PORTFOLIO OPTIMIZATION WITH CONDITIONAL VALUE-AT-RISK OBJECTIVE AND CONSTRAINTS

Pavlo Krokhmal[1], Jonas Palmquist[2], and Stanislav Uryasev[1]

Date: September 25, 2001

Correspondence should be addressed to: Stanislav Uryasev

[1]University of Florida, Dept. of Industrial and Systems Engineering, PO Box 116595, 303 Weil Hall, Gainesville, FL 32611-6595, Tel.: (352) 392–3091, E-mail: uryasev@ise.ufl.edu, URL: http://www.ise.ufl.edu/uryasev

[2]McKinsey & Company, Stockholm, Sweden, E-mail: jonas_palmquist@mckinsey.com

Abstract

Recently, a new approach for optimization of Conditional Value-at-Risk (CVaR) was suggested and tested with several applications. For continuous distributions, CVaR is defined as the expected loss exceeding Value-at Risk (VaR). However, generally, CVaR is the weighted average of VaR and losses exceeding VaR. Central to the approach is an optimization technique for calculating VaR and optimizing CVaR simultaneously. This paper extends this approach to the optimization problems with CVaR constraints. In particular, the approach can be used for maximizing expected returns under CVaR constraints. Multiple CVaR constraints with various confidence levels can be used to shape the profit/loss distribution. A case study for the portfolio of S&P 100 stocks is performed to demonstrate how the new optimization techniques can be implemented.

1 Introduction

Portfolio optimization has come a long way from Markowitz (1952) seminal work which introduces return/variance risk management framework. Developments in portfolio optimization are stimulated by two basic requirements: (1) adequate modeling of utility functions, risks, and constraints; (2) efficiency, i.e., ability to handle large numbers of instruments and scenarios.

Current regulations for finance businesses formulate some of the risk management requirements in terms of percentiles of loss distributions. An upper percentile of the loss distribution is called Value-at-Risk (VaR)[1]. For instance, 95%-VaR is an upper estimate of losses which is exceeded with 5% probability. The popularity of VaR is mostly related to a simple and easy to understand representation of high losses. VaR can be quite efficiently estimated and managed when underlying risk factors are normally (log-normally) distributed. For comprehensive introduction to risk management using VaR, we refer the reader to (Jorion, 1997). However, for non-normal distributions, VaR may have undesirable properties (Artzner at al., 1997, 1999) such as lack of sub-additivity, i.e., VaR of a portfolio with two instruments may be greater than the sum of individual VaRs of these two instruments[2]. Also, VaR is difficult to control/optimize for discrete distributions, when it is calculated using scenarios. In this case, VaR is non-convex (see definition of convexity in (Rockafellar, 1970) and non-smooth as a function of positions, and has multiple local extrema. An extensive description of various methodologies for the modeling of VaR can be seen, along with related resources, at URL http://www.gloriamundi.org/. Mostly, approaches to calculating VaR rely on linear approximation of the portfolio risks and assume a joint normal (or log-normal) distribution of the underlying market parameters (Duffie and Pan (1997), Jorion (1996), Pritsker (1997), RiskMetrics (1996), Simons (1996), Stublo Beder (1995), Stambaugh (1996)). Also, historical or Monte Carlo simulation-based tools are used when the portfolio contains nonlinear instruments such as options (Jorion (1996), Mauser and Rosen (1991), Pritsker (1997), RiskMetrics (1996), Stublo Beder (1995), Stambaugh (1996)). Discussions of optimization problems involving VaR can be found in Litterman (1997a, 1997b), Kast at al. (1998), Lucas and Klaassen (1998).

[1]By definition, VaR is the percentile of the loss distribution, i.e., with a specified confidence level α, the α-VaR of a portfolio is the lowest amount ζ such that, with probability α, the loss is less or equal to ζ. Regulations require that VaR should be a fraction of the available capital.

[2]When returns of instruments are normally distributed, VaR is sub-additive, i.e., diversification of the portfolio reduces VaR. For non-normal distributions, e.g., for discrete distribution, diversification of the portfolio may increase VaR.

Although risk management with percentile functions is a very important topic and in spite of significant research efforts (Andersen and Sornette (1999), Basak and Shapiro (1998), Emmer at al. (2000), Gaivoronski and Pflug (2000), Gourieroux et al. (2000), Grootweld and Hallerbach (2000), Kast at al. (1998), Puelz (1999), Tasche (1999)), efficient algorithms for optimization of percentiles for reasonable dimensions (over one hundred instruments and one thousand scenarios) are still not available. On the other hand, the existing efficient optimization techniques for portfolio allocation[3] do not allow for direct controlling[4] of percentiles of distributions (in this regard, we can mention the mean absolute deviation approach (Konno and Yamazaki, 1991), the regret optimization approach (Dembo and Rosen, 1999), and the minimax approach (Young, 1998)). This fact stimulated our development of the new optimization algorithms presented in this paper.

This paper suggests to use, as a supplement (or alternative) to VaR, another percentile risk measure which is called Conditional Value-at-Risk. The CVaR risk measure is closely related to VaR. For continuous distributions, CVaR is defined as the conditional expected loss under the condition that it exceeds VaR, see Rockafellar and Uryasev (2000). For continuous distributions, this risk measure also is known as Mean Excess Loss, Mean Shortfall, or Tail Value-at-Risk. However, for general distributions, including discrete distributions, CVaR is defined as the weighted average of VaR and losses strictly exceeding VaR, see Rockafellar and Uryasev (2000). Recently, Acerbi et al. (2001), Acerbi and Tasche (2001) redefined expected shortfall similarly to CVaR.

For general distributions, CVaR, which is a quite similar to VaR measure of risk has more attractive properties than VaR. CVaR is sub-additive and convex (Rockafellar and Uryasev, 2000). Moreover, CVaR is a *coherent* measure of risk in the sense of Artzner et al. (1997, 1999). Coherency of CVaR was first proved by Pflug (2000); see also Rockafellar and Uryasev (2001), Acerbi et al. (2001), Acerbi and Tasche (2001). Although CVaR has not become a standard in the finance industry, CVaR is gaining in the insurance industry (Embrechts at al., 1997). Similar

[3]High efficiency of these tools can be attributed to using linear programming (LP) techniques. LP optimization algorithms are implemented in number of commercial packages, and allow for solving of very large problems with millions of variables and scenarios. Sensitivities to parameters are calculated automatically using dual variables. Integer constraints can also be relatively well treated in linear problems (compared to quadratic or other nonlinear problems). However, recently developed interior point algorithms work equally well both for portfolios with linear and quadratic performance functions, see for instance Duarte (1999). The reader interested in various applications of optimization techniques in the finance area can find relevant papers in Ziemba and Mulvey (1998).

[4]It is impossible to impose constraints in VaR terms for general distributions without deteriorating the efficiency of these algorithms.

to CVaR measures have been introduced earlier in stochastic programming literature, but not in financial mathematics context. The conditional expectation constraints and integrated chance constraints described in (Prekopa, 1997) may serve the same purpose as CVaR.

Numerical experiments indicate that usually the minimization of CVaR also leads to near optimal solutions in VaR terms because VaR never exceeds CVaR (Rockafellar and Uryasev, 2000). Therefore, portfolios with low CVaR must have low VaR as well. Moreover, when the return-loss distribution is normal, these two measures are equivalent (Rockafellar and Uryasev, 2000), i.e., they provide the same optimal portfolio. However for very skewed distributions, CVaR and VaR risk optimal portfolios may be quite different. Moreover, minimizing of VaR may stretch the tail exceeding VaR because VaR does not control losses exceeding VaR, see Larsen at al. (2002). Also, Gaivoronski and Pflug (2000) have found that in some cases optimization of VaR and CVaR may lead to quite different portfolios.

Rockafellar and Uryasev (2000) demonstrated that linear programming techniques can be used for optimization of the Conditional Value-at-Risk (CVaR) risk measure. A simple description of the approach for minimizing CVaR and optimization problems with CVaR constraints can be found in (Uryasev, 2000). Several case studies showed that risk optimization with the CVaR performance function and constraints can be done for large portfolios and a large number of scenarios with relatively small computational resources. A case study on the hedging of a portfolio of options using the CVaR minimization technique is included in (Rockafellar and Uryasev, 2000). This problem was first studied in the paper by Mauser and Rosen (1991) with the minimum expected regret approach. Also, the CVaR minimization approach was applied to credit risk management of a portfolio of bonds, see Andersson at al. (1999).

This paper extends the CVaR minimization approach (Rockafellar and Uryasev, 2000) to other classes of problems with CVaR functions. We show that this approach can be used also for maximizing reward functions (e.g., expected returns) under CVaR constraints, as opposed to minimizing CVaR. Moreover, it is possible to impose many CVaR constraints with different confidence levels and shape the loss distribution according to the preferences of the decision maker. These preferences are specified directly in percentile terms, compared to the traditional approach, which specifies risk preferences in terms of utility functions. For instance, we may require that the mean values of the worst 1%, 5% and 10% losses are limited by some values. This approach provides a new efficient and flexible risk management tool.

The next section briefly describes the CVaR minimization approach from (Rockafellar and Uryasev, 2000) to lay the foundation for the further extensions. In Section 3, we formulated a general theorem on various equivalent representations of efficient frontiers with concave reward and convex risk functions. This equivalence is well known for mean-variance, see for instance, Steinbach (1999), and for mean-regret, (Dembo and Rosen, 1999), performance functions. We have shown that it holds for any concave reward and convex risk function, in particular for the CVaR risk function considered in this paper. In Section 4, using auxiliary variables, we formulated a theorem on reducing the problem with CVaR constraints to a much simpler convex problem. A similar result is also formulated for the case when both the reward and CVaR are included in the performance function. As it was earlier identified in (Rockafellar and Uryasev, 2000), the optimization automatically sets the auxiliary variable to VaR, which significantly simplifies the problem solution. Further, when the distribution is given by a fixed number of scenarios and the loss function is linear, we showed how the CVaR function can be replaced by a linear function and an additional set of linear constraints. In section 7, we developed a one-period model for optimizing a portfolio of stocks using historical scenario generation. A case study on the optimization of S&P100 portfolio of stocks with CVaR constraints is presented in the last section. We compared the return-CVaR and return-variance efficient frontiers of the portfolios. Finally, formal proofs of theorems are included in the appendix.

