k \right]^{\frac{1}{2}}.$$

1.3.1.5 Calculation of Standard Risk Using Mean Matrix and Covariance Smatrix (st_risk_d)

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x}, \vec{\theta}) = L(\vec{x}, \theta_0, \theta_1 \ldots, \theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent random values.

$\vec{\mu} = (\mu_0, \mu_1, \ldots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0, 1, \ldots, I;$

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0, \theta_0) & cov(\theta_0, \theta_1) & \ldots & cov(\theta_0, \theta_I) \\ cov(\theta_1, \theta_0) & cov(\theta_1, \theta_1) & \ldots & cov(\theta_1, \theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I, \theta_0) & cov(\theta_I, \theta_1) & \ldots & cov(\theta_I, \theta_I) \end{pmatrix}.$$

These parameters should be presented in form of two matrices: matrix of means and covariance Smatrix. Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Mean and variance of the loss function are calculated using elements of Mean Matrix, ***A***, and Covariance Smatrix, ***V***, as follows:

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i\mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_ix_k.$$

Given Mean Matrix, and Covariance Smatrix, the Standard Risk is calculated as follows:

$$\text{st\_risk\_d}\left(L(\vec{x},\vec{\theta})\right) = \sigma_L + \mu_L.$$

1.3.1.6 Calculation of Standard Risk Using Mean Matrix and Variance Matrix (st_risk_i)

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1\ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I}\theta_i x_i \quad .$$

All coefficients are independent random values with parameters:

$$\mu_i = E[\theta_i],\ \sigma_i^2 = E[(\theta_i - \mu_i)^2],\ i = 0,1,\ldots,I.$$

These parameters should be presented in form of two matrices: matrix of means and variance matrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Variance matrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

Mean and variance of the loss function are calculated using elements of Mean Matrix, ***A***, and variance matrix, ***V*** , as follows:

$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;$
$\sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2 \sigma_I^2.$

Given Mean Matrix, and Variance Matrix, the Standard Risk is calculated as follows:

$$\text{st\_risk\_i}\left(L(\vec{x},\vec{\theta})\right) = \sigma_L + \mu_L.$$

1.3.1.7 Calculation of Standard Gain Using Mean Matrix and Covariance Smatrix (st_risk_d_g)

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \ldots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

The corresponding Gain Function is

$$G(\vec{x},\vec{\theta}) = L(\vec{x},-\vec{\theta}) = -L(\vec{x},\vec{\theta}) = -\theta_0 + \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent random values.

$\vec{\mu} = \left(\mu_0, \mu_1, \ldots, \mu_I\right)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\ldots,I;$

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

These parameters should be presented in form of two matrices: matrix of means and covariance Smatrix. Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \ldots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \ldots & cov(\theta_1,\theta_I) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \ldots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Mean and variance of the loss function are calculated using elements of Mean Matrix, ***A***, and Covariance Smatrix, ***V***, as follows:

$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;$
$\sigma_L^2 = cov(\theta_0, \theta_0) - 2\sum_{i=1}^{I} cov(\theta_0, \theta_i) x_i + \sum_{i=1}^{I} \sum_{k=1}^{I} cov(\theta_i, \theta_k) x_i x_k.$

Given Mean Matrix, and Covariance Smatrix, the Standard Gain is calculated as follows:

$$\text{st\_risk\_d\_g}\left(G\left(\vec{x}, \vec{\theta}\right)\right) = \text{st\_risk\_d}\left(L\left(\vec{x}, -\vec{\theta}\right)\right) = \sigma_L - \mu_L.$$

1.3.1.8 Calculation of Standard Gain Using Mean Matrix and Variance Matrix (st_risk_i_g)

Let $\vec{x} = (x_1, x_2, \ldots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \ldots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x}, \vec{\theta}) = L(\vec{x}, \theta_0, \theta_1 \ldots, \theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

The corresponding Gain Function is

$$G(\vec{x}, \vec{\theta}) = L(\vec{x}, -\vec{\theta}) = -L(\vec{x}, \vec{\theta}) = -\theta_0 + \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent random values with parameters:

$$\mu_i = E[\theta_i], \ \sigma_i^2 = E[(\theta_i - \mu_i)^2], \ i = 0, 1, \ldots, I.$$

These parameters should be presented in form of two matrices: matrix of means and variance matrix.
Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \mu_0 & \mu_1 & \ldots & \mu_I \end{pmatrix}.$$

Variance matrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \ldots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \ldots & \sigma_I^2 \end{pmatrix}.$$

Mean and variance of the loss function are calculated using elements of Mean Matrix, ***A*,** and variance matrix, ***V*** , as follows:

$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;$
$\sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2 \sigma_I^2.$

