Risk measures for CDOs

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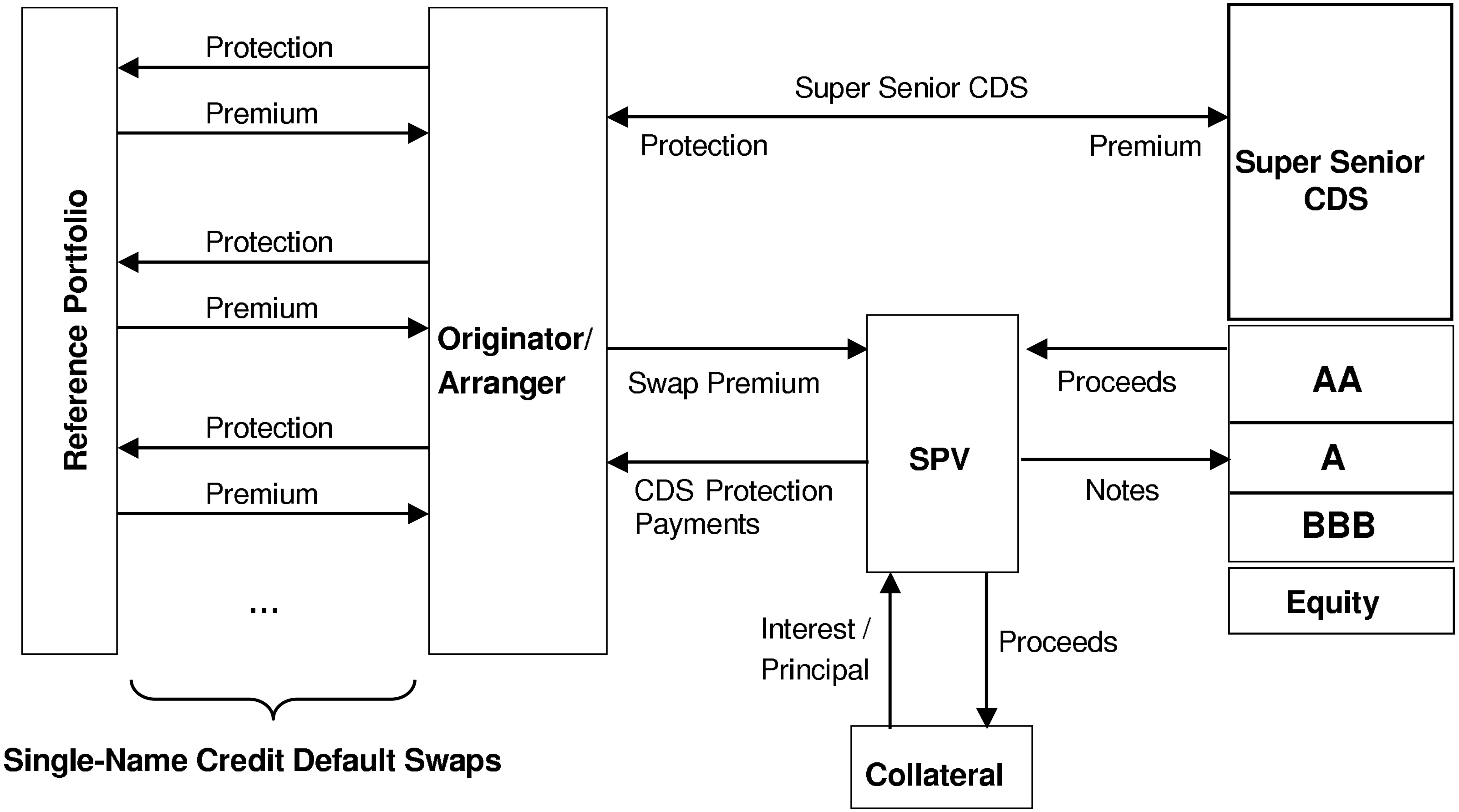
Agenda

- Basic structure of Collaterized Debt Obligations
- Single Transaction Risk Measures
- Portfolio Risk Measures

Synthetic transaction based on credit default swaps (CSO)