2 Conditional Value-at-Risk

The approach developed in (Rockafellar and Uryasev, 2000) provides the foundation for the analysis conducted in this paper. First, following (Rockafellar and Uryasev, 2000), we formally define CVaR and present several theoretical results which are needed for understanding this paper. Let $f(\mathbf{x}, \mathbf{y})$ be the loss associated with the decision vector[5] $\mathbf{x}$, to be chosen from a certain subset $\mathbf{X}$ of $\mathbb{R}^n$, and the random vector $\mathbf{y}$ in $\mathbb{R}^m$. The vector $\mathbf{x}$ can be interpreted as a portfolio, with $\mathbf{X}$ as the set of available portfolios (subject to various constraints), but other interpretations could be made as well. The vector $\mathbf{y}$ stands for the uncertainties, e.g., market prices, that can affect the loss. Of course the loss might be negative and thus, in effect, constitute a gain.

For each $\mathbf{x}$, the loss $f(\mathbf{x}, \mathbf{y})$ is a random variable having a distribution in $\mathbb{R}$ induced by that of $\mathbf{y}$. The underlying probability distribution of $\mathbf{y}$ in $\mathbb{R}^m$ will be assumed for convenience to have density, which we denote by $p(\mathbf{y})$. This assumption is not critical for the considered approach. The paper by Rockafellar and Uryasev (2001) defines CVaR for general distributions; however, here, for simplicity, we assume that the distribution has density. The probability of $f(\mathbf{x}, \mathbf{y})$ not

[5]We use boldface font for vectors to distinguish them from scalars.

exceeding a threshold ζ is then given by

$$\Psi(\mathbf{x},\zeta) \quad = \quad \int_{f(\mathbf{x},\mathbf{y})\leq\zeta} p(\mathbf{y})\, d\mathbf{y}. \tag{1}$$

As a function of ζ for fixed $\mathbf{x}$, $\Psi(\mathbf{x},\zeta)$ is the cumulative distribution function for the loss associated with $\mathbf{x}$. It completely determines the behavior of this random variable and is fundamental in defining VaR and CVaR.

The function $\Psi(\mathbf{x},\zeta)$ is nondecreasing with respect to (w.r.t.) ζ and we assume that $\Psi(\mathbf{x},\zeta)$ is everywhere continuous w.r.t. ζ. This assumption, like the previous one about density in $\mathbf{y}$, is made for simplicity. In some common situations, the required continuity follows from properties of the loss $f(\mathbf{x},\mathbf{y})$ and the density $p(\mathbf{y})$; see (Uryasev 1995).

The α-VaR and α-CVaR values for the loss random variable associated with $\mathbf{x}$ and any specified probability level α in $(0,1)$ will be denoted by $\zeta_\alpha(\mathbf{x})$ and $\phi_\alpha(\mathbf{x})$. In our setting they are given by

$$\zeta_\alpha(\mathbf{x}) \;=\; \min\{\,\zeta\in\mathbb{R}:\; \Psi(\mathbf{x},\zeta)\geq\alpha\,\} \tag{2}$$

and

$$\phi_\alpha(\mathbf{x}) \;=\; (1-\alpha)^{-1}\int_{f(\mathbf{x},\mathbf{y})\geq\zeta_\alpha(\mathbf{x})} f(\mathbf{x},\mathbf{y})p(\mathbf{y})\, d\mathbf{y}. \tag{3}$$

In the first formula, $\zeta_\alpha(\mathbf{x})$ comes out as the left endpoint of the nonempty interval[6] consisting of the values ζ such that actually $\Psi(\mathbf{x},\zeta)=\alpha$. In the second formula, the probability that $f(\mathbf{x},\mathbf{y})\geq\zeta_\alpha(\mathbf{x})$ is therefore equal to $1-\alpha$. Thus, $\phi_\alpha(\mathbf{x})$ comes out as the conditional expectation of the loss associated with $\mathbf{x}$ relative to that loss being $\zeta_\alpha(\mathbf{x})$ or greater.

The key to the approach is a characterization of $\phi_\alpha(\mathbf{x})$ and $\zeta_\alpha(\mathbf{x})$ in terms of the function F_α on $X\times\mathbb{R}$ that we now define by

$$F_\alpha(\mathbf{x},\zeta) \;=\; \zeta + (1-\alpha)^{-1}\int_{\mathbf{y}\in\mathbf{R}^n} [f(\mathbf{x},\mathbf{y})-\zeta]^+\, p(\mathbf{y})\, d\mathbf{y}, \tag{4}$$

where $[t]^+ = \max\{t,0\}$. The crucial features of F_α, under the assumptions made above, are as follows (Rockafellar and Uryasev, 2000).

Theorem 1. *As a function of ζ, $F_\alpha(\mathbf{x},\zeta)$ is convex and continuously differentiable. The α-CVaR of the loss associated with any $\mathbf{x}\in X$ can be determined from the formula*

$$\phi_\alpha(\mathbf{x}) \;=\; \min_{\zeta\in\mathbf{R}} F_\alpha(\mathbf{x},\zeta). \tag{5}$$

[6] This follows from $\Psi(\mathbf{x},\zeta)$ being continuous and nondecreasing w.r.t. ζ. The interval might contain more than a single point if Ψ has "flat spots."

In this formula, the set consisting of the values of ζ for which the minimum is attained, namely

$$A_\alpha(\mathbf{x}) \;=\; \operatorname*{argmin}_{\zeta\in\mathbb{R}} F_\alpha(\mathbf{x},\zeta), \tag{6}$$

is a nonempty, closed, bounded interval (perhaps reducing to a single point), and the α-VaR of the loss is given by

$$\zeta_\alpha(\mathbf{x}) \;=\; \textit{left endpoint of } A_\alpha(\mathbf{x}). \tag{7}$$

In particular, one always has

$$\zeta_\alpha(\mathbf{x}) \in \operatorname*{argmin}_{\zeta\in\mathbb{R}} F_\alpha(\mathbf{x},\zeta) \quad \textit{and} \quad \phi_\alpha(\mathbf{x}) = F_\alpha(\mathbf{x},\zeta_\alpha(\mathbf{x})). \tag{8}$$

For background on convexity, which is a key property in optimization that in particular eliminates the possibility of a local minimum being different from a global minimum, see, for instance, Rockafellar (1970). Other important advantages of viewing VaR and CVaR through the formulas in Theorem 1 are captured in the next theorem, also proved in (Rockafellar and Uryasev, 2000).

Theorem 2. *Minimizing the α-CVaR of the loss associated with $\mathbf{x}$ over all $\mathbf{x}\in\mathbf{X}$ is equivalent to minimizing $F_\alpha(\mathbf{x},\zeta)$ over all $(\mathbf{x},\zeta)\in\mathbf{X}\times\mathbb{R}$, in the sense that*

$$\min_{\mathbf{x}\in\mathbf{X}} \phi_\alpha(\mathbf{x}) \;=\; \min_{(\mathbf{x},\zeta)\in\mathbf{X}\times\mathbb{R}} F_\alpha(\mathbf{x},\zeta), \tag{9}$$

where moreover a pair $(\mathbf{x}^*,\zeta^*)$ achieves the right hand side minimum if and only if $\mathbf{x}^*$ achieves the left hand side minimum and $\zeta^*\in A_\alpha(\mathbf{x}^*)$. In particular, therefore, in circumstances where the interval $A_\alpha(\mathbf{x}^*)$ reduces to a single point (as is typical), the minimization of $F(\mathbf{x},\zeta)$ over $(\mathbf{x},\zeta)\in\mathbf{X}\times\mathbb{R}$ produces a pair $(\mathbf{x}^*,\zeta^*)$, not necessarily unique, such that $\mathbf{x}^*$ minimizes the α-CVaR and ζ^* gives the corresponding α-VaR.

Furthermore, $F_\alpha(\mathbf{x},\zeta)$ is convex w.r.t. $(\mathbf{x},\zeta)$,and $\phi_\alpha(\mathbf{x})$ is convex w.r.t. $\mathbf{x}$, when $f(\mathbf{x},\mathbf{y})$ is convex with respect to $\mathbf{x}$, in which case, if the constraints are such that $\mathbf{X}$ is a convex set, the joint minimization is an instance of convex programming.

According to Theorem 2, it is not necessary, for the purpose of determining a vector $\mathbf{x}$ that yields the minimum α-CVaR, to work directly with the function $\phi_\alpha(\mathbf{x})$, which may be hard to do because of the nature of its definition in terms of the α-VaR value $\zeta_\alpha(\mathbf{x})$ and the often troublesome mathematical properties of that value. Instead, one can operate on the far simpler expression $F_\alpha(\mathbf{x},\zeta)$ with its convexity in the variable ζ and even, very commonly, with respect to $(\mathbf{x},\zeta)$.

3 Efficient Frontier: Different Formulations

The paper by Rockafellar and Uryasev (2000) considered minimizing CVaR, while requiring a minimum expected return. By considering different expected returns, we can generate an efficient frontier. Alternatively, we also can maximize returns while not allowing large risks. We, therefore, can swap the CVaR function and the expected return in the problem formulation (compared to (Rockafellar and Uryasev, 2000), thus minimizing the negative expected return with a CVaR constraint. By considering different levels of risks, we can generate the efficient frontier.

We will show in a general setting that there are three equivalent formulations of the optimization problem. They are equivalent in the sense that they produce the same efficient frontier. The following theorem is valid for general functions satisfying conditions of the theorem.