Given Mean Matrix, and Variance Matrix, the Standard Gain is calculated as follows:

$$\text{st\_risk\_i\_g}\left(G\left(\vec{x}, \vec{\theta}\right)\right) = \text{st\_risk\_i}\left(L\left(\vec{x}, -\vec{\theta}\right)\right) = \sigma_L - \mu_L.$$

1.3.1.9 Calculation of Mean Square Penalty Using Products Smatrix (meansquare)

The Mean Square Penalty is calculated as follows:

$$\textbf{meansquare}(\vec{x}, A) = a_{00} - 2\sum_{j=1}^{I} a_{0j} x_j + \sum_{i=1}^{I}\sum_{j=1}^{I} a_{ij} x_i x_j \ ,$$

where $x_i,\ i = 1,\dots,I$, are components of the Point $\vec{x}$, **and** $a_{ij},\ i = 0,1,\dots,I;\ j = 0,1,\dots,I,$ are coefficients of the following Smatrix:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & a_{00} & a_{01} & \cdots & a_{0I} \\ 2 & a_{10} & a_{11} & \cdots & a_{1I} \\ \hdashline I+1 & a_{I0} & a_{I,1} & \cdots & a_{II} \end{pmatrix}$$

Smatrix can be imported to PSG or generated inside of PSG Shell Environment from a Matrix of Scenarios (see the PSG Help section "Symmetric Matrix", subsections "**Conversion of Matrix of Scenarios to Covariance Matrix**", and "**Conversion of Matrix of Scenarios to Matrix of Expectations of Products**").

Particularly, if elements of the Smatrix are expectations of products

$$a_{ik} = E[\theta_i \cdot \theta_k] = \sum_{j=1}^{J} p_j \theta_{ji} \theta_{jk} \quad , \quad i,k = 0,1,\dots,I,$$

then the Mean Square Penalty is calculated as follows:

$$\textbf{meansquare}(\vec{x}, \vec{\theta}) = E\left[\theta_0 \cdot \theta_0\right] - 2\sum_{i=1}^{I} E\left[\theta_0 \cdot \theta_i\right] x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} E\left[\theta_i \cdot \theta_k\right] x_i x_k \ .$$

1.3.1.10 Calculation of Mean Square Penalty Using Mean Matrix and Covariance Smatrix (meansquare_d)

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x}, \vec{\theta}) = L(\vec{x}, \theta_0, \theta_1 \dots, \theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are mutually dependent random values.

$\vec{\mu} = (\mu_0, \mu_1, \dots, \mu_I)$ is the vector of means: $\mu_i = E\theta_i, i = 0,1,\dots,I;$

Σ is the covariance matrix:

$$\Sigma = \begin{pmatrix} cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots \\ cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

These parameters should be presented in form of two matrices: matrix of means and covariance Smatrix. Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_0 & \mu_1 & \dots & \mu_I \end{pmatrix}.$$

Covariance Smatrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & cov(\theta_0,\theta_0) & cov(\theta_0,\theta_1) & \dots & cov(\theta_0,\theta_I) \\ 2 & cov(\theta_1,\theta_0) & cov(\theta_1,\theta_1) & \dots & cov(\theta_1,\theta_I) \\ \dots & \dots & \dots & \dots & \dots \\ I+1 & cov(\theta_I,\theta_0) & cov(\theta_I,\theta_1) & \dots & cov(\theta_I,\theta_I) \end{pmatrix}.$$

Mean and variance of the loss function are calculated using elements of Mean Matrix, ***A***, and Covariance Smatrix, ***V***, as follows:

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;$$

$$\sigma_L^2 = cov(\theta_0,\theta_0) - 2\sum_{i=1}^{I} cov(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I} cov(\theta_i,\theta_k)x_i x_k.$$

Given Mean Matrix, and Covariance Smatrix, the Mean Square Penalty is calculated as follows:

1.3.1.11 Calculation of Mean Square Penalty Using Mean Matrix and Variance Matrix (meansquare_i)

Let $\vec{x} = (x_1, x_2, \dots, x_I)$ be a decision vector; $\vec{\theta} = (\theta_0, \theta_1, \dots \theta_I)$ be a vector of random coefficients for Loss Function

$$L(\vec{x},\vec{\theta}) = L(\vec{x},\theta_0,\theta_1 \dots,\theta_I) = \theta_0 - \sum_{i=1}^{I} \theta_i x_i \quad .$$

All coefficients are independent random values with parameters:

$$\mu_i = E[\theta_i],\ \sigma_i^2 = E[(\theta_i - \mu_i)^2],\ i = 0,1,\dots,I.$$

These parameters should be presented in form of two matrices: matrix of means and variance matrix. Matrix of means has the following form:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \mu_0 & \mu_1 & \dots & \mu_I \end{pmatrix}.$$

Variance matrix has the following form:

$$V = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & \sigma_0^2 & \sigma_1^2 & \dots & \sigma_I^2 \end{pmatrix}.$$

Mean and variance of the loss function are calculated using elements of Mean Matrix, ***A,*** and variance matrix, ***V*** , as follows:

$$\mu_L = \mu_0 - \sum_{i=1}^{I} x_i \mu_i;$$
$$\sigma_L^2 = \sigma_0^2 + \sum_{i=1}^{I} x_I^2 \sigma_I^2.$$

Given Mean Matrix, and Variance Matrix, the Mean Square Penalty is calculated as follows:

$$\text{meansquare\_i}\left(L(\vec{x}, \vec{\theta})\right) = \sigma_L^2 + \mu_L^2.$$

1.3.1.12 Calculation of Variance Using Smatrix (variance)

The Variance is calculated as follows:

$$\text{variance}(\vec{x}, A) = a_{00} - 2\sum_{j=1}^{I} a_{0j} x_j + \sum_{i=1}^{I}\sum_{j=1}^{I} a_{ij} x_i x_j \ ,$$

where $x_i, i = 1,\dots,I$, are components of the Point $\vec{x}$,

and $a_{ij}, i = 0,1,\dots,I;\ j = 0,1,\dots,I,$ are coefficients of the following Smatrix:

$$A = \begin{pmatrix} id & scenario\_benchmark & name1 & \dots & nameI \\ 1 & a_{00} & a_{01} & \dots & a_{0I} \\ 2 & a_{10} & a_{11} & \dots & a_{1I} \\ \hline I+1 & a_{I0} & a_{I,1} & \dots & a_{II} \end{pmatrix}$$

Smatrix can be imported to PSG or generated inside of PSG Shell Environment from a Matrix of Scenarios (see the PSG Help section "Symmetric Matrix", subsections "**Conversion of Matrix of Scenarios to Covariance Matrix**", and "**Conversion of Matrix of Scenarios to Matrix of Expectations of Products**").

Particularly, if elements of the Smatrix are covariances:

$$a_{ik} = \text{cov}(\theta_i, \theta_k) = \sum_{j=1}^{J} p_j(\theta_{ji} - E[\theta_i])(\theta_{jk} - E[\theta_k]) = \sum_{j=1}^{J} p_j \theta_{ji}\theta_{jk} - E[\theta_i]E[\theta_k],$$

$$i, k = 0, 1, \dots, I,$$

$$E[\theta_i] = \sum_{j=1}^{J} p_j \theta_{ji}\ ,\ i = 1, 2, \dots, I,$$

then the Variance is calculated as follows:

$$\mathbf{variance}(\vec{x},\vec{\theta}) = \mathbf{cov}(\theta_0,\theta_0) - 2\sum_{i=1}^{I}\mathbf{cov}(\theta_0,\theta_i)x_i + \sum_{i=1}^{I}\sum_{k=1}^{I}\mathbf{cov}(\theta_i,\theta_k)x_i x_k .$$

1.3.1.13 Properties of Standard Group

Functions from the Standard group are calculated with double precision. The name of any function from this group may contain up to 128 symbols. The name of the Standard Penalty function calculated with Products Smatrix should begin with the string "st_pen_", the name of the Standard Penalty function calculated with Mean Matrix and Covariance Smatrix should begin with the string "st_pen_d_", the name of the Standard Penalty function calculated with Mean Matrix and Variance Matrix should begin with the string "st_pen_i_", the name of the Standard Deviation function should begin with the string "st_dev_", the name of the Standard Risk function calculated with Mean Matrix and Covariance Smatrix should begin with the string "st_risk_d_", the name of the Standard Risk function calculated with Mean Matrix and Variance Matrix should begin with the string "st_risk_i_", the name of the Standard Gain function calculated with Mean Matrix and Covariance Smatrix should begin with the string "st_risk_d_g_", the name of the Standard Gain function calculated with Mean Matrix and Variance Matrix should begin with the string "st_risk_i_g_", the name of the Mean Square Penalty function calculated with Products Smatrix should begin with the string "meansquare_", the name of the Mean Square Penalty function calculated with Mean Matrix and Covariance Smatrix should begin with the string "meansquare_d_", the name of the Mean Square Penalty function calculated with Mean Matrix and Variance Matrix should begin with the string "meansquare_i_", the name of the Variance function should begin with the string "variance_". The names of these functions may include only alphabetic characters, numbers, and the underscore sign, "_". The names of these functions are "insensitive" to the case, i.e. there is no difference between low case and upper case in these names.

Index

- A -

- B -

- C -

- D -

- E -

- F -

- L -

- M -

- N -

- P -

- Q -

- R -

- S -

- U -

- V -

Back Cover