VaR vs CVaR in Risk Management and Optimization
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Joint presentation with
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Agenda

- Compare Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR)
  - definitions of VaR and CVaR
  - basic properties of VaR and CVaR
  - axiomatic definition of Risk and Deviation Measures
  - reasons affecting the choice between VaR and CVaR
  - risk management/optimization case studies conducted with Portfolio Safeguard package by AORDA.com
Risk Management

- Risk Management is a procedure for shaping a loss distribution
- Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) are popular functions for measuring risk
- The choice between VaR and CVaR is affected by:
  - differences in mathematical properties,
  - stability of statistical estimation,
  - simplicity of optimization procedures,
  - acceptance by regulators

- Conclusions from these properties are contradictory
Risk Management

- Key observations:
  - CVaR has *superior mathematical properties* versus VaR
  - *Risk management* with CVaR functions can be done very *efficiently*
  - VaR *does not control* scenarios exceeding VaR
  - CVaR *accounts* for losses exceeding VaR
  - *Deviation* and *Risk* are different risk management concepts
  - *CVaR Deviation* is a strong *competitor to the Standard Deviation*
VaR and CVaR Representation

![Graph showing VaR and CVaR Representation](image)

- **VaR (Value at Risk)**
- **CVaR (Conditional Value at Risk)**
- **Loss**
- **Frequency**
- **Mean**
- **Probability** $1 - \alpha$
- **Maximum Loss**

*Note: The graph illustrates the distribution of losses with VaR and CVaR at specific confidence levels.*
VaR, CVaR, CVaR+ and CVaR−
Value-at-Risk

$X$ a *loss* random variable

$$\text{VaR}_\alpha(X) = \min\{z \mid F_X(z) \geq \alpha\} \quad \text{for} \quad \alpha \in ]0,1[$$

- $\text{VaR}_\alpha(X)$ is non convex and discontinuous function of the confidence level $\alpha$ for discrete distributions
- $\text{VaR}_\alpha(X)$ is non-sub-additive
- Difficult to control/optimize for non-normal distributions: VaR has many extremums for discrete distributions
Conditional Value-at-Risk


For \( \alpha \in ]0, 1[ \)

\[
CVaR_\alpha (X) = \int_{-\infty}^{+\infty} z dF_X^\alpha (z)
\]

where

\[
F_X^\alpha (z) = \begin{cases} 
0 & \text{when } z < VaR_\alpha (X) \\
\frac{F_X(z) - \alpha}{1 - \alpha} & \text{when } z \geq VaR_\alpha (X)
\end{cases}
\]
**Conditional Value-at-Risk**

- **CVaR\(^{+}\) (Upper CVaR):** expected value of \(X\) strictly exceeding VaR (also called Mean Excess Loss and Expected Shortfall)

  \[
  CVaR_{\alpha}^{+}(X) = E[X \mid X > VaR_{\alpha}(X)]
  \]

- **CVaR\(^{-}\) (Lower CVaR):** expected value of \(X\) weakly exceeding VaR (also called Tail VaR)

  \[
  CVaR_{\alpha}^{-}(X) = E[X \mid X \geq VaR_{\alpha}(X)]
  \]

**Property:** \(CVaR_{\alpha}(X)\) is weighted average of \(CVaR_{\alpha}^{+}(X)\) and \(VaR_{\alpha}(X)\)

\[
CVaR_{\alpha}(X) = \begin{cases} 
\lambda_{\alpha}(X) VaR_{\alpha}(X) + (1 - \lambda_{\alpha}(X)) CVaR_{\alpha}^{+}(X) & \text{if } F_{X}(VaR_{\alpha}(X)) < 1 \\
VaR_{\alpha}(X) & \text{if } F_{X}(VaR_{\alpha}(X)) = 1 
\end{cases}
\]

\[
\lambda_{\alpha}(X) = \frac{F_{X}(VaR_{\alpha}(X)) - \alpha}{1 - \alpha}
\]

zero for continuous distributions!!!
Conditional Value-at-Risk

- Definition on previous page is a major innovation

- $CVaR^+_\alpha(X)$ and $VaR^\alpha(X)$ for general loss distributions are discontinuous functions

- CVaR is continuous with respect to $\alpha$

- CVaR is convex in $X$

- VaR, CVaR$^-$, CVaR$^+$ may be non-convex

- $VaR \leq CVaR^- \leq CVaR \leq CVaR^+$
VaR, CVaR, CVaR+ and CVaR−
CVaR: Discrete Distributions

- $\alpha$ does not “split” atoms: $\text{VaR} < \text{CVaR}^- < \text{CVaR} = \text{CVaR}^+$,
  $\lambda = (\Psi - \alpha)/(1 - \alpha) = 0$

Six scenarios, $p_1 = p_2 = \cdots = p_6 = \frac{1}{6}$, $\alpha = \frac{2}{3} = \frac{4}{6}$

$\text{CVaR} = \text{CVaR}^+ = \frac{1}{2} f_5 + \frac{1}{2} f_6$

![Diagram showing six scenarios with probabilities and losses]
\textbf{CVaR: Discrete Distributions}

- $\alpha$ “splits” the atom: $\text{VaR} < \text{CVaR}^- < \text{CVaR} < \text{CVaR}^+$, \\
  \[ \lambda = (\Psi - \alpha)/(1 - \alpha) > 0 \]

Six scenarios, $p_1 = p_2 = \cdots = p_6 = \frac{1}{6}$, $\alpha = \frac{7}{12}$

\[ \text{CVaR} = \frac{1}{5} \text{VaR} + \frac{4}{5} \text{CVaR}^+ = \frac{1}{5} f_4 + \frac{2}{5} f_5 + \frac{2}{5} f_6 \]