Reference Portfolio

Protection

Premium

Protection

Premium

Protection

Premium

…

Single-Name Credit Default Swaps

Originator/ Arranger

Super Senior CDS

Protection

Premium

Super Senior CDS

Swap Premium

CDS Protection Payments

SPV

Proceeds

Notes

AA

A

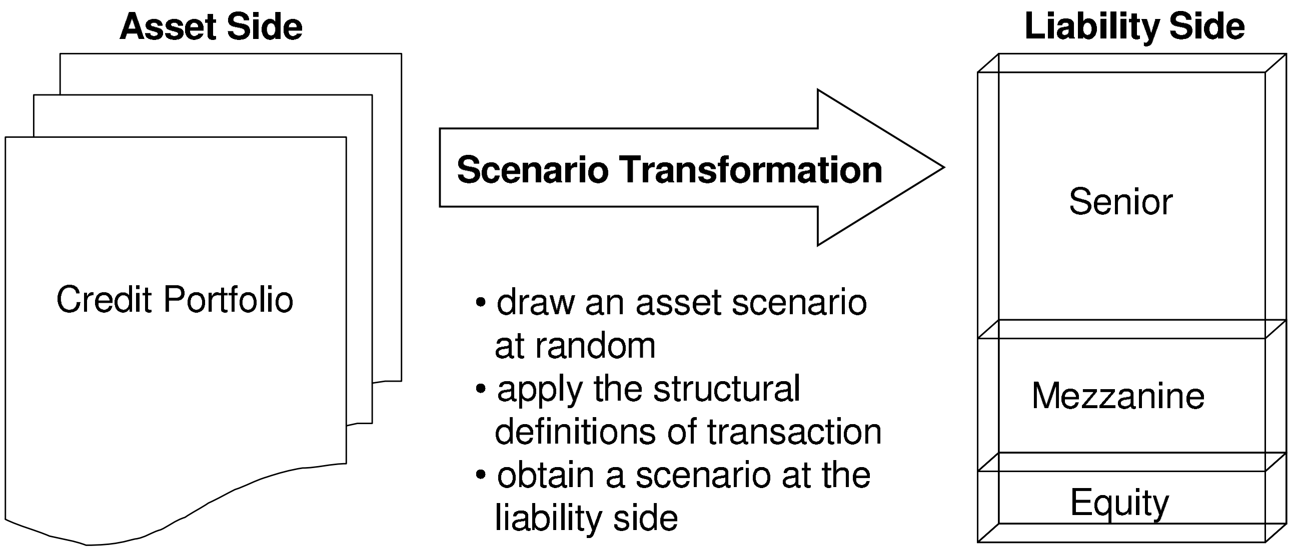
BBB

Equity

Interest / Principal

Proceeds

Collateral



Asset Side

Credit Portfolio

Scenario Transformation

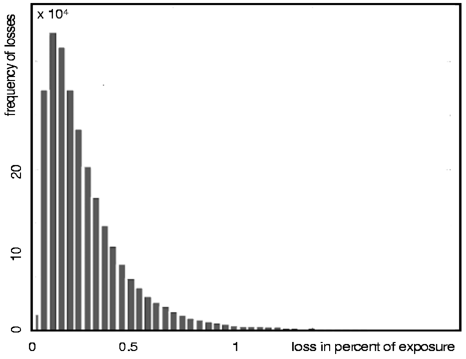
- draw an asset scenario at random
- apply the structural definitions of transaction
- obtain a scenario at the liability side

Liability Side

Senior

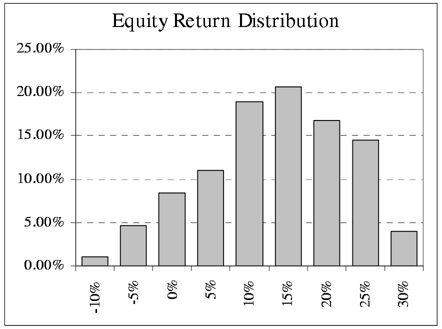
Mezzanine

Equity

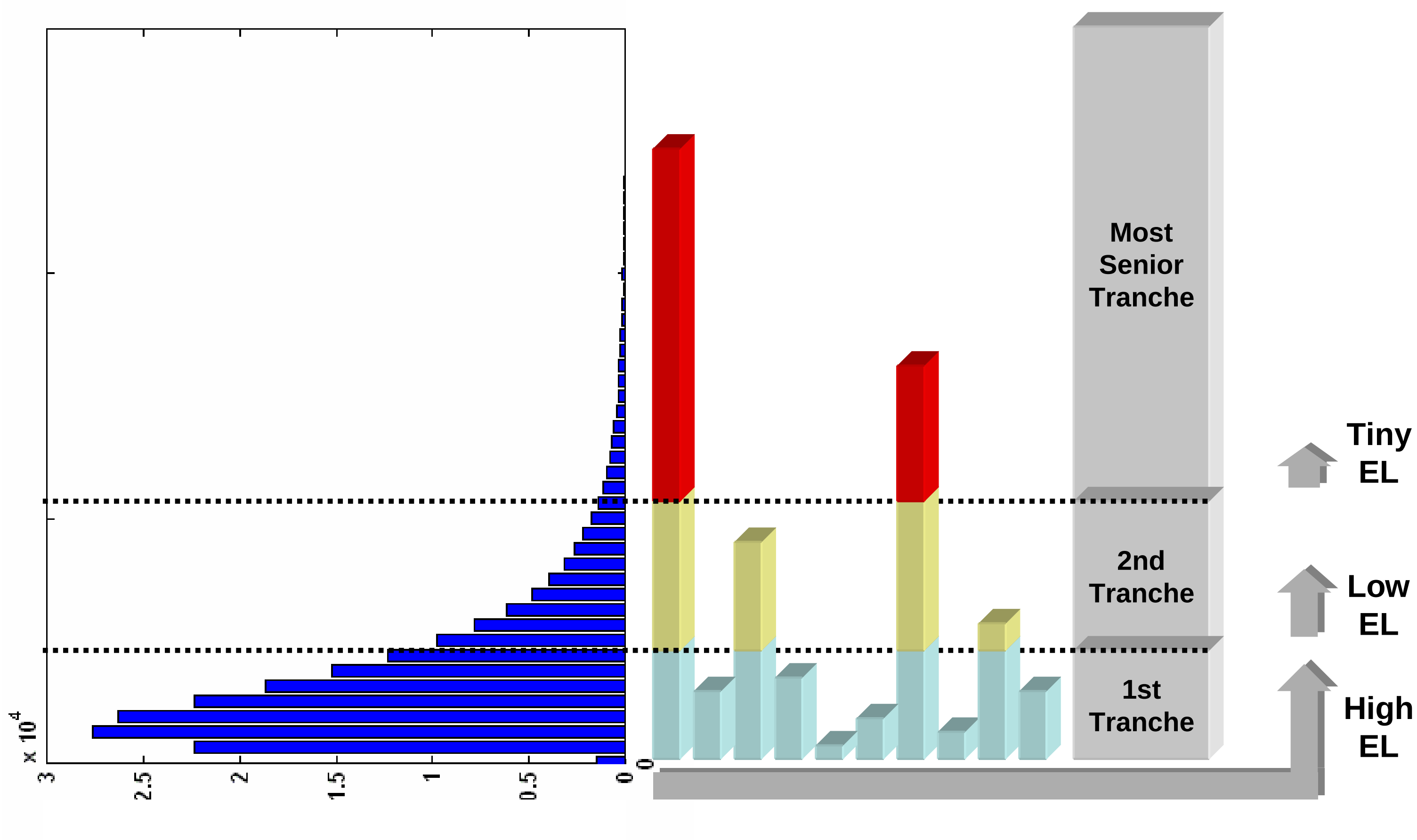


Example:

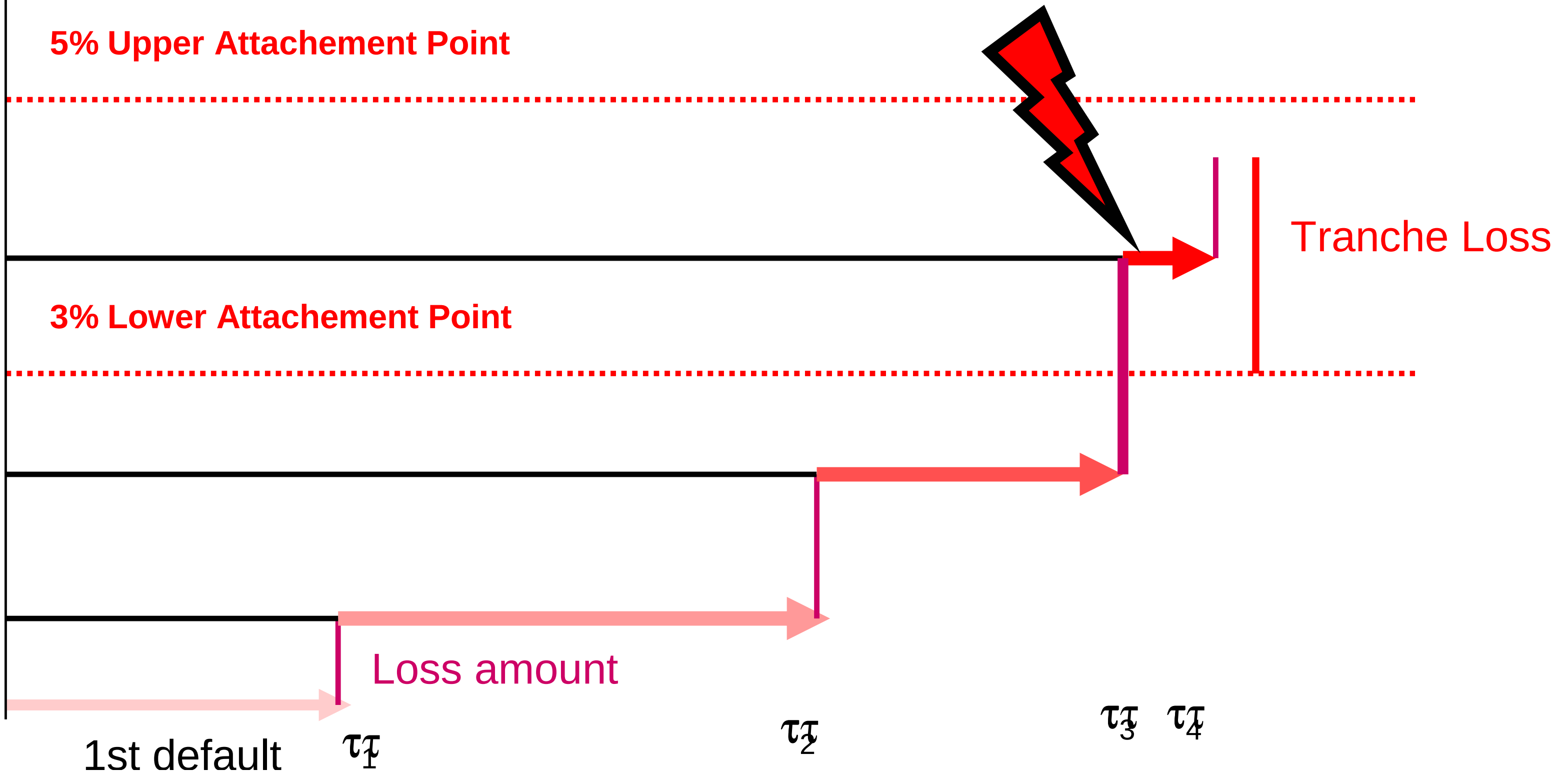
obtain the equity return distribution of a CDO by transforming 100,000 asset scenarios and then aggregating accordingly



Static Model



Dynamic?



5% Upper Attachement Point

3% Lower Attachement Point

Tranche Loss

Loss amount

1st default

τ_1 τ_2 τ_3 τ_4

Default time model

Starting point: default times vector $\boldsymbol{\tau} = (\tau_1, ..., \tau_m)$ with $p_i(t) = P[\tau_i \leq t]$ strictly monotone and continuous. Default quote path $(L_t)_{0 \leq t \leq T}$ with

$$L_t = \sum_{k=1}^{m} \mathbf{1}_{\{\tau_k \leq t\}} \tag{1}$$

The entire multivariate distribution with $C = C_{\tau_1,..\tau_m}$ and $F_i(t) = F_{\tau_i}(t) = p_i(t)$, the single default probability curve is then retrieved by

$$F(t_1, .., t_m) = C(F_1(t_1), .., F_m(t_m)) \tag{2}$$

Valuation

A CDO-tranch, with lower attachment point K and upper attachment point U, has the loss function, assuming non-random exposure l_i and recovery rate R_i for obligor i

$$L_t^{CDO}(\omega) \;=\; \min(U, \max(K, \sum_{i:\tau_i(\omega)\leq t} l_i(1-R_i))) - K$$

or equivalently as a difference of two stop-loss strategies

$$L_t^{CDO} = \min(U, \sum_{i:\tau_i\leq t} l_i(1-R_i)) - \min(K, \sum_{i:\tau_i\leq t} l_i(1-R_i))).$$

To value the CDO at a given point in time it is sufficent to calculate the expected value of this random variable. (=market price, if expectation or default probabilities are taken under the risk neutral measure)

Normal Copula Approach

Factor model formulation: Y d-dimensional standard normal, ϵ_i standard normal, factor loadings $0 \leq \beta_i, \beta_i^T \beta_i \leq 1$

$$\begin{aligned} A_i &= \beta_i^T \cdot Y + \sqrt{1 - \beta_i^T \beta_i} \epsilon_i \\ \tau_i &= p_i^{-1}(N(A_i)) \end{aligned}$$

Y might be called state-of-economy and the correlation (and hence dependency) structure is fully given by the βs.

Conditional Independence Concept

Let us denote for $I \in \mathcal{P}(\{1,..,m\})$ and $K \leq 0$, the expected value of $\sum_{i \in I} l_i \mathbf{1}_{\{A_i \leq N^{-1}(p_i(t))\}}$ given Y by $S(t,K,I,Y)$. Then we have the following recursion

$$\begin{aligned} & S(t,K,I,Y) \\ = \; & p_k(t,Y) S(t, K - l_k, I \setminus \{k\}, Y) + (1 - p_k(t,Y)) S(t,K,I \setminus \{k\}, Y) \end{aligned} \tag{3}$$

Here $p_k(t,Y)$ denotes the conditional probability of counterparty k defaulting before time t given Y, namely

$$p_k(t,Y) \; = \; N\left(\frac{N^{-1}(p_i(t)) - \beta_i^T Y}{\sqrt{1 - \beta_i^T \beta_i}}\right)$$

Recursion

Formula (3) provides

- A recursion of calculating the conditional expected value, and then to integrate over the normal variable Y
- A linear dependency of the conditional expected value from the conditional default probabilities of counterparties. Very useful for the calculation of sensitivities with respect to changes in default probabilities.

Exposure?

Important question for risk management application:

What is an ”exposure equivalent”? E.g. if bank lends also 10 Million to DCX, and there is 20 Million DCX in an underlying pool of an CDO, how can these two exposures compare?

The answer might depend on one of the following measures!

Stand-alone Risk Measures

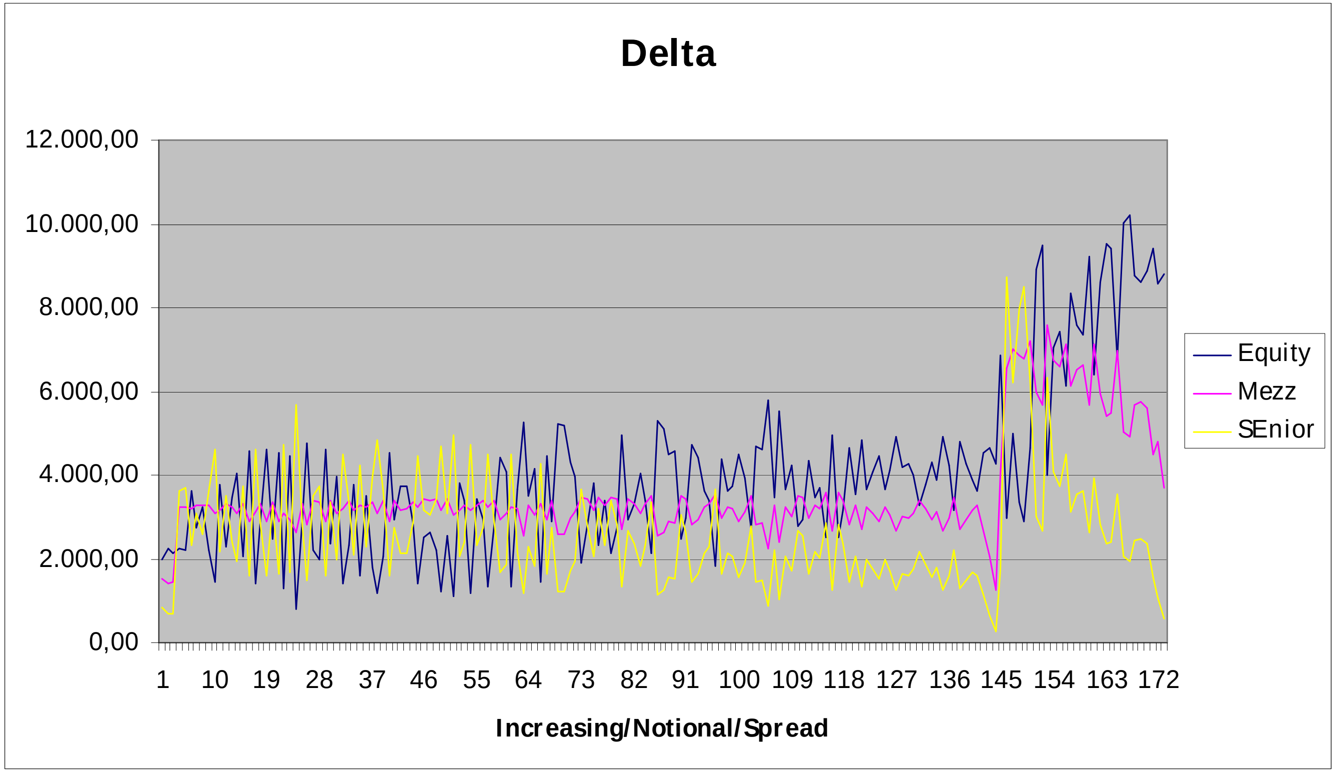
For each single name in the underlying portfolio, risk manager should calculate

- Delta: $\frac{\partial}{\partial p_i(\cdot)} E[L_t^{CDO}]$.
  Parallel shift of the default probability curve by 1 bp. Calculate change in value. Convexity only comes from integrating over systematic factor Y in (3).
- Gap-Risk: Increase credit curve (i.e. $p_i(t)$) by fixed amount like 0.3%, 1% or 4% or by a percentage shift.
- Jump-to-default: Assume that counterparty defaults and recalculate the value of the tranches again.

All three calculations can be based on the recursion above.

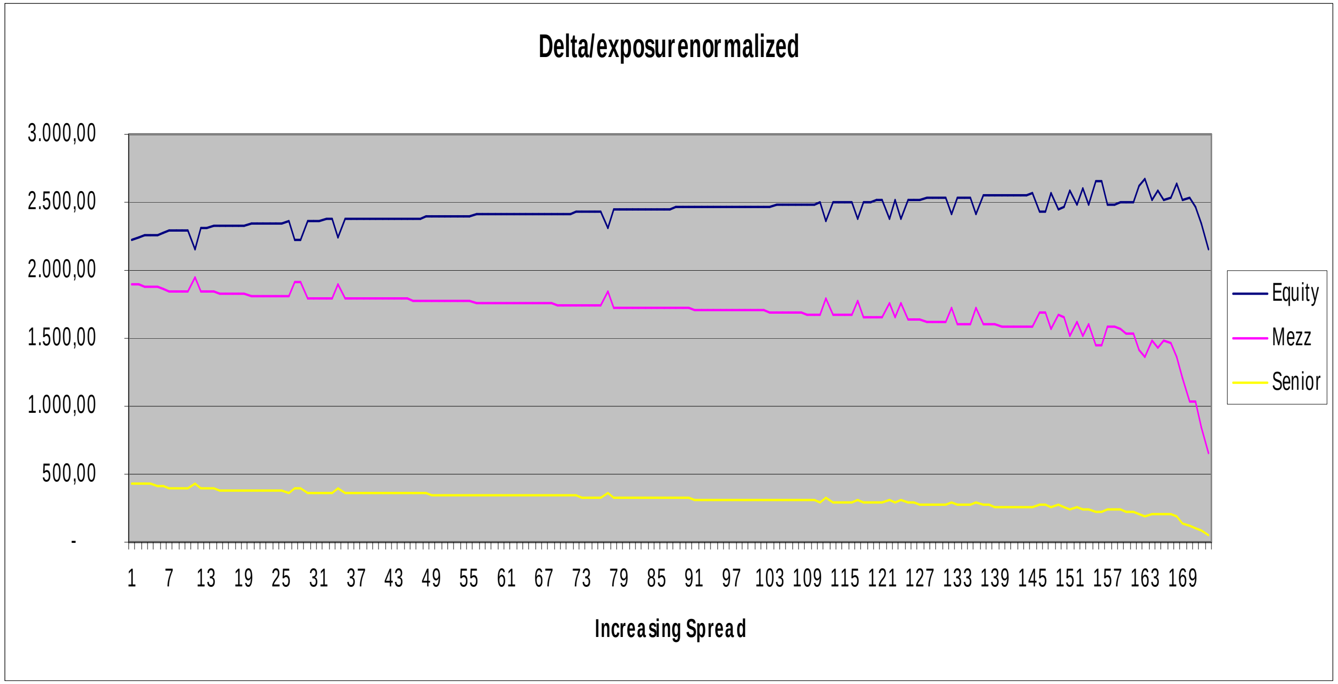
Delta

Real Portfolio of 120 names 3 to 6 tranch



Stylized Portfolio

All the correlation and adjusted to the exposure



Expected Loss Contributions

Decompose $E[L_t^{CDO}]$ into contributions to each single name. Follows from a decomposition of L_t^{CDO} into contribution to each single name.

For a given i we denote by $\alpha(i)(\omega)$ the number such that $\tau_{[\alpha(i)(\omega)]}(\omega) = \tau_i(\omega)$. Hence counterparty i caused the $\alpha(i)(\omega)$s.

The loss contributed to counterparty i for each ω turns out to be $V(i)(\omega) =$

$$\begin{cases} 0 & \text{if} \quad \sum_{k=1}^{\alpha(i)(\omega)} l_{\alpha^{-1}(k)(\omega)} < K \vee \sum_{k=1}^{\alpha(i)(\omega)-1} l_{\alpha^{-1}(k)(\omega)} > U \\ l_i & \text{if} \quad K < \sum_{k=1}^{\alpha(i)(\omega)-1} l_{\alpha^{-1}(k)(\omega)} < \sum_{k=1}^{\alpha(i)(\omega)} C([k]) < U \\ \sum_{k=1}^{\alpha(i)(\omega)} l_{\alpha^{-1}(k)(\omega)} - K & \text{if} \quad \sum_{k=1}^{\alpha(i)(\omega)-1} l_{\alpha^{-1}(k)(\omega)} < K < \sum_{k=1}^{\alpha(i)(\omega)} l_{\alpha^{-1}(k)(\omega)} < U \\ U - \sum_{k=1}^{\alpha(i)(\omega)-1} l_{\alpha^{-1}(k)(\omega)} & \text{if} \quad K < \sum_{k=1}^{\alpha(i)(\omega)-1} C([k]) < U < \sum_{k=1}^{\alpha(i)(\omega)} l_{\alpha^{-1}(k)(\omega)} \\ U - K & \text{if} \quad \sum_{k=1}^{\alpha(i)(\omega)-1} l_{\alpha^{-1}(k)(\omega)} < K < U < \sum_{k=1}^{\alpha(i)(\omega)} l_{\alpha^{-1}(k)(\omega)} \end{cases}$$

The most important property of $V(i)$ is that