Theorem 3. *Let us consider the functions $\phi(\mathbf{x})$ and $R(\mathbf{x})$ dependent on the decision vector $\mathbf{x}$, and the following three problems:*

$$\text{(P1)} \qquad \min_{x} \quad \phi(\mathbf{x}) - \mu_1 R(\mathbf{x}), \quad \mathbf{x} \in \mathbf{X}, \quad \mu_1 \geq 0,$$

$$\text{(P2)} \qquad \min_{x} \quad \phi(\mathbf{x}), \quad R(\mathbf{x}) \geq \rho, \quad \mathbf{x} \in \mathbf{X},$$

$$\text{(P3)} \qquad \min_{x} \quad -R(\mathbf{x}), \quad \phi(\mathbf{x}) \leq \omega, \quad \mathbf{x} \in \mathbf{X}.$$

Suppose that constraints $R(\mathbf{x}) \geq \rho$, $\phi(\mathbf{x}) \leq \omega$ have internal points.[7] *Varying the parameters μ_1, ρ, and ω, traces the efficient frontiers for the problems (P1)-(P3), accordingly. If $\phi(\mathbf{x})$ is convex, $R(\mathbf{x})$ is concave and the set $\mathbf{X}$ is convex, then the three problems, (P1)-(P3), generate the same efficient frontier.*

The proof of Theorem 3 is furnished in Appendix A.

The equivalence between problems (P1)-(P3) is well known for mean-variance (Steinbach, 1999) and mean-regret (Dembo and Rosen, 1999) efficient frontiers. We have shown that it holds for any concave reward and convex risk functions with convex constraints.

Further, we consider that the loss function $f(\mathbf{x}, \mathbf{y})$ is linear w.r.t. $\mathbf{x}$, therefore Theorem 2 implies that the CVaR risk function $\phi_\alpha(\mathbf{x})$ is convex w.r.t. $\mathbf{x}$. Also, we suppose that the reward function, $R(\mathbf{x})$ is linear and the constraints are linear. The conditions of Theorem 3 are satisfied for the CVaR risk function $\phi_\alpha(\mathbf{x})$ and the reward function $R(\mathbf{x})$. Therefore, maximizing the reward under a CVaR constraint, generates the same efficient frontier as the minimization of CVaR under a constraint on the reward.

[7] This condition can be replaced by some other regularity conditions used in duality theorems.

4 Equivalent Formulations with Auxiliary Variables

Theorem 3 implies that we can use problem formulations (P1), (P2), and (P3) for generating the efficient frontier with the CVaR risk function $\phi_\alpha(\mathbf{x})$ and the reward function $R(\mathbf{x})$. Theorem 2 shows that the function $F_\alpha(\mathbf{x},\zeta)$ can be used instead of $\phi_\alpha(\mathbf{x})$ to solve problem (P2). Further, we demonstrate that, similarly, the function $F_\alpha(\mathbf{x},\zeta)$ can be used instead of $\phi_\alpha(\mathbf{x})$ in problems (P1) and (P3).

Theorem 4. *The two minimization problems below*

$$\text{(P4)} \qquad \min_{\mathbf{x}\in\mathbf{X}} \quad -R(\mathbf{x}), \quad \phi_\alpha(\mathbf{x}) \le \omega, \quad \mathbf{x}\in\mathbf{X}$$

and

$$\text{(P4}'\text{)} \qquad \min_{(\zeta,\mathbf{x})\in\mathbf{X}\times\mathrm{I\!R}} \quad -R(\mathbf{x}), \quad F_\alpha(\mathbf{x},\zeta) \le \omega, \quad \mathbf{x}\in\mathbf{X}$$

are equivalent in the sense that their objectives achieve the same minimum values. Moreover, if the CVaR constraint in (P4) is active, a pair $(\mathbf{x}^*,\zeta^*)$ achieves the minimum of (P4$'$) if and only if $\mathbf{x}^*$ achieves the minimum of (P4) and $\zeta^*\in A_\alpha(\mathbf{x}^*)$. In particular, when the interval $A_\alpha(\mathbf{x}^*)$ reduces to a single point, the minimization of $-R(\mathbf{x})$ over $(\mathbf{x},\zeta)\in\mathbf{X}\times\mathrm{I\!R}$ produces a pair $(\mathbf{x}^*,\zeta^*)$ such that $\mathbf{x}^*$ maximizes the return and ζ^* gives the corresponding α-VaR.

Theorem 5. *The two minimization problems below*

$$\text{(P5)} \qquad \min_{\mathbf{x}\in\mathbf{X}} \quad \phi_\alpha(\mathbf{x}) - \mu_1 R(\mathbf{x}), \quad \mu_1 \ge 0, \quad \mathbf{x}\in\mathbf{X}$$

and

$$\text{(P5}'\text{)} \qquad \min_{(\mathbf{x},\zeta)\in\mathbf{X}\times\mathrm{I\!R}} \quad F_\alpha(\mathbf{x},\zeta) - \mu_1 R(\mathbf{x}), \quad \mu_1 \ge 0, \quad \mathbf{x}\in\mathbf{X}$$

are equivalent in the sense that their objectives achieve the same minimum values. Moreover, a pair $(\mathbf{x}^*,\zeta^*)$ achieves the minimum of (P5$'$) if and only if $\mathbf{x}^*$ achieves the minimum of (P5) and $\zeta^*\in A_\alpha(\mathbf{x}^*)$. In particular, when the interval $A_\alpha(\mathbf{x}^*)$ reduces to a single point, the minimization of $F_\alpha(\mathbf{x},\zeta)-\mu_1 R(\mathbf{x})$ over $(\mathbf{x},\zeta)\in\mathbf{X}\times\mathrm{I\!R}$ produces a pair $(\mathbf{x}^*,\zeta^*)$ such that $\mathbf{x}^*$ minimizes $\phi_\alpha(\mathbf{x})-\mu_1 R(\mathbf{x})$ and ζ^* gives the corresponding α-VaR.

The proof of Theorems 4 and 5 are furnished in Appendix B.

5 Discretization

The equivalent problem formulations presented in Theorems 2, 4 and 5 can be combined with ideas for approximating the integral in $F_\alpha(\mathbf{x},\zeta)$, see (4). This offers a rich range of possibilities.

The integral in $F_\alpha(\mathbf{x},\zeta)$ can be approximated in various ways. For example, this can be done by sampling the probability distribution of $\mathbf{y}$ according to its density $p(\mathbf{y})$. If the sampling generates a collection of vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_J$, then the corresponding approximation to

$$F_\alpha(\mathbf{x},\zeta) = \zeta + (1-\alpha)^{-1} \int_{\mathbf{y}\in\mathbb{R}^n} [f(\mathbf{x},\mathbf{y}) - \zeta]^+ p(\mathbf{y})\, d\mathbf{y}$$

is

$$\tilde{F}_\alpha(\mathbf{x},\zeta) \;=\; \zeta + (1-\alpha)^{-1} \sum_{j=1}^{J} \pi_j \, [f(\mathbf{x},\mathbf{y}_j) - \zeta]^+ \;, \tag{10}$$

where π_j are probabilities of scenarios $\mathbf{y}_j$. If the loss function $f(\mathbf{x},\mathbf{y})$ is linear w.r.t. $\mathbf{x}$, then the function $\tilde{F}_\alpha(\mathbf{x},\zeta)$ is convex and piecewise linear.

6 Linearization

The function $F_\alpha(\mathbf{x},\zeta)$ in optimization problems in Theorems 2, 4, and 5 can be approximated by the function $\tilde{F}_\alpha(\mathbf{x},\zeta)$. Further, by using dummy variables z_j, $j = 1, ..., J$, the function $\tilde{F}_\alpha(\mathbf{x},\zeta)$ can be replaced by the linear function $\zeta + (1-\alpha)^{-1} \sum_{j=1}^{J} \pi_j \, z_j$ and the set of linear constraints

$$z_j \geq f(\mathbf{x},\mathbf{y}_j) - \zeta, \qquad z_j \geq 0, \quad j = 1, ..., J, \qquad \zeta \in \mathbb{R}.$$

For instance, by using Theorem 4 we can replace the constraint

$$\phi_\alpha(\mathbf{x}) \leq \omega$$

in optimization problem (P4) by the constraint

$$F_\alpha(\mathbf{x},\zeta) \leq \omega \;.$$

Further, the above constraint can be approximated by

$$\tilde{F}_\alpha(\mathbf{x},\zeta) \leq \omega \;, \tag{11}$$

and reduced to the following system of linear constraints

$$\zeta + (1-\alpha)^{-1} \sum_{j=1}^{J} \pi_j \, z_j \leq \omega, \tag{12}$$

$$z_j \geq f(\mathbf{x}, \mathbf{y}_j) - \zeta, \qquad z_j \geq 0, \quad j = 1, ..., J, \qquad \zeta \in \mathrm{I\!R}. \tag{13}$$

Similarly, approximations by linear functions can be done in the optimization problems in Theorems 2 and 5.

7 One Period Portfolio Optimization Model with Transaction Costs

7.1 Loss and Reward Functions

Let us consider a portfolio of n, $(i = 1, ..., n)$ different financial instruments in the market, among which there is one risk-free instrument (cash, or bank account etc). Let $\mathbf{x}^0 = (x_1^0, x_2^0, ..., x_n^0)^T$ be the positions, i.e., number of shares, of each instrument in the initial portfolio, and let $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ be the positions in the optimal portfolio that we intend to find using the algorithm. The initial prices for the instruments are given by $\mathbf{q} = (q_1, q_2, ..., q_n)^T$. The inner product $\mathbf{q}^T\mathbf{x}^0$ is thus the initial portfolio value. The scenario-dependent prices for each instrument at the end of the period are given by $\mathbf{y} = (y_1, y_2, ..., y_n)^T$. The loss function over the period is

$$f(\mathbf{x}, \mathbf{y}; \mathbf{x}^0, \mathbf{q}) = -\mathbf{y}^T\mathbf{x} + \mathbf{q}^T\mathbf{x}^0. \tag{14}$$

The reward function $R(\mathbf{x})$ is the expected value of the portfolio at the end of the period,

$$R(\mathbf{x}) = \mathrm{I\!E}[\, \mathbf{y}^T\mathbf{x}\,] = \sum_{i=1}^{n} \mathrm{I\!E}[y_i]x_i \ . \tag{15}$$

Evidently, defined in this way, the reward function $R(\mathbf{x})$ and the loss function $f(\mathbf{x}, \mathbf{y})$ are related as

$$R(\mathbf{x}) = -\mathrm{I\!E}[f(\mathbf{x}, \mathbf{y})] + \mathbf{q}^T\mathbf{x}^0.$$

The reward function $R(\mathbf{x})$ is linear (and therefore concave) in $\mathbf{x}$.

7.2 CVaR Constraint

Current regulations impose capital requirements on investment companies, proportional to the VaR of a portfolio. These requirements can be enforced by constraining portfolio CVaR at different confidence levels, since CVaR $\geq$ VaR. The upper bound on CVaR can be chosen as the maximum VaR. According to this, we find it meaningful to present the risk constraint in the form

$$\phi_\alpha(\mathbf{x}) \leq \omega\, \mathbf{q}^T\mathbf{x}^0, \tag{16}$$

where the risk function $\phi_\alpha(\mathbf{x})$ is defined as the α–CVaR for the loss function given by (14), and ω is a percentage of the initial portfolio value $\mathbf{q}^T\mathbf{x}^0$, allowed for risk exposure. The loss function given by (14) is linear (and therefore convex) in $\mathbf{x}$, therefore, the α-CVaR function $\phi_\alpha(\mathbf{x})$ is also convex in $\mathbf{x}$. The set of linear constraints corresponding to (16), is

$$\zeta + (1-\alpha)^{-1} \sum_{j=1}^{J} \pi_j \, z_j \leq \omega \sum_{i=1}^{n} q_i \, x_i^0 \ , \tag{17}$$

$$z_j \geq \sum_{i=1}^{n} (-y_{ij} x_i + q_i x_i^0) - \zeta \ , \qquad z_j \geq 0, \qquad j = 1, ..., J. \tag{18}$$

7.3 Transaction costs

We assume a linear transaction cost, proportional to the total dollar value of the bought/sold assets. For a treatment of non-convex transaction costs, see Konno and Wijayanayake (1999). With every instrument, we associate a transaction cost c_i. When buying or selling instrument i, one pays c_i times the amount of transaction. For cash we set $c_{\text{cash}} = 0$. That is, one only pays for buying and selling the instrument, and not for moving the cash in and out of the account.

According to that, we consider a balance constraint that maintains the total value of the portfolio including transaction costs

$$\sum_{i=1}^{n} q_i x_i^0 = \sum_{i=1}^{n} c_i q_i \, |x_i^0 - x_i| + \sum_{i=1}^{n} q_i x_i \ .$$

This equality can be reformulated using the following set of linear constraints[8]

$$\sum_{i=1}^{n} q_i x_i^0 = \sum_{i=1}^{n} c_i q_i \, (u_i^+ + u_i^-) + \sum_{i=1}^{n} q_i x_i \ ,$$

$$x_i - x_i^0 = u_i^+ - u_i^-, \qquad i = 1, ..., n,$$

$$u_i^+ \geq 0, \ u_i^- \geq 0, \qquad i = 1, ..., n.$$

7.4 Value Constraint

We do not allow for an instrument i to constitute more than a given percent, ν_i, of the total portfolio value

$$q_i x_i \leq \nu_i \sum_{k=1}^{n} x_k q_k \ .$$

This constraint makes sense only when short positions are not allowed.

[8]The nonlinear constraint $u_i^+ \, u_i^- = 0$ can be omitted since simultaneous buying and selling of the same instrument, i, can never be optimal.

7.5 Change in Individual Positions (Liquidity Constraints) and Bounds on Positions

We consider that position changes can be bounded. This bound could be, for example, a fixed number or be proportional to the initial position in the instrument

$$0 \leq u_i^- \leq \underline{u}_i^-, \quad 0 \leq u_i^+ \leq \overline{u}_i^+, \qquad i = 1, ..., n.$$

These constraints may reflect limited liquidity of instruments in the portfolio (large transactions may significantly affect the price q_i).

We, also, consider that the positions themselves can be bounded

$$\underline{x}_i \leq x_i \leq \overline{x}_i, \qquad i = 1, ..., n. \tag{19}$$

7.6 The Optimization Problem

Below we present the problem formulation, which optimizes the reward function subject to constraints described in sections (7.2)–(7.5).

$$\min_{\mathbf{x},\zeta} \sum_{i=1}^{n} -\mathbb{E}[y_i] x_i, \tag{20}$$

subject to

$$\zeta + (1-\alpha)^{-1} \sum_{j=1}^{J} \pi_j \, z_j \leq \omega \sum_{k=1}^{n} q_k x_k^0, \tag{21}$$

$$z_j \geq \sum_{i=1}^{n} (-y_{ij} x_i + q_i x_i^0) - \zeta, \qquad z_j \geq 0, \qquad j = 1, ..., J, \tag{22}$$

$$q_i x_i \leq \nu_i \sum_{k=1}^{n} q_k x_k, \qquad i = 1, ..., n, \tag{23}$$

$$\sum_{i=1}^{n} q_i x_i^0 = \sum_{i=1}^{n} c_i q_i \, (u_i^+ + u_i^-) + \sum_{i=1}^{n} q_i x_i, \tag{24}$$

$$x_i - x_i^0 = u_i^+ - u_i^-, \qquad i = 1, ..., n, \tag{25}$$

$$0 \leq u_i^- \leq \underline{u}_i^-, \quad 0 \leq u_i^+ \leq \overline{u}_i^+, \qquad i = 1, ..., n, \tag{26}$$

$$\underline{x}_i \le x_i \le \overline{x}_i, \qquad i = 1, ..., n. \tag{27}$$

By solving this problem, we get the optimal vector $\mathbf{x}^*$, the corresponding VaR, which equals[9] ζ^*, and the maximum expected return, which equals $\mathbb{E}[\mathbf{y}]\mathbf{x}^*/(\mathbf{q}^T\mathbf{x}^0)$. The efficient return-CVaR frontier is obtained by taking different risk tolerance levels ω.

7.7 Scenario Generation

With our approach, the integral in the CVaR function is approximated by the weighed sum over all scenarios. This approach can be used with different schemes for generating scenarios. For example, one can assume a joint distribution for the price-return process for all instruments and generate scenarios in a Monte Carlo simulation. Also, the approach allows for using historical data without assuming a particular distribution. In our case study, we used historical returns over a certain time period for the scenario generation, with length Δt of the period equal to the portfolio optimization period. For instance, when minimizing over a one day period, we take the ratio of the closing prices of two consecutive days, p^{t_j} and p^{t_j+1}. Similarly, for a two week period, we consider historical returns p^{t_j+10}/p^{t_j}. In such a fashion, we represent the scenario set for random variable y_i, which is the end-of-period price of instrument i, with the set of J historical returns multiplied by the current price q_i,

$$y_{ij} = q_i \, p_i^{t_j+\Delta t}/p_i^{t_j} \ , \quad j = 1, \ldots, J,$$

where $t_1, \ldots, t_J$ are closing times for J consecutive business days. Further, in the numerical simulations, we consider a two week period, $\Delta t = 10$. The expected end-of-period price of instrument i is

$$\mathbb{E}[y_i] = \sum_{j=1}^{J} \pi_j \, y_{ij} = J^{-1} \sum_{j=1}^{J} y_{ij} \ ,$$

where we assumed that all scenarios y_{ij} are equally probable, i.e. $\pi_j = 1/J$.

8 Case Study: Portfolio of S&P100 Stocks

We now proceed with a case study and construct the efficient frontier of a portfolio consisting of stocks in the S&P100 index. We maximized the portfolio value subject to various constraints on

[9] If there are many optimal solutions, VaR equals the lowest optimal value ζ^*.

CVaR. The algorithm was implemented in C++ and we used the CPLEX 7.0 Callable Library to solve the LP problem.

This case study is designed as a demonstration of the methodology, rather than a practical recommendation for investments. We have used historical data for scenario generation (10-day historical returns). While there is some estimation error in the risk measure, this error is much greater for expected returns. The historical returns over a 10-day period provide very little information on the actual "to-be-realized out-of-sample" returns; i.e., historical returns have little "forecasting power." These issues are discussed in many academic studies, including (Jorion 1996, 2000, Michaud, 1989). The primary purpose of the presented case study is the demonstration of the novel CVaR risk management methodology and the possibility to apply it to portfolio optimization. This technology can be combined with more adequate scenario generation procedures utilizing expert opinions and advanced statistical forecasting techniques, such as neural networks. The suggested model is designed as one stage of the multistage investment model to be used in a realistic investment environment. However, discussing this multistage investment model and the scenario generation procedures used for this model is beyond the scope of this paper.

The set of instruments to invest in was set to the stocks in the S&P100 as of the first of September 1999. Due to insufficient data, six of the stocks were excluded[10]. The optimization was run for two-week period, ten business days. For scenario generation, we used closing prices for five hundreds of the overlapping two-week periods (July 1, 1997 - July 8, 1999). In effect, this was an in-sample optimization using 500 overlapping returns measured over 10 business days.

The initial portfolio contained only cash, and the algorithm should determine an optimal investment decision subject to risk constraints. The limits on the positions were set to $\underline{x}_i = 0$ and $\overline{x}_i = \infty$ respectively, i.e., short positions were not allowed. The limits on the changes in the individual positions, $\underline{u}^-$ and $\overline{u}^+$, were both set to infinity. The limit on how large a part of the total portfolio value one single asset can constitute, ν_i, was set to 20% for all i. The return on cash was set to 0.16% over two weeks. We made calculations with various values of the parameter ω in CVaR constraint[11].

[10] Citigroup Inc., Hartford Financial Svc.Gp., Lucent Technologies, Mallinckrodt Inc., Raytheon Co., U.S. Bancorp.

[11] ω was set as some percentage of the initial portfolio value.

8.1 Efficient Frontier and Portfolio Configuration

Fig. 1 shows the efficient frontier of the portfolio with the CVaR constraint. The values on the Risk scale represent the tolerance level ω, i.e. the percentage of the initial portfolio value which is allowed for risk exposure. For example, setting Risk = 10% ($\omega = 0.10$) and $\alpha = 0.95$ implies that the average loss in 5% worst cases must not exceed 10% of the initial portfolio value. Naturally, higher risk tolerance levels ω in CVaR constraint (21) allow for achieving higher expected returns. It is also apparent from Fig. 1 that for every value of risk confidence level α there exists some value ω, after which the CVaR constraint becomes inactive (i.e., not binding). A higher expected return cannot be attained without loosening other constraints in problem (20)–(27), or without adding new instruments to the optimization set. In this numerical example, the maximum rate of return that can be achieved for the given set of instruments and constraints equals 2.96% over two weeks. However, very small values of risk tolerance ω cause the optimization problem (20)–(27) to be infeasible; in other words, there is no such combination of assets that would satisfy CVaR constraints (21)–(22) and the constraints on positions (23)–(27) simultaneously.

Table 1 presents the portfolio configuration for different risk levels ($\alpha = 0.90$). Recall that we imposed the constraint on the percentage ν of the total portfolio value that one stock can constitute (23). We set $\nu = 0.2$, i.e., a single asset cannot constitute more than 20% of the total portfolio value. Table 1 shows that for higher levels of allowed risk, the algorithm reduces the number of the instruments in the portfolio in order to achieve a higher return (due to the imposed constraints, the minimal number of instruments in the portfolio, including risk-free cash, equals five). This confirms the well-known fact that "diversifying" the portfolio reduces the risk. Relaxing the constraint on risk allows the algorithm to choose only the most profitable stocks. As we tighten the risk tolerance level, the number of instruments in the portfolio increases, and for more "conservative" investing (2% risk), we obtain a portfolio with more than 15 assets, including the risk-free asset (cash). The instruments not shown in the table have zero portfolio weights for all risk levels.

Transaction costs need to be taken into account when employing an active trading strategy. Transaction costs account for a fee paid to the broker/market, bid-ask spreads, and poor liquidity. To examine the impact of the transaction costs, we calculated the efficient frontier with the following transaction costs, $c = 0\%, 0.25\%$, and 1%. Fig. 2 shows that the transaction costs nonlinearly lower the expected return. Since transaction costs are incorporated into the optimization problem, they also affect the choice of stocks.

8.2 Comparison with Mean-Variance Portfolio Optimization

In this section, we illustrate the relation of the developed approach to the standard Markowitz mean-variance (MV) framework. It was shown in (Rockafellar and Uryasev, 2000) that for normally distributed loss functions these two methodologies are equivalent in the sense that they generate the same efficient frontier. However, in the case of non-normal, and especially non-symmetric distributions, CVaR and MV portfolio optimization approaches may reveal significant differences. Indeed, the CVaR optimization technique aims at reshaping one tail of the loss distribution, which corresponds to high losses, and does not account for the opposite tail representing high profits. On the contrary, the Markowitz approach defines the risk as the variance of the loss distribution, and since the variance incorporates information from both tails, it is affected by high gains as well as by high losses.

Here, we used historical returns as a scenario input to the model, without making any assumptions about the distribution of the scenario variables.We compared the CVaR methodology with the MV approach by running the optimization algorithms on the same set of instruments and scenarios. The MV optimization problem was formulated as follows (see Markowitz, 1952):

$$\min_{\mathbf{x}} \sum_{i=1}^{n} \sum_{k=1}^{n} \sigma_{ik}\, x_i x_k, \tag{28}$$

subject to

$$\sum_{i=1}^{n} x_i = 1, \tag{29}$$

$$\sum_{i=1}^{n} \mathbb{E}[r_i]\, x_i = r_p, \tag{30}$$

$$0 \le x_i \le \nu_i, \quad i = 1, \ldots, n, \tag{31}$$

where x_i are portfolio weights, unlike problem (20)–(27), where x_i are numbers of shares of corresponding instruments. r_i is the rate of return of instrument i, and σ_{ik} is the covariance between returns of instruments i and k: $\sigma_{ik} = \mathrm{cov}(r_i, r_k)$. The first constraint (29) is the budget constraint; (30) requires portfolio's expected return to be equal to a prescribed value r_p; finally, (31) imposes bounds on portfolio weights, where ν_i are the same as in (23). The set of constraints (29)–(31) is identical to (23)–(27), except for transaction cost constraints. The expectations and covariances in (28), (30) are computed using the 10-day historical returns, which were used for scenario generation in the CVaR optimization model:

$$r_{ij} = p_i^{t_j+10}/p_i^{t_j} - 1, \quad \mathbb{E}[r_i] = \frac{1}{J}\sum_{j=1}^{J} r_{ij}, \quad \sigma_{ik} = \frac{1}{J-1}\sum_{j=1}^{J}(r_{ij} - \mathbb{E}[r_i])(r_{kj} - \mathbb{E}[r_k]).$$

Figure 3 displays the CVaR–efficient portfolios in Return/CVaR scales for the risk confidence level $\alpha = 0.95$ (continuous line). Also, for each return it displays the CVaR of the MV optimal portfolio (dashed line). Note, that for a given return, the MV optimal portfolio has a higher CVaR risk level than the efficient Return/CVaR portfolio. Figure 4 displays similar graphs for $\alpha = 0.99$. The discrepancy between CVaR and MV solutions is higher for the higher confidence level.

Figure 5 displays the efficient frontier for Return/MV efficient portfolios (continuous line). Also, for each return it displays the standard deviation of the CVaR optimal portfolio with confidence level $\alpha = 0.95$ (dashed line). As expected, for a given return, the CVaR optimal portfolio has a higher standard deviation than the efficient Return/MV portfolio. Similar graphs are displayed in Figure 6 for $\alpha = 0.99$. The discrepancy between CVaR and MV solutions is higher for the higher confidence level, similar to Figures 3, 4.

However, the difference between the MV and CVaR approaches is not very significant. Relatively close graphs of CVaR– and MV–optimal portfolios indicate that a CVaR optimal portfolio is "near optimal" in MV–sense, and vice versa, a MV–optimal portfolio is "near optimal" in CVaR–sense. This agreement between the two solutions should not, however, be misleading in deciding that the discussed portfolio management methodologies "are the same". The obtained results are dataset-specific, and the closeness of solutions of CVaR and MV optimization problems is caused by apparently "close-to-normal" distributions of the historical returns used in our case study. Including options in the portfolio or credit risk with skewed return distributions may lead to quite different optimal solutions of the efficient MV and CVaR portfolios (Mausser and Rosen, 1999, Larsen at al., 2002).

9 Concluding Remarks

The paper extends the approach for portfolio optimization (Rockafellar and Uryasev, 2000), which simultaneously calculates VaR and optimizes CVaR. We first showed (Theorem 3) that for risk-return optimization problems with convex constraints, one can use different optimization formulations. This is true in particular for the considered CVaR optimization problem. We then showed (Theorems 4 and 5) that the approach by Rockafellar and Uryasev (2000) can be extended to the reformulated problems with CVaR constraints and the weighted return-CVaR performance function. The optimization with multiple CVaR constrains for different time frames and at different confidence levels allows for shaping distributions according to the decision maker's preferences. We developed a model for optimizing portfolio returns with CVaR constraints using historical scenarios and conducted a case study on optimizing portfolio of S&P100 stocks. The case study showed that the optimization algorithm, which is based on linear programming techniques, is very stable and efficient. The approach can handle large number of instruments and scenarios. CVaR risk management constraints (reduced to linear constraints) can be used in various applications to bound percentiles of loss distributions.

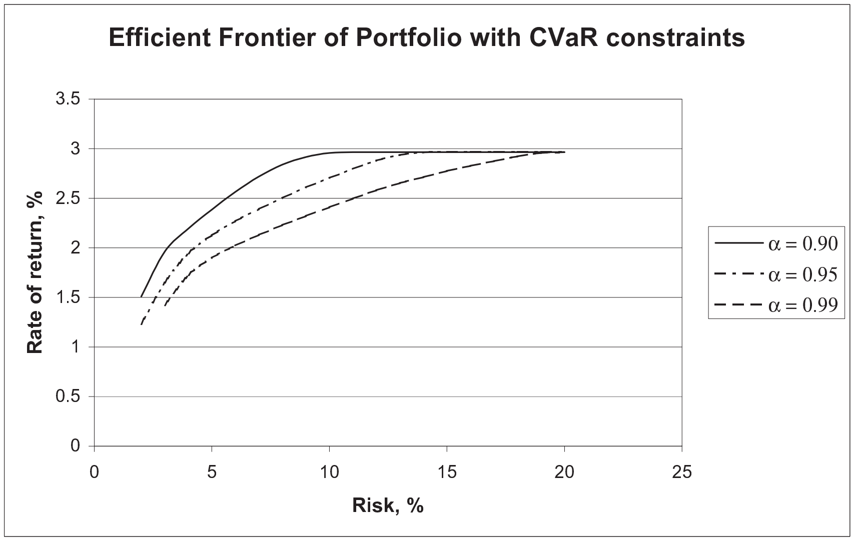

Figure 1: Efficient frontier (optimization with CVaR constraints). Rate of Return is the expected rate of return of the optimal portfolio during a 2 week period. The Risk scale displays the risk tolerance level ω in the CVaR risk constraint as the percentage of the initial portfolio value.

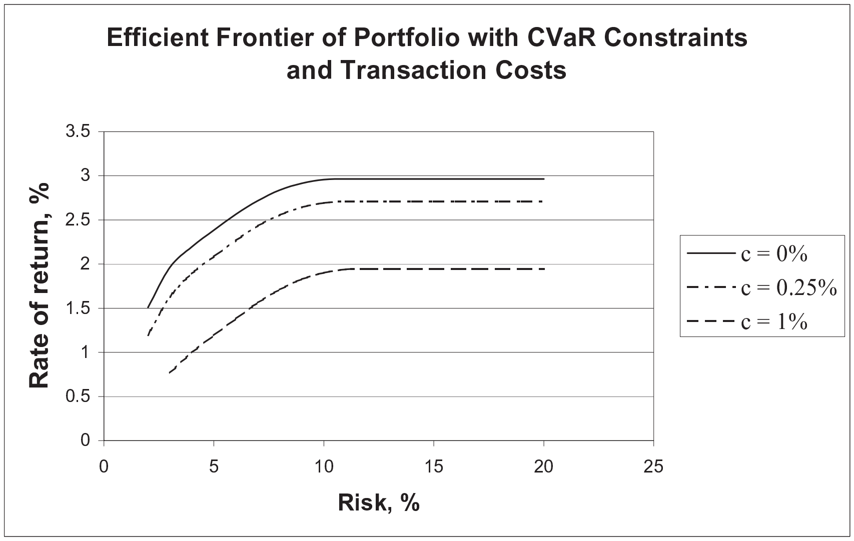

Figure 2: Efficient frontier of optimal portfolio with CVaR constraints in presence of transaction costs $c = 0\%, 0.25\%$, and 1%. Rate of Return is the expected rate of return of the optimal portfolio during a 2 week period. The Risk scale displays the risk tolerance level ω in the CVaR risk constraint ($\alpha = 0.90$) as the percentage of the initial portfolio value.

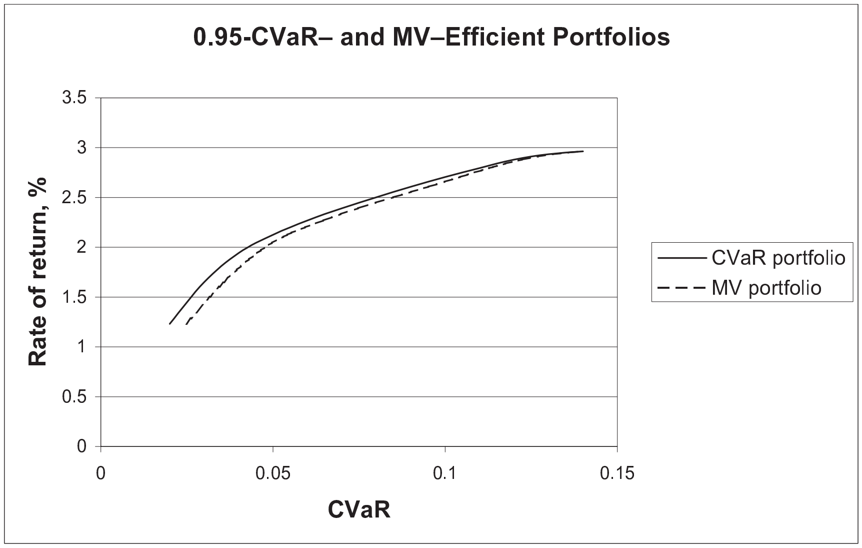

Figure 3: Efficient frontiers of CVaR– and MV–optimal portfolios. The CVaR–optimal portfolio was obtained by maximizing expected returns subject to the constraint on portfolio's CVaR with 95%–confidence level ($\alpha = 0.95$). The horizontal and vertical scales respectively display CVaR and expected rate of return of a portfolio over a two week period.

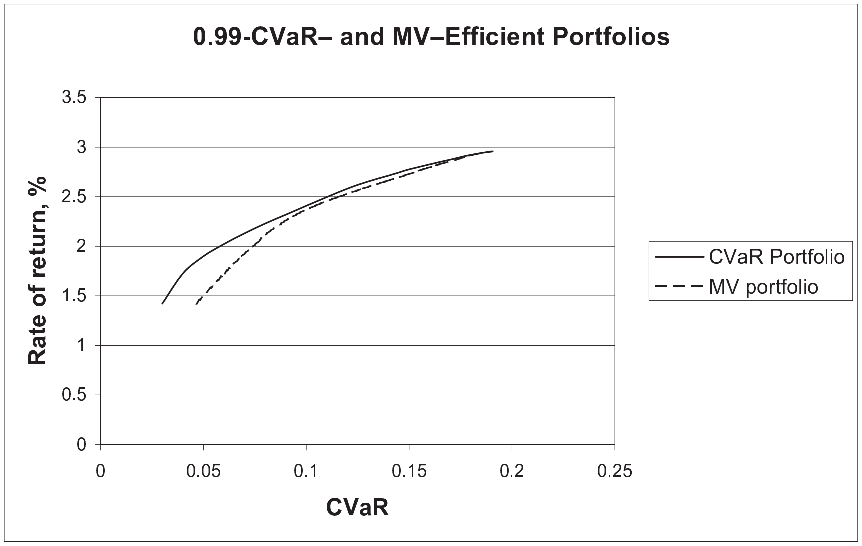

Figure 4: Efficient frontiers of CVaR– and MV–optimal portfolios. The CVaR–optimal portfolio was obtained by maximizing expected returns subject to the constraint on portfolio's CVaR with 99%–confidence level ($\alpha = 0.99$). The horizontal and vertical scales respectively display CVaR and expected rate of return of a portfolio over a two week period.

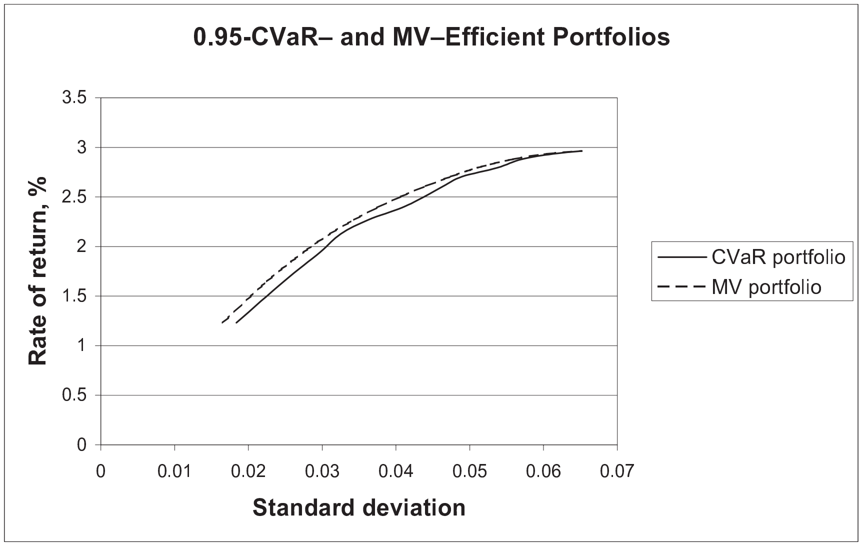

Figure 5: Efficient frontiers of CVaR– and MV–optimal portfolios. The CVaR–optimal portfolio was obtained by maximizing expected returns subject to the constraint on portfolio's CVaR with 95%–confidence level ($\alpha = 0.95$). The horizontal and vertical scales respectively display the standard deviation and expected rate of return of a portfolio over a two week period.

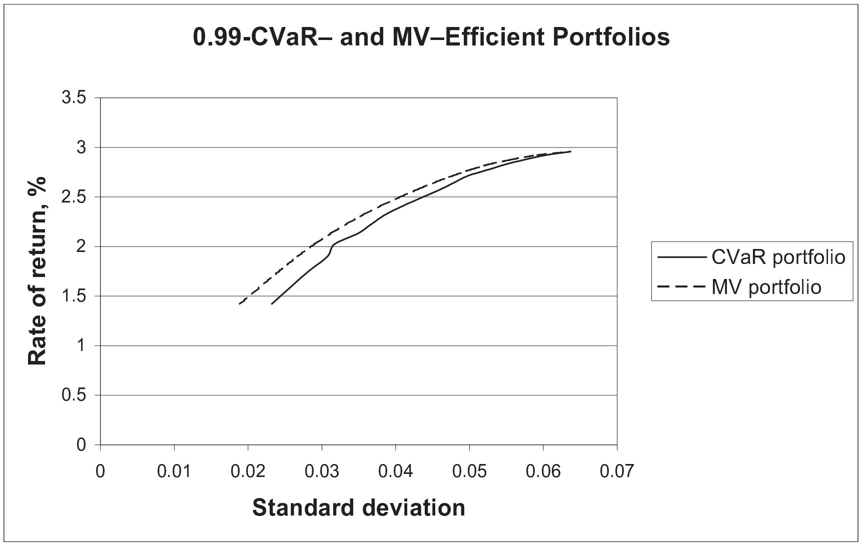

Figure 6: Efficient frontiers of CVaR– and MV–optimal portfolios. The CVaR–optimal portfolio was obtained by maximizing expected returns subject to the constraint on portfolio's CVaR with 99%–confidence level ($\alpha = 0.99$). The horizontal and vertical scales respectively display the standard deviation and expected rate of return of a portfolio over a two week period.

Table 1: Portfolio configuration: assets' weights (%) in the optimal portfolio depending on the risk level (the instruments not included in the table have zero portfolio weights).

Risk ω, %	2	3	4	5	6	7	8	9	10
Exp.Ret,%	1.508	1.962	2.195	2.384	2.565	2.719	2.838	2.915	2.956
St.Dev.	0.0220	0.0290	0.0333	0.0385	0.0439	0.0486	0.0532	0.0586	0.0637
CVaR	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Cash	7.7	0	0	0	0	0	0	0	0
AA	1.1	0	0	0	0	0	0	0	0
AIT	7.2	11.3	14.4	20.0	19.1	13.1	0	0	0
BEL	2.0	0.8	0	0	0	0	0	0	0
CGP	0.2	0	0	0	0	0	0	0	0
CSC	1.0	0	0	0	0	0	0	0	0
CSCO	1.0	0.4	9.4	13.3	14.5	20.0	20.0	20.0	20.0
ETR	5.0	0	0	0	0	0	0	0	0
GD	10.0	9.9	3.9	0	0	0	0	0	0
IBM	13.7	13.7	7.9	1.7	0	1.2	1.4	0	0
LTD	3.6	3.3	0	0	0	0	0	0	0
MOB	4.2	0	0	0	0	0	0	0	0
MSFT	0	0	0	0	0	0	0	0	13.8
SO	3.7	0	0	0	0	0	0	0	0
T	10.7	20.0	20.0	20.0	20.0	20.0	20.0	10.5	0
TAN	8.4	9.5	12.4	14.6	20.0	20.0	20.0	20.0	20.0
TXN	0.4	1.7	0.4	0	0	1.4	9.3	20.0	20.0
UCM	20.0	20.0	20.0	13.8	6.4	0	0	0	0
UIS	0.2	6.3	11.6	16.7	20.0	20.0	20.0	20.0	20.0
WMT	0	3.1	0	0	0	4.3	9.2	9.5	6.2

References

Acerbi, C., Nordio C. and C. Sirtori (2001) Expected shortfall as a tool for financial risk management. *Working paper*, (download from `http://www.gloriamundi.org`).

Acerbi, C. and D. Tasche (2001) On the coherence of expected shortfall. *Working paper*, (download from `http://www.gloriamundi.org`).

Andersen, J. V., and D. Sornette (1999) Have Your Cake and Eat It Too: Increasing Returns While Lowering Large Risks. *Working Paper.* University of Los Angeles (can be downloaded: `http://www.gloriamundi.org`).

Andersson, F., Mausser, H., Rosen, D. and S. Uryasev (1999) Credit Risk Optimization with Conditional Value–at–Risk. *Mathematical Programming*, Series B, December 2000.

Artzner, P., Delbaen F., Eber, J. M. and D. Heath (1997) Thinking Coherently. *Risk,* 10, November, 68–71.

Artzner, P., Delbaen F., Eber, J. M. and D. Heath (1999) Coherent Measures of Risk. *Mathematical Finance*, 9, 203-228.

Basak, S. and A. Shapiro (1998) Value-at-Risk Based Management: Optimal Policies and Asset Prices. *Working Paper.* Wharton School, University of Pennsylvania (can be downloaded: `http://www.gloriamundi.org`.

Dembo, R. S. and D. Rosen (1999) The Practice of Portfolio Replication: A Practical Overview of Forward and Inverse Problems. *Annals of Operations Research*, 85, 267–284.

Duarte, A. (1999) Fast Computation of Efficient Portfolios. *Journal of Risk*, 1(4), 71–94.

Duffie, D. and J. Pan (1997) An Overview of Value-at-Risk. *Journal of Derivatives*, 4, 7–49.

Embrechts, P., Kluppelberg, S., and T. Mikosch (1997) *Extremal Events in Finance and Insurance.* Springer Verlag.

Emmer S., Kluppelberg, C. and R. Korn (2000) Optimal Portfolios with Bounded Capital at Risks. *Working paper.* Munich University of Technology, June.

Gaivoronski, A. A. and G. Pflug (2000) Value at Risk in Portfolio Optimization: Properties and Computational Approach. *Working Paper 00/2, Norwegian University of Science and*

Technology.

Gourieroux, C., Laurent, J. P. and O. Scaillet (2000) Sensitivity Analysis of Values-at-Risk. *Working paper.* Universite de Louvan, January (can be downloaded: `http://www.gloriamundi.org`).

Grootweld H. and W. G. Hallerbach (2000) Upgrading VaR from Diagnostic Metric to Decision Variable: A Wise Thing to Do? *Report 2003* Erasmus Center for Financial Research, June.

Jorion, Ph. (1992) Portfolio Optimization in Practice. *Financial Analysts Journal*, 48 (January), 68–74.

Jorion, Ph. (1996) *Value at Risk : A New Benchmark for Measuring Derivatives Risk.* Irwin Professional Pub.

Jorion, Ph. (1997) *Value at Risk: The New Benchmark for Controlling Market Risk.* McGraw-Hill.

Jorion, Ph. (2000) Risk Management Lessons form LTCM. *European Financial Management*, 6, 277-300. (can be downloaded: `http://www.gsm.uci.edu/ jorion/papers/ltcm.pdf`)

Kast, R., Luciano, E., and L. Peccati (1998) *VaR and Optimization: 2nd International Workshop on Preferences and Decisions.* Trento, July 1–3 1998.

Konno, H. and A. Wijayanayake (1999) *Mean-Absolute Deviation Portfolio Optimization Model under Transaction Costs.* Departement of IE and Management, Tokyo Institute of Technology.

Konno, H. and H. Yamazaki (1991) Mean Absolute Deviation Portfolio Optimization Model and Its Application to Tokyo Stock Market. *Management Science*, 37, 519–531.

Larsen, N., Mausser H., and S. Uryasev (2002) Algorithms for Optimization of Value-at-Risk. *In:* P. Pardalos and V. K. Tsitsiringos, (Eds) *Financial Engineering, e-commerce and Supply Chain.* Kluwer Academic Publishers, 129-157 (can be downloaded: `http://www.ise.ufl.edu/uryasev/VaR-minimization.pdf`).

Litterman, R. (1997a) Hot Spots and Hedges (I). *Risk*, 10(3), 42–45.

Litterman, R. (1997b) Hot Spots and Hedges (II). *Risk*, 10(5), 38–42.

Lucas, A., and P. Klaassen (1998) Extreme Returns, Downside Risk, and Optimal Asset Allocation. *Journal of Portfolio Management*, 25(1), 71–79.

Markowitz, H.M. (1952) Portfolio Selection. *Journal of finance*, 7(1), 77–91.

Mausser, H. and D. Rosen (1991) Beyond VaR: From Measuring Risk to Managing Risk. *ALGO Research Quarterly*, 1(2), 5–20. (can be downloaded: `http://www.algorithmics.com/`

research/dec98/arq-beyondvar.pdf).

Mausser, H. and D. Rosen (1999) Efficient Risk/Return Frontiers for Credit Risk. *ALGO Research Quarterly*, 2(4), 35–48. (can be downloaded: www.algorithmics.com/research/dec99/arq2-4-frontiers.pdf).

Michaud, R. (1989) The Markowitz Optimization Enigma: Is Optimized Optimal? *Financial Analysts Journal*, 45, 31–42.

Pflug, G. (2000) Some Remarks on the Value-at-Risk and the Conditional Value-at-Risk. *In:* S. Uryasev (Ed) *Probabilistic Constrained Optimization: Methodology and Applications.* Kluwer Academic Publishers.

Prekopa, A. (1995) *Stochastic Programming.* Kluwer Academic Publishers.

Pritsker, M. (1997) Evaluating Value at Risk Methodologies. *Journal of Financial Services Research*, 12(2/3), 201–242.

Pshenichnyi, B. N. (1971) *Necessary Conditions for an Extremum.* Dekker, New York.

Puelz, A. (1999) Value-at-Risk Based Portfolio Optimization. *Working paper.* Southern Methodist University, November.

RiskMetrics^{TM} (1996) *Technical Document*, 4–th Edition, New York, NY, J. P. Morgan Inc., December.

Rockafellar, R. T. (1970) *Convex Analysis.* Princeton Mathematics, Vol. 28, Princeton Univ. Press.

Rockafellar, R. T. and S. Uryasev (2000) Optimization of Conditional Value-at-Risk. *Journal of Risk*, 2, 21–41.

Rockafellar, R. T. and S. Uryasev (2001) Conditional Value-at-Risk for General Loss Distributions. *Research Report 2001-5.* ISE Dept., University of Florida, April, 2001 (Can be downloaded: http://www.ise.ufl.edu/uryasev/cvar2.pdf)

Simons, K. (1996) Value-at-Risk New Approaches to Risk Management. *New England Economic Review*, Sept/Oct, 3–13.

Stambaugh, F. (1996) Risk and Value-at-Risk. *European Management Journal*, 14(6), 612–621.

Steinbach, M. C. (1999) Markowitz Revisited: Single–Period and Multi–period Mean–Variance Models. *Working Paper.* Konrad–Zuse–Zentrum für Informationstechnik Berlin, Preprint SC-99-

30, Aug.

Stublo Beder, T. (1995): VAR: Seductive but Dangerous. *Financial Analysts Journal*, Sep.-Oct., 12–24.

Tasche, D. (1999) Risk contributions and performance measurement. *Working paper.* Munich University of Technology, July.

Uryasev, S. (1995) Derivatives of Probability Functions and Some Applications. *Annals of Operations Research*, 56, 287–311.

Uryasev, S. (2000) Conditional Value-at-Risk: Optimization Algorithms and Applications *Financial Engineering News*, 14, February, 2000.

Young, M. R. (1998) A Minimax Portfolio Selection Rule with Linear Programming Solution. *Management Science*, 44(5), 673–683.

Ziemba, W. T. and J. M. Mulvey (Eds) (1998) *Worldwide Asset and Liability Modeling.* Cambridge Univ. Pr.

Appendix A. Proof of Theorem 3

The proof of Theorem 3 is based on the Kuhn-Tucker necessary and sufficient conditions stated in the following theorem.

Theorem A1 (Kuhn-Tacker, Theorem 2.5, (Pshenichnyi, 1971)). *Consider the problem*

$$\min \psi_0(\mathbf{x}),$$

$$\psi_i(\mathbf{x}) \leq 0 \qquad i = -m, ..., -1,$$

$$\psi_i(\mathbf{x}) = 0 \qquad i = 1, ..., n,$$

$$\mathbf{x} \in \mathbf{X}.$$

Let $\psi_i(\mathbf{x})$ be functionals on a linear space, E, such that $\psi_i(\mathbf{x})$ are convex for $i \leq 0$ and linear for $i \geq 0$ and $\mathbf{X}$ is some given convex subset of E. Then in order that $\psi_0(\mathbf{x})$ achieves its minimum point at $\mathbf{x}^* \in E$ it is necessary that there exists constants $\lambda_i, \quad i = -m, ..., n$, such that

$$\sum_{i=-m}^{n} \lambda_i \psi_i(\mathbf{x}^*) \leq \sum_{i=-m}^{n} \lambda_i \psi_i(\mathbf{x})$$

for all $\mathbf{x} \in \mathbf{X}$. Moreover, $\lambda_i \geq 0$ for each $i \leq 0$, and $\lambda_i \psi_i(\mathbf{x}_0) = 0$ for each $i \neq 0$. If $\lambda_0 \geq 0$, then the conditions are also sufficient.

Let us write down the necessary and sufficient Kuhn-Tacker conditions for problems (P1),(P2), and (P3). After some equivalent transformations these conditions can be stated as follows:

Kuhn-Tacker conditions for (P1) are, actually, a definition of the minimum point.
K-T conditions for (P1)

(KT1) $$\phi(\mathbf{x}^*) - \mu_1 R(\mathbf{x}^*) \ \leq \ \phi(\mathbf{x}) - \mu_1 R(\mathbf{x}) \,, \quad \mu_1 \geq 0 \,, \quad \mathbf{x} \in \mathbf{X} \,.$$

K-T conditions for (P2)

$$\lambda_0^2 \phi(\mathbf{x}^*) + \lambda_1^2 (\, \rho - R(\mathbf{x}^*)) \ \leq \ \lambda_0^2 \phi(\mathbf{x}) + \lambda_1^2 (\, \rho - R(\mathbf{x})) \,,$$

$$\lambda_1^2 (\, \rho - R(\mathbf{x})) = 0 \,, \ \ \lambda_0^2 > 0 \,, \ \ \lambda_1^2 \geq 0 \,, \quad \mathbf{x} \in \mathbf{X} \,.$$

$$\Downarrow$$

(KT2) $$\phi(\mathbf{x}^*) - \mu_2 R(\mathbf{x}^*) \ \le \ \phi(\mathbf{x}) - \mu_2 R(\mathbf{x})\,,$$

$$\mu_2(\,\rho - R(\mathbf{x}^*)) = 0\,, \ \ \mu_2 \ge 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

<u>K-T conditions for (P3)</u>

$$\lambda_0^3(-R(\mathbf{x}^*)) + \lambda_1^3(\phi(\mathbf{x}^*) - \omega) \ \le \ \lambda_0^3(-R(\mathbf{x})) + \lambda_1^3(\,\phi(\mathbf{x}) - \omega)\,,$$

$$\lambda_1^3(\phi(\mathbf{x}^*) - \omega) = 0\,, \ \ \lambda_0^3 > 0\,, \ \ \lambda_1^3 \ge 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

$$\Downarrow$$

(KT3) $$-R(\mathbf{x}^*) + \mu_3 \phi(\mathbf{x}^*) \ \le \ -R(\mathbf{x}) + \mu_3 \phi(\mathbf{x})\,,$$

$$\mu_3(\phi(\mathbf{x}^*) - \omega) = 0\,, \ \ \mu_3 \ge 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

Following (Steinbach, 1999), we call μ_2 in (KT2) the optimal reward multiplier, and μ_3 in (KT3) the risk multiplier. Further, using conditions (KT1) and (KT2), we show that a solution of problem (P1) is also a solution of (P2) and vice versa, a solution of problem (P2) is also a solution of (P1).

Lemma A1. *If a point $\mathbf{x}^*$ is a solution of (P1), then the point $\mathbf{x}^*$ is a solution of (P2) with parameter $\rho = R(\mathbf{x}^*)$. Also, stated in the other direction, if $\mathbf{x}^*$ is a solution of (P2) and μ_2 is the optimal reward multiplier in (KT2), then $\mathbf{x}^*$ is a solution of (P1) with $\mu_1 = \mu_2$.*

Proof of Lemma A1. Let us prove the first statement of Lemma A1. If $\mathbf{x}^*$ is a solution of (P1), then it satisfies condition (KT1). Evidently, this solution $\mathbf{x}^*$ satisfies (KT2) with $\rho = R(\mathbf{x}^*)$ and $\mu_2 = \mu_1$.

Now, let us prove the second statement of Lemma A1. Suppose that $\mathbf{x}^*$ is a solution of (P2) and (KT2) is satisfied. Then, (KT1) is satisfied with parameter $\mu_1 = \mu_2$ and $\mathbf{x}^*$ is a solution of (P1). Lemma A1 is proved. $\diamond$

Further, using conditions (KT1) and (KT3), we show that a solution of problems (P1) is also a solution of (P3) and vice versa, a solution of problems (P3) is also a solution of (P1).

Lemma A2. *If a point* $\mathbf{x}^*$ *is a solution of (P1), then the point* $\mathbf{x}^*$ *is a solution of (P3) with the parameter* $\omega = \phi(\mathbf{x})$*. Also, stated in other direction, if* $\mathbf{x}^*$ *is a solution of (P3) and* μ_3 *is a positive risk multiplier in (KT3), then* $\mathbf{x}^*$ *is a solution of (P1) with* $\mu_1 = 1/\mu_3$*.*

Proof of Lemma A2. Let us prove the first statement of Lemma A2. If $\mathbf{x}^*$ is a solution of (P1), then it satisfies the condition (KT1). If $\mu_1 > 0$, then this solution $\mathbf{x}^*$ satisfies (KT3) with $\mu_3 = 1/\mu_1$ and $\omega = \phi(\mathbf{x})$.

Now, let us prove the second statement of Lemma A2. Suppose that $\mathbf{x}^*$ is a solution of (P3) and (KT3) is satisfied with $\mu_3 > 0$. Then, (KT1) is satisfied with parameter $\mu_1 = 1/\mu_3$ and $\mathbf{x}^*$ is a solution of (P1). Lemma A2 is proved. $\diamond$

Lemma A1 implies that the efficient frontiers of problems (P1) and (P2) coincide. Similar, Lemma A2 implies that the efficient frontiers of problems (P1) and (P3) coincide. Consequently, efficient frontiers of problems (P1), (P2), and (P3) coincide. Theorem 3 is proved. $\diamond$

Appendix B. Proofs of Theorems 4 and 5.

Proof of Theorems 4. With Theorem A1, the necessary and sufficient conditions for the problem (P4') are stated as follows

$$(\mathrm{KT3'}) \qquad -R(\mathbf{x}^*) + \mu_3 F_\alpha(\mathbf{x}^*, \zeta^*) \ \le\ -R(\mathbf{x}) + \mu_3 F_\alpha(\mathbf{x}, \zeta)\,,$$

$$\mu_3(F_\alpha(\mathbf{x}^*, \zeta^*) - \omega) = 0\,, \ \ \mu_3 \ge 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

First, suppose that $\mathbf{x}^*$ is a solution of (P4) and $\zeta^* \in A_\alpha(\mathbf{x}^*)$. Let us show that $(\mathbf{x}^*, \zeta^*)$ is a solution of (P4'). Using necessary and sufficient conditions (KT3) and Theorem 1 we have

$$-R(\mathbf{x}^*) + \mu_3 F_\alpha(\mathbf{x}^*, \zeta^*) \ =\ -R(\mathbf{x}^*) + \mu_3 \phi_\alpha(\mathbf{x}^*)$$

$$\le\ -R(\mathbf{x}) + \mu_3 \phi_\alpha(\mathbf{x}) \ =\ -R(\mathbf{x}) + \mu_3 \min_\zeta F_\alpha(\mathbf{x}, \zeta)$$

$$\le\ -R(\mathbf{x}) + \mu_3 F_\alpha(\mathbf{x}, \zeta)\,,$$

and

$$\mu_3(F_\alpha(\mathbf{x}^*, \zeta^*) - \omega) = \mu_3(\phi_\alpha(\mathbf{x}^*) - \omega) = 0\,, \ \ \mu_3 \ge 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

Thus, (KT3') conditions are satisfied and $(\mathbf{x}^*, \zeta^*)$ is a solution of (P4').

Now, let us suppose that $(\mathbf{x}^*, \zeta^*)$ achieves the minimum of (P4') and $\mu_3 > 0$. For fixed $\mathbf{x}^*$, the point ζ^* minimizes the function $-R(\mathbf{x}^*) + \mu_3 F_\alpha(\mathbf{x}^*, \zeta)$, and, consequently, the function $F_\alpha(\mathbf{x}^*, \zeta)$. Then, Theorem 1 implies that $\zeta^* \in A_\alpha(\mathbf{x}^*)$. Further, since $(\mathbf{x}^*, \zeta^*)$ is a solution of (P4'), conditions (KT3') and Theorem 1 imply that

$$-R(\mathbf{x}^*) + \mu_3 \phi_\alpha(\mathbf{x}^*) \ =\ -R(\mathbf{x}^*) + \mu_3 F_\alpha(\mathbf{x}^*, \zeta^*)$$

$$\le\ -R(\mathbf{x}) + \mu_3 F_\alpha(\mathbf{x}, \zeta_\alpha(\mathbf{x})) = -R(\mathbf{x}) + \mu_3 \phi_\alpha(\mathbf{x})$$

and

$$\mu_3(\phi_\alpha(\mathbf{x}^*) - \omega) = \mu_3(F_\alpha(\mathbf{x}^*, \zeta^*) - \omega) = 0\,, \ \ \mu_3 \ge 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

We proved that conditions (KT3) are satisfied, i.e., $\mathbf{x}^*$ is a solution of (P4). Theorem 4 is proved. $\diamond$

Proof of Theorems 5. Let $\mathbf{x}^*$ is a solution of (P5), i.e.,

$$\phi_\alpha(\mathbf{x}^*) - \mu_1 R(\mathbf{x}^*) \ \le\ \phi_\alpha(\mathbf{x}) - \mu_1 R(\mathbf{x})\,, \ \ \mu_1 \ge 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

and $\zeta^* \in A_\alpha(\mathbf{x}^*)$. Using Theorem 1 we have

$$F_\alpha(\mathbf{x}^*, \zeta^*) - \mu_1 R(\mathbf{x}^*) \ = \ \phi_\alpha(\mathbf{x}^*) - \mu_1 R(\mathbf{x}^*)$$

$$\leq \ \phi_\alpha(\mathbf{x}) - \mu_1 R(\mathbf{x}) \ = \ \min_\zeta F_\alpha(\mathbf{x}, \zeta) - \mu_1 R(\mathbf{x})$$

$$\leq \ F_\alpha(\mathbf{x}, \zeta) - \mu_1 R(\mathbf{x}) \, , \quad \mathbf{x} \in \mathbf{X} \, ,$$

i.e, $(\mathbf{x}^*, \zeta^*)$ is a solution of problem (P5').

Now, let us consider that $(\mathbf{x}^*, \zeta^*)$ is a solution of problem (P5'). For the fixed point $\mathbf{x}^*$, the point ζ^* minimizes the functions $F_\alpha(\mathbf{x}^*, \zeta) - \mu_1 R(\mathbf{x}^*)$ and, consequently, the point ζ^* minimizes the function $F_\alpha(\mathbf{x}^*, \zeta)$. Then, Theorem 1 implies that $\zeta^* \in A_\alpha(\mathbf{x}^*)$. Further, since $(\mathbf{x}^*, \zeta^*)$ is a solution of (P5'), Theorem 1 implies

$$\phi_\alpha(\mathbf{x}^*) - \mu_1 R(\mathbf{x}^*) \ = \ F_\alpha(\mathbf{x}^*, \zeta^*) - \mu_1 R(\mathbf{x}^*)$$

$$\leq F_\alpha(\mathbf{x}, \zeta_\alpha(\mathbf{x})) - \mu_1 R(\mathbf{x}) \ = \ \phi_\alpha(\mathbf{x}) - \mu_1 R(\mathbf{x}) \, , \quad \mathbf{x} \in \mathbf{X} \, .$$

Theorem 5 is proved. $\diamond$