Probability

\[
\begin{array}{ccc}
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\hline
f_1 & f_2 & f_3 \\
\end{array}
\]

Loss

\[
\begin{array}{cccc}
\frac{1}{6} & \frac{1}{12} & \frac{1}{6} & \frac{1}{6} \\
\hline
f_4 & f_5 & f_6 \\
\end{array}
\]

CVaR

\[
\begin{array}{c}
\text{VaR} \quad \text{CVaR}^- \\
\frac{1}{2} f_5 + \frac{1}{2} f_6 = \text{CVaR}^+ \\
\end{array}
\]
CVaR: Discrete Distributions

- \( \alpha \) “splits” the last atom: \( \text{VaR} = \text{CVaR}^- = \text{CVaR}+ \), \( \text{CVaR}^+ \) is not defined, \( \lambda = (\Psi - \alpha)/(1 - \alpha) > 0 \)

Four scenarios, \( p_1 = p_2 = p_3 = p_4 = \frac{1}{4}, \ \alpha = \frac{7}{8} \)

\[ \text{CVaR} = \text{VaR} = f_4 \]
CVaR: Equivalent Definitions

- **Pflug** defines CVaR via an optimization problem, as in Rockafellar and Uryasev (2000)

\[
CVaR_\alpha(X) = \min_c \left\{ c + \frac{1}{1 - \alpha} E[X - c]^+ \right\}
\]

- **Acerbi** showed that CVaR is equivalent to Expected Shortfall defined by

\[
CVaR_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha V\alpha R_\beta(X) d\beta
\]

RISK MANAGEMENT: INSURANCE

Accident lost

Payment

Premium

Deductible

PDF

∞

∞

Payment

Payment

PDF

∞

∞

Deductible

Premium

Payment
TWO CONCEPTS OF RISK

- Risk as a possible loss

  Minimum amount of cash to be added to make a portfolio (or project) sufficiently safe

Example 1. MaxLoss

  - Three equally probable outcomes, \{ -4, 2, 5 \};
    \text{MaxLoss} = -4; \text{Risk} = 4
  - Three equally probable outcomes, \{ 0, 6, 9 \};
    \text{MaxLoss} = 0; \text{Risk} = 0

Risk as an uncertainty in outcomes

Some measure of deviation in outcomes

Example 2. Standard Deviation

  - Three equally probable outcomes, \{ 0, 6, 9 \}; \text{Standard Deviation} > 0
Risk Measures: axiomatic definition

- A functional $\mathcal{R}: \mathcal{L}^2 \to ]-\infty, \infty]$ is a **coherent risk measure in the extended sense** if:

  R1: $\mathcal{R}(C) = C$ for all constant $C$

  R2: $\mathcal{R}((1 - \lambda)X + \lambda X') \leq (1 - \lambda)\mathcal{R}(X) + \lambda\mathcal{R}(X')$ for $\lambda \in ]0, 1[$ (convexity)

  R3: $\mathcal{R}(X) \leq \mathcal{R}(X')$ when $X \leq X'$ (monotonicity)

  R4: $\mathcal{R}(X) \leq 0$ when $||X^k - X||_2 \to 0$ with $\mathcal{R}(X^k) \leq 0$ (closedness)

- A functional $\mathcal{R}: \mathcal{L}^2 \to ]-\infty, \infty]$ is a **coherent risk measure in the basic sense** if it satisfies axioms R1, R2, R3, R4 and R5:

  R5: $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for $\lambda > 0$ (positive homogeneity)
Risk Measures: axiomatic definition

- A functional $\mathcal{R}: \mathcal{L}^2 \to ]-\infty, \infty]$ is an **averse risk measure in the extended sense** if it satisfies axioms $R1$, $R2$, $R4$ and $R6$:
  
  $R6: \mathcal{R}(X) > EX$ for all nonconstant $X$ (aversity)

- A functional $\mathcal{R}: \mathcal{L}^2 \to ]-\infty, \infty]$ is an **averse risk measure in the basic sense** if it satisfies axioms $R1$, $R2$, $R4$, $R6$ and $R5$

- Aversity has the interpretation that the risk of loss in a nonconstant random variable $X$ **cannot be acceptable** unless $EX<0$

- $R2 + R5 \quad \rightarrow \quad \mathcal{R}(X + X') \leq \mathcal{R}(X) + \mathcal{R}(X')$ (subadditivity)
Risk Measures: axiomatic definition

- Examples of coherent risk measures:
  - \( R(X) = E[X] \)
  - \( R(X) = \sup X \)

- Examples of risk measures not coherent:
  - \( R(X) = E[X] + \lambda \sigma(X) \), \( \lambda > 0 \), violates R3 (monotonicity)
  - \( R(X) = \text{VaR}_\alpha(X) \) violates subadditivity

- \( R(X) = \text{CVaR}_\alpha(X) \) for \( \alpha \in ]0,1] \) is a coherent measure of risk in the basic sense and it is an averse measure of risk !!!

- Averse measure of risk might not be coherent, a coherent measure might not be averse
A functional $\mathcal{D}: \mathcal{L}^2 \to [0, \infty]$ is called a deviation measure in the extended sense if it satisfies:

D1: $\mathcal{D}(C) = 0$ for constant $C$, but $\mathcal{D}(X) > 0$ for nonconstant $X$

D2: $\mathcal{D}\left((1 - \lambda)X + \lambda X'\right) \leq (1 - \lambda)\mathcal{D}(X) + \lambda \mathcal{D}(X')$ for $\lambda \in ]0, 1[$ (convexity)

D3: $\mathcal{D}(X) \leq d$ when $||X^k - X||_2 \to 0$ with $\mathcal{D}(X^k) \leq d$ (closedness)

A functional $\mathcal{D}: \mathcal{L}^2 \to [0, \infty]$ is called a deviation measure in the basic sense if it satisfies axioms D1, D2, D3 and D4:

D4: $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ (positive homogeneity)

A deviation measure in extended or basic sense is also coherent if it additionally satisfies D5:

D5: $\mathcal{D}(X) \leq \sup X - E[X]$ (upper range boundedness)
Deviation Measures: axiomatic definition

- Examples of deviation measures in the basic sense:
  - Standard Deviation
  - Standard Semideviations
  - Mean Absolute Deviation

- \( \alpha \)-Value-at-Risk Deviation measure:
  \[
  \text{VaR}_\alpha^\Delta(X) = \text{VaR}_\alpha(X - EX)
  \]
  \( \text{VaR}_\alpha \) Dev does not satisfy convexity axiom D2 \( \rightarrow \) it is not a deviation measure

- \( \alpha \)-Conditional Value-at-Risk Deviation measure:
  \[
  \text{CVaR}_\alpha^\Delta(X) = \text{CVaR}_\alpha(X - EX)
  \]
  Coherent deviation measure in basic sense !!!
- CVaR Deviation Measure is a coherent deviation measure in the basic sense
Rockafellar et al. (2006) showed the existence of a *one-to-one correspondence* between deviation measures in the extended sense and averse risk measures in the extended sense:

\[ D(X) = R(X - EX) \]
\[ R(X) = D(X) + EX \]

\( R \) is coherent \( \iff \) \( D \) is coherent

\( R \) is positive homogeneous \( \iff \) \( D \) is positive homogeneous

## Risk vs Deviation Measures

<table>
<thead>
<tr>
<th>Deviation Measure</th>
<th>Counterpart Risk Measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(X)$</td>
<td>$EX + \sigma(X)$</td>
</tr>
<tr>
<td>$CVaR_\alpha^A(X)$</td>
<td>$CVaR_\alpha(X)$</td>
</tr>
<tr>
<td>Mixed $- CVaR_\alpha^A(X)$</td>
<td>Mixed $- CVaR_\alpha(X)$</td>
</tr>
</tbody>
</table>

\[
Mixed - CVaR_\alpha^A(X) = \sum_{k=1}^{K} \lambda_k CVaR_\alpha^A(X_k)
\]

where

\[
Mixed - CVaR_\alpha(X) = \sum_{k=1}^{K} \lambda_k CVaR_\alpha(X_k)
\]

$\lambda_k \geq 0$, $\sum_{k=1}^{K} \lambda_k = 1$ and $\alpha_k$ in $]0,1[$
Chance and VaR Constraints

- Let $f_i(x, \omega), i = 1,..,m$ be some random loss function.
- By definition: $VaR_\alpha(x) = \min \{ \varepsilon : \Pr \{ f(x, \omega) \leq \varepsilon \} \geq \alpha \}$
- Then the following holds:
  $$\Pr \{ f(x, \omega) \leq \varepsilon \} \geq \alpha \iff VaR_\alpha(X) \leq \varepsilon$$
- In general $VaR_\alpha(x)$ is nonconvex w.r.t. $x$, (e.g., discrete distributions)
- $VaR_\alpha(X) \leq \varepsilon$ and $\Pr \{ f(x, \omega) \leq \varepsilon \} \geq \alpha$ may be nonconvex constraints
VaR vs CVaR in optimization

- VaR is difficult to optimize numerically when losses are not normally distributed
- PSG package allows VaR optimization
- In optimization modeling, CVaR is superior to VaR:
  - For elliptical distribution minimizing VaR, CVaR or Variance is equivalent
  - CVaR can be expressed as a minimization formula (Rockafellar and Uryasev, 2000)
  - CVaR preserve convexity
CVaR optimization

\[ F_\alpha(x, \zeta) = \zeta + \frac{1}{1-\alpha} E\{[f(x, \xi) - \zeta]^+] \] 

**Theorem 1:**

1. \( F_\alpha(x, \zeta) \) is convex w.r.t. \( \alpha \)

2. \( \alpha \)-VaR is a minimizer of \( F \) with respect to \( \zeta \):
   \[ \text{VaR}_\alpha(f(x, \xi)) = \zeta_\alpha(f(x, \xi)) = \arg \min_{\zeta} F_\alpha(x, \zeta) \]

3. \( \alpha \)-CVaR equals minimal value (w.r.t. \( \zeta \)) of function \( F \):
   \[ \text{CVaR}_\alpha(f(x, \xi)) = \min_{\zeta} F_\alpha(x, \zeta) \]
CVaR optimization

- Preservation of convexity: if \( f(x, \xi) \) is convex in \( x \) then \( CVaR_\alpha(X) \) is convex in \( x \)
- If \( f(x, \xi) \) is convex in \( x \) then \( F_\alpha(x, \zeta) \) is convex in \( x \) and \( \zeta \)
- \( \min_{x} CVaR_\alpha(x) = \min_{(x, \xi)} F_\alpha(x, \xi) \)
- If \( f(x^*, \zeta^*) \) minimizes \( F_\alpha \) over \( X \times \mathbb{R} \) then \( CVaR_\alpha(x^*) = F_\alpha(x^*, \zeta^*) \)
- \( \min_{x \in X} g(x) \) s. t. \( CVaR_\alpha(x) \leq \omega_i, i = 1, \ldots, I \) is equivalent to
  \[
  \min_{x, \xi_1, \ldots, \xi_I \in X \times \mathbb{R} \times \ldots \times \mathbb{R}} g(x) \\
  \text{s. t. } F_{\alpha_i}(x, \zeta_i) \leq \omega_i, i = 1, \ldots, I
  \]
CVaR optimization

- In the case of discrete distributions:

\[ F_\alpha(x, \zeta) = \zeta + (1 - \alpha)^{-1} \sum_{k=1}^{N} p_k \left[ f(x, \xi^k) - \zeta \right]^+ \]

\[ z^+ = \max \{ z, 0 \} \]

- The constraint \( F_\alpha(x, \zeta) \leq \omega \) can be replaced by a system of inequalities introducing additional variables \( \eta_k \):

\[ \eta_k \geq 0, \quad f(x, y_k) - \zeta - \eta_k \leq 0, \quad k = 1, \ldots, N \]

\[ \zeta + \frac{1}{1-\alpha} \sum_{k=1}^{N} p_k \eta_k \leq \omega \]
Generalized Regression Problem

- Approximate random variable $Y$ by random variables $X_1, X_2, ..., X_n$.

$$\min \mathcal{E}(Y - [c_0 + c_1X_1 + \cdots + c_nX_n])$$

- Error measure = satisfies axioms
  
  (E1) $\mathcal{E}(0) = 0$, $\mathcal{E}(X) > 0$, for $X \neq 0$

  (E2) $\mathcal{E}(\lambda X) = \lambda \mathcal{E}(X)$ for $\lambda \geq 0$

  (E3) $\mathcal{E}(X + X') \leq \mathcal{E}(X) + \mathcal{E}(X')$

  (E4) Lower semicontinuity

Rockafellar, R. T., Uryasev, S. and M. Zabarankin:

"Risk Tuning with Generalized Linear Regression", accepted for publication in Mathematics of Operations Research, 2008
For an error measure $\varepsilon =:$

- the corresponding deviation measure $< \text{ is }$

$$D(X) = \min_{c} \varepsilon(X - C)$$

- the corresponding statistic $K \text{ is }$

$$S(X) = \arg\min_{c} \varepsilon(X - C)$$
Theorem: Separation Principle

General regression problem

\[
\min_{c_0, c_1, \ldots, c_n} \varepsilon(Y - [c_0 + c_1X_1 + \cdots + c_nX_n])
\]

is equivalent to

\[
\min_{c_1, \ldots, c_n} D(Y - [c_1X_1 + \cdots + c_nX_n])
\text{ s.t. } c_0 \in S(Y - [c_1X_1 + \cdots + c_nX_n])
\]
Percentile Regression and CVaR Deviation

Error: \( \varepsilon_{KB}^\alpha = E[X_+] + (\alpha^{-1} - 1)E[X_-] \)

Deviation: \( D(X) = CVaR_\alpha(X - EX) \)

Risk: \( R(X) = CVaR_\alpha(X) \)

Statistic: \( S(X) = -VaR_\alpha(X) \)

\[
\min_{C \in R} \left( E[X - C]^+ + (\alpha^{-1} - 1)E[X - C]^-_\right) = CVaR_\alpha(X - EX)
\]

\[
\arg \min_{C \in R} \left( E[X - C]^+ + (\alpha^{-1} - 1)E[X - C]^-_\right) = VaR_\alpha(X)
\]

Stability of Estimation

- VaR and CVaR with same confidence level measure “different parts” of the distribution
- For a specific distribution the confidence levels $\alpha_1$ and $\alpha_2$ for comparison of VaR and CVaR should be found from the equation

$$VaR_{\alpha_1} (X) = CVaR_{\alpha_2} (X)$$

- Yamai and Yoshiha (2002), for the same confidence level:
  - VaR estimators are more stable than CVaR estimators
  - The difference is more prominent for fat-tailed distributions
  - Larger sample sizes increase accuracy of CVaR estimation
  - More research needed to compare stability of estimators for the same part of the distribution.
Decomposition According to Risk Factors Contributions

- For continuous distributions the following decompositions of VaR and CVaR hold:

\[
VaR_{\alpha}(X) = \sum_{i=1}^{n} \frac{\partial VaR_{\alpha}(X)}{\partial z_i} z_i = E[X_i \mid X = VaR_{\alpha}(X)] z_i
\]

\[
CVaR_{\alpha}(X) = \sum_{i=1}^{n} \frac{\partial CVaR_{\alpha}(X)}{\partial z_i} z_i = E[X_i \mid X \geq VaR_{\alpha}(X)] z_i
\]

- When a distribution is modeled by scenarios it is easier to estimate \( E[X_i \mid X \geq VaR_{\alpha}(X)] \) than \( E[X_i \mid X = VaR_{\alpha}(X)] \)

- Estimators of \( \frac{\partial CVaR_{\alpha}(X)}{\partial z_i} \) are more stable than estimators of \( \frac{\partial VaR_{\alpha}(X)}{\partial z_i} \)
Generalized Master Fund Theorem and CAPM

Assumptions:

- Several groups of investors each with utility function $U_j(ER_j, D_j(R_j))$
- Utility functions are concave w.r.t. mean and deviation increasing w.r.t. mean decreasing w.r.t. deviation
- Investors maximize utility functions s.t. budget constraint.

Rockafellar, R.T., Uryasev, S., Zabarankin, M. “Master Funds in Portfolio Analysis with General Deviation Measures”, JBF, 2005
Efficient Set: Classical Theory

One Fund Theorem

Auxiliary Problem

\[
\min_y \sigma(Y_p)
\]
\[\text{s. t. } EY_p = \zeta \quad y^T e = 1\]
Efficient Sets: General Deviation

Master fund of negative type

\[-d(-1,-\zeta)\]

\[d(1,\zeta)\]

Master fund of positive type

\[r_0^+ \quad r_0^- \quad r_0\]

convex slopes non-smooth non-symmetric threshold interval
Generalized Master Fund Theorem and CAPM

- Equilibrium exists w.r.t. $\mathcal{D}_j$
- Each investor has *its own master fund* and invests in its own master fund and in the risk-free asset
- **Generalized CAPM** holds:

$$\bar{r}_{ij} - r_0 = \beta_{ij} (\bar{r}_{iM} - r_0)$$

$$\beta_{ij} = \frac{\text{Covar}(G_j, r_{ij})}{\mathcal{D}(-r_{jM})}$$

$\text{Covar}(G_j, r_{ij}) = E[(G_j - EG_j)(r_{ij} - Er_{ij})]$  

$\bar{r}_{ij}$ is expected return of asset $i$ in group $j$  
$r_0$ is risk-free rate  
$\bar{r}_{iM}$ is expected return of market fund for investor group $j$  
$G_j$ is the risk identifier for the market fund $j$
Generalized Master Fund Theorem and CAPM

When $\mathcal{D}(X) = \sigma(X)$ then

$$\beta_i = \frac{\text{Covar}(r_i, r_M)}{\sigma^2(r_M)}$$

When $\mathcal{D}(X) = \sigma_-(X)$ then

$$\beta_i = \frac{\text{Covar}(r_i, r_M)}{\sigma^2(r_M)}$$

When $\mathcal{D}(X) = CVaR^\Delta_{\alpha}(X)$ then

$$\beta_i = \frac{E[r_i - \bar{r}_M | - r_M \geq VaR_{\alpha}(-r_M)]}{CVaR^\Delta_{\alpha}(-r_M)}$$
All investors have the same risk preferences: standard deviation

Discounting by risk-free rate with adjustment for uncertainties (derived in PhD dissertation of Sarykalin)

\[
\pi_i = \frac{E\zeta_i}{1 + r_0 + \beta_i(r_M - r_0)}
\]

\[
\pi_i = \frac{1}{1 + r_0} \left( E\zeta_i - \beta_i(r_M - r_0) \right)
\]

\[
\beta_i = \frac{\text{cov}(r_i, r_M)}{\sigma_M^2}
\]
Generalized CAPM

- There are different groups of investors $k=1,\ldots,K$
- Risk attitude of each group of investors can be expressed through its deviation measure $D_k$

Consequently:

Each group of investors invests its own Master Fund $M$
Generalized CAPM

\[
\beta_i = \frac{\text{cov}(-r_i, Q_M^D)}{D(r_M)}
\]

\[
\pi_i = \frac{E \zeta_i}{1 + r_0 + \beta_i (r_M - r_0)}
\]

\[
\pi_i = \frac{1}{1 + r_0} \left( E \zeta_i - \beta_i (r_M - r_0) \right)
\]

\[D = \text{deviation measure}\]

\[Q_M^P = \text{risk identifier for } D \text{ corresponding to } M\]
Investors Buying Out-of-the-money S&P500 Put Options

- Group of investors buys S&P500 options
- Risk preferences are described by mixed CVaR deviation

\[ D(X) = \sum_{i=1}^{n} \lambda_i CVaR_{\alpha_i}^\Delta(X) \]

- Assume that S&P500 is their Master Fund
- Out-of-the-money put option is an investments in low tail of price distribution. CVaR deviations can capture the tail.
Mixed CVaR Deviation Risk Envelope

\[ D(X) = \sup_{Q \in \mathcal{Q}} XQ - EX \]

\[ D(X) = \text{CVaR}^\Delta_{\alpha}(X) \]

\[ Q_X(\omega) = 1\{X(\omega) \geq \text{VaR}_\alpha(X)\} \]

\[ D(X) = \sum_{i=1}^{n} \lambda_i \text{CVaR}^\Delta_{\alpha_i}(X) \]

\[ Q_X(\omega) = \sum_{i=1}^{n} \lambda_i 1\{X(\omega) \geq \text{VaR}_{\alpha_i}(X)\} \]
Data

- Put options prices on Oct, 20 2009 maturing on Nov, 20 2009
- 2490 monthly returns (overlapping daily):
  every trading day \( t: \quad r_t = \ln S_t - \ln S_{t-21} \)
- Mean return adjusted to 6.4% annually.
- 2490 scenarios of S&P500 option payoffs on Nov, 20 2009
CVaR Deviation

- Risk preferences of Put Option buyers:

\[ D(X) = \sum_{j=1}^{5} \lambda_j CVaR_{\alpha_j}^\Delta (X) \]

\[ \alpha_1 = 99\% \quad \alpha_2 = 95\% \quad \alpha_3 = 85\% \quad \alpha_4 = 75\% \quad \alpha_5 = 50\% \]

- We want to estimate values for coefficients \( \lambda_j \).
Prices $\iff$ Implied Volatilities

- Option prices with different strike prices vary very significantly

- Black-Scholes formula: prices $\iff$ implied volatilities
Graphs: \[ D(X) = CVaR_{50\%}^\Delta (X) \]
Graphs: \[ D(X) = CVaR^\Delta_{75\%}(X) \]
Graphs: \[ D(X) = CVaR_{85\%}^\Delta(X) \]
Graphs: \[ D(X) = CVaR^\Delta_{95\%}(X) \]
Graphs: \[ D(X) = \sum_{j=1}^{5} \lambda_j CVaR_{\alpha_j}^\Delta (X) \]
**VaR: Pros**

1. VaR is a relatively **simple** risk management concept and has a **clear** interpretation
2. Specifying VaR for all confidence levels **completely defines** the distribution (superior to $\sigma$)
3. VaR focuses on the **part of the distribution** specified by the confidence level
4. Estimation procedures are **stable**
5. VaR can be estimated with **parametric models**
VaR: Cons

1. VaR does not account for properties of the distribution **beyond** the confidence level
2. Risk control using VaR may lead to **undesirable results** for skewed distributions
3. VaR is a *non-convex* and *discontinuous* function for discrete distributions
CVaR: Pros

1. VaR has a clear engineering interpretation
2. Specifying CVaR for all confidence levels completely defines the distribution (superior to $\sigma$)
3. CVaR is a coherent risk measure
4. CVaR is continuous w.r.t. $\alpha$
5. $CVaR_\alpha (w_1 X_1 + ... + w_n X_n)$ is a convex function w.r.t. $(w_1, ..., w_n)$
6. CVaR optimization can be reduced to convex programming and in some cases to linear programming
**CVaR: Cons**

1. CVaR is more *sensitive* than VaR to estimation errors

2. CVaR accuracy is *heavily affected* by accuracy of tail modeling
VaR or CVaR in financial applications?

- VaR is not restrictive as CVaR with the same confidence level → a trader may prefer VaR
- A company owner may prefer CVaR; a board of director may prefer reporting VaR to shareholders
- VaR may be better for portfolio optimization when good models for the tails are not available
- CVaR should be used when good models for the tails are available
- CVaR has superior mathematical properties
- CVaR can be easily handled in optimization and statistics
- Avoid comparison of VaR and CVaR for the same level $\alpha$
Adequate accounting for risks, classical and downside risk measures:

- Value-at-Risk (VaR)
- Conditional Value-at-Risk (CVaR)
- Drawdown
- Maximum Loss
- Lower Partial Moment
- Probability (e.g. default probability)
- Variance
- St.Dev.

and many others

Various data inputs for risk functions: scenarios and covariances

- Historical observations of returns/prices
- Monte-Carlo based simulations, e.g. from RiskMetrics or S&P CDO evaluator
- Covariance matrices, e.g. from Barra factor models
Powerful and robust optimization tools:

- four environments:
  - Shell (Windows-dialog)
  - MATLAB
  - C++
  - Run-file

- simultaneous constraints on many functions at various times (e.g., multiple constraints on standard deviations obtained by resampling in combination with drawdown constraints)
PSG: Risk Functions

- Matrix of scenarios
  - or
  - Covariance matrix

  Risk function
  (e.g., St.Dev, VaR, CVaR, Mean, ...)

- Matrix with one row

  Linear function
PSG: Example

INPUT DATA: Matrix of Scenarios

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>Scenario Benchmark</th>
<th>Scenario Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{11}$</td>
<td>$\theta_{12}$</td>
<td>$\theta_{01}$</td>
<td>$p_1$</td>
</tr>
<tr>
<td>$\theta_{21}$</td>
<td>$\theta_{22}$</td>
<td>$\theta_{02}$</td>
<td>$p_2$</td>
</tr>
</tbody>
</table>

Loss Vector

<table>
<thead>
<tr>
<th>Loss Function</th>
<th>Scenario Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{01} - (\theta_{11} \times X + \theta_{21} \times Y)$</td>
<td>$p_1$</td>
</tr>
<tr>
<td>$\theta_{02} - (\theta_{12} \times X + \theta_{22} \times Y)$</td>
<td>$p_2$</td>
</tr>
</tbody>
</table>

Function Evaluation

<table>
<thead>
<tr>
<th>Function Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean $p_1 \times [\theta_{01} - (\theta_{11} \times X + \theta_{21} \times Y)] + p_2 \times [\theta_{02} - (\theta_{12} \times X + \theta_{22} \times Y)]$</td>
</tr>
<tr>
<td>Max Loss $\max[\theta_{01} - (\theta_{11} \times X + \theta_{21} \times Y), [\theta_{02} - (\theta_{12} \times X + \theta_{22} \times Y)]$</td>
</tr>
</tbody>
</table>
PSG: Operations with Functions
Case Study: Risk Control using VaR

- Risk control using VaR may lead to paradoxical results for skewed distributions
- Undesirable feature of VaR optimization: VaR minimization may increase the extreme losses

- Case Study main result: minimization of 99%-VaR Deviation leads to 13% increase in 99%-CVaR compared to 99%-CVaR of optimal 99%-CVaR Deviation portfolio

- Consistency with theoretical results: CVaR is coherent, VaR is not coherent

Case Study:  Risk Control using VaR

\[
\begin{align*}
\text{P.1} & & \text{Pb.2} \\
\min CVaR^\Delta_{\alpha} (x) & \quad \text{s.t.} & \sum_{i=1}^{n} r_i x_i & \geq \bar{r} \\
\sum_{i=1}^{n} x_i & = 1 & \sum_{i=1}^{n} x_i & = 1 \\
\min VaR^\Delta_{\alpha} (x) & \quad \text{s.t.} & \sum_{i=1}^{n} r_i x_i & \geq \bar{r} \\
\sum_{i=1}^{n} x_i & = 1 & \sum_{i=1}^{n} x_i & = 1
\end{align*}
\]

<table>
<thead>
<tr>
<th>Example</th>
<th>(\text{min CVaR}^\Delta_{0.99})</th>
<th>(\text{min VaR}^\Delta_{0.99})</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>CVaR(0.99)</td>
<td>0.0073</td>
<td>0.0083</td>
<td>1.130</td>
</tr>
<tr>
<td>CVaR(\Delta)(0.99)</td>
<td>0.0363</td>
<td>0.0373</td>
<td>1.026</td>
</tr>
<tr>
<td>VaR(0.99)</td>
<td>0.0023</td>
<td>0.0005</td>
<td>0.231</td>
</tr>
<tr>
<td>VaR(\Delta)(0.99)</td>
<td>0.0313</td>
<td>0.0295</td>
<td>0.944</td>
</tr>
<tr>
<td>Max Loss = CVaR(1)</td>
<td>0.0133</td>
<td>0.0148</td>
<td>1.116</td>
</tr>
<tr>
<td>Max Loss Deviation = CVaR(\Delta)(1)</td>
<td>0.0423</td>
<td>0.0438</td>
<td>1.036</td>
</tr>
</tbody>
</table>

Columns 2, 3 report value of risk functions at optimal point of Problem 1 and 2; Column “Ratio” reports ratio of Column 3 to Column 2.
Case Study: Linear Regression-Hedging

- Investigate performance of optimal hedging strategies based on different deviation measures.
- Determining optimal hedging strategy is a linear regression problem:
  \[ \hat{\theta} = x_1 \theta_1 + \ldots + x_I \theta_I \]
- Benchmark portfolio value is the response variable, replicating financial instruments values are predictors, portfolio weights are coefficients of the predictors to be determined.
- Coefficients \( x_1, \ldots, x_I \) chosen to minimize a replication error function depending upon the residual \( \theta_0 - \hat{\theta} \).
Case Study: Linear Regression-Hedging

Loss Function = \( L(x, \theta) = L(x_1, ..., x_i, \theta_0, ..., \theta_l) = \theta_0 - \sum_{i=1}^{l} \theta_i x_i \)

Two Tail \( \alpha \%-VaR \) Deviation = \( TwoTailVaR_{\alpha}^A(L(x, \theta)) = VaR_{\alpha}(L(x, \theta)) + VaR_{\alpha}(-L(x, \theta)) \)

\[
\begin{align*}
(1) & \quad \min_{x} \quad CVaR_{0.9}^A(L(x, \theta)) \\
(2) & \quad \min_{x} \quad MAD(L(X, \theta)) \\
(3) & \quad \min_{x} \quad \sigma(L(x, \theta)) \\
(4) & \quad \min_{x} \quad TwoTailVaR_{0.75}^A(L(X, \theta)) \\
(5) & \quad \min_{x} \quad TwoTailVaR_{0.9}^A(L(X, \theta))
\end{align*}
\]

- Out-of-sample performance of hedging strategies significantly depends on the skewness of the distribution
- Two-Tailed 90%-VaR has the best out-of-sample performance
- Standard deviation has the worst out-of-sample performance
## Case Study: Linear Regression-Hedging

<table>
<thead>
<tr>
<th>Optimal Points</th>
<th>CVaR_{0.9}^{\Delta}</th>
<th>MAD</th>
<th>σ</th>
<th>TwoTailVaR_{0.75}^{\Delta}</th>
<th>TwoTailVaR_{0.9}^{\Delta}</th>
</tr>
</thead>
<tbody>
<tr>
<td>CVaR_{0.9}^{\Delta}</td>
<td>0.690</td>
<td>0.815</td>
<td>1.961</td>
<td>0.275</td>
<td>1.122</td>
</tr>
<tr>
<td>MAD</td>
<td>1.137</td>
<td>0.714</td>
<td>1.641</td>
<td>0.379</td>
<td>1.880</td>
</tr>
<tr>
<td>σ</td>
<td>1.405</td>
<td>0.644</td>
<td>1.110</td>
<td>0.979</td>
<td>1.829</td>
</tr>
<tr>
<td>TwoTailVaR_{0.75}^{\Delta}</td>
<td>1.316</td>
<td>0.956</td>
<td>1.955</td>
<td>0.999</td>
<td>1.557</td>
</tr>
<tr>
<td>TwoTailVaR_{0.9}^{\Delta}</td>
<td>0.922</td>
<td>0.743</td>
<td>1.821</td>
<td>0.643</td>
<td>1.256</td>
</tr>
</tbody>
</table>

Out-of-sample performance of different deviation measures evaluated at optimal points of the 5 different hedging strategies (e.g., the first row is for $CVaR_{0.9}^{\Delta}$ hedging strategy).

<table>
<thead>
<tr>
<th>Optimal Points</th>
<th>Max Loss</th>
<th>CVaR_{0.9}</th>
<th>VaR_{0.9}</th>
</tr>
</thead>
<tbody>
<tr>
<td>CVaR_{0.9}^{\Delta}</td>
<td>-18.01</td>
<td>-18.05</td>
<td>-18.08</td>
</tr>
<tr>
<td>MAD</td>
<td>-16.49</td>
<td>-17.44</td>
<td>-17.88</td>
</tr>
<tr>
<td>σ</td>
<td>-13.31</td>
<td>-15.29</td>
<td>-15.60</td>
</tr>
<tr>
<td>TwoTailVaR_{0.75}^{\Delta}</td>
<td>-15.31</td>
<td>-16.19</td>
<td>-16.71</td>
</tr>
<tr>
<td>TwoTailVaR_{0.9}^{\Delta}</td>
<td>-18.02</td>
<td>-18.51</td>
<td>-18.66</td>
</tr>
</tbody>
</table>

Each row reports value of different risk functions evaluated at optimal points of the 5 different hedging strategies (e.g., the first raw is for $CVaR_{0.9}^{\Delta}$ hedging strategy)
Example: Chance and VaR constraints equivalence

We illustrate numerically the equivalence:

\[ \text{Prob}\{L(x, \theta) > \epsilon\} \leq 1 - \alpha \quad \Leftrightarrow \quad \text{VaR}_\alpha(L(x, \theta)) \leq \epsilon, \]

Problem 1:

\[
\begin{align*}
\text{max} & \quad E[-L(x, \theta)] \\
\text{s.t.} & \quad \text{Prob}\{L(x, \theta) > \epsilon\} \leq 1 - \alpha = 0.05 \\
& \quad v_i \leq x_i \leq u_i, \quad i = 1, \ldots, l, \\
& \quad \sum_{i=1}^l x_i = 1.
\end{align*}
\]

Problem 2:

\[
\begin{align*}
\text{max} & \quad E[-L(x, \theta)] \\
\text{s.t.} & \quad \text{VaR}_\alpha(L(x, \theta)) \leq \epsilon, \\
& \quad v_i \leq x_i \leq u_i, \quad i = 1, \ldots, l \\
& \quad \sum_{i=1}^l x_i = 1.
\end{align*}
\]

<table>
<thead>
<tr>
<th>Optimal Weights</th>
<th>Prob \leq 0.05</th>
<th>VaR \leq \epsilon</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
<td>.051</td>
<td>.051</td>
</tr>
<tr>
<td>(x_2)</td>
<td>.055</td>
<td>.055</td>
</tr>
<tr>
<td>(x_3)</td>
<td>.071</td>
<td>.071</td>
</tr>
<tr>
<td>(x_4)</td>
<td>.053</td>
<td>.053</td>
</tr>
<tr>
<td>(x_5)</td>
<td>.079</td>
<td>.079</td>
</tr>
<tr>
<td>(x_6)</td>
<td>.289</td>
<td>.289</td>
</tr>
<tr>
<td>(x_7)</td>
<td>.020</td>
<td>.020</td>
</tr>
<tr>
<td>(x_8)</td>
<td>.300</td>
<td>.300</td>
</tr>
<tr>
<td>(x_9)</td>
<td>.063</td>
<td>.063</td>
</tr>
<tr>
<td>(x_{10})</td>
<td>.020</td>
<td>.020</td>
</tr>
</tbody>
</table>

At optimality the two problems selected the same portfolio with the same objective function value.
Case Study: Portfolio Rebalancing Strategies, Risks and Deviations

- We consider a portfolio rebalancing problem:

\[
\begin{align*}
\min & \quad R(x, \theta) - k \ast E[-L(x, \theta)] \\
\text{s.t.} & \quad \sum_{i=1}^{l} x_i = 1, \\
& \quad v_i \leq (x_i) \leq u_i, \quad i=1,..,l
\end{align*}
\]

- We used as risk functions VaR, CVaR, VaR Deviation, CVaR Deviation, Standard Deviation

- We evaluated Sharpe ratio and mean value of each sequence of portfolios

- We found a good performance of VaR and VaR Deviation minimization

- Standard Deviation minimization leads to inferior results
Results depend on the scenario dataset and on $k$

In the presence of mean reversion the tails of historical distribution are not good predictors of the tail in the future

VaR disregards the unstable part of the distribution thus may lead to good out-of-sample performance

<table>
<thead>
<tr>
<th>$k$</th>
<th>VaR</th>
<th>CVaR</th>
<th>VaR Deviation</th>
<th>CVaR Deviation</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>−1</td>
<td>1.2710</td>
<td>1.2609</td>
<td>1.2588</td>
<td>1.2693</td>
<td>1.2380</td>
</tr>
<tr>
<td>−3</td>
<td>1.2711</td>
<td>1.2667</td>
<td>1.2762</td>
<td>1.2652</td>
<td>1.2672</td>
</tr>
<tr>
<td>−5</td>
<td>1.2712</td>
<td>1.2666</td>
<td>1.2721</td>
<td>1.2743</td>
<td>1.2628</td>
</tr>
</tbody>
</table>

Sharpe ratio for the rebalancing strategy when different risk functions are used in the objective for different values of the parameter $k$
Conclusions: key observations

- CVaR has superior mathematical properties: CVaR is coherent, CVaR of a portfolio is a continuous and convex function with respect to optimization variables.

- CVaR can be optimized and constrained with convex and linear programming methods; VaR is relatively difficult to optimize.

- VaR does not control scenarios exceeding VaR.

- VaR estimates are statistically more stable than CVaR estimates.

- VaR may lead to bearing high uncontrollable risk.

- CVaR is more sensitive than VaR to estimation errors.

- CVaR accuracy is heavily affected by accuracy of tail modeling.
Conclusions: key observations

- There is a **one-to-one correspondence** between Risk Measures and Deviation Measures

- **CVaR Deviation** is a strong competitor of Standard Deviation

- **Mixed CVaR Deviation** should be used when tails are not modeled correctly. Mixed CVaR Deviation gives different weight to different parts of the distribution

- Master Fund Theorem and CAPM **can be generalized** with the for different deviation measure.
Conclusions: Case Studies

- **Case Study 1**: risk control using VaR may lead to paradoxical results for skewed distribution. Minimization of VaR may lead to a stretch of the tail of the distribution exceeding VaR.

- **Case Study 2**: determining optimal hedging strategy is a linear regression problem. Out-of-sample performance based on different Deviation Measures depends on the skewness of the distribution, we found standard deviation have the worst performance.

- **Case Study 3**: chance constraints and percentiles of a distribution are closely related, VaR and Chance constraints are equivalent.

- **Case Study 4**: the choice of the risk function to minimize in a portfolio rebalancing strategy depends on the scenario dataset. In the presence of mean reversion VaR neglecting tails may lead to good out-of-sample performance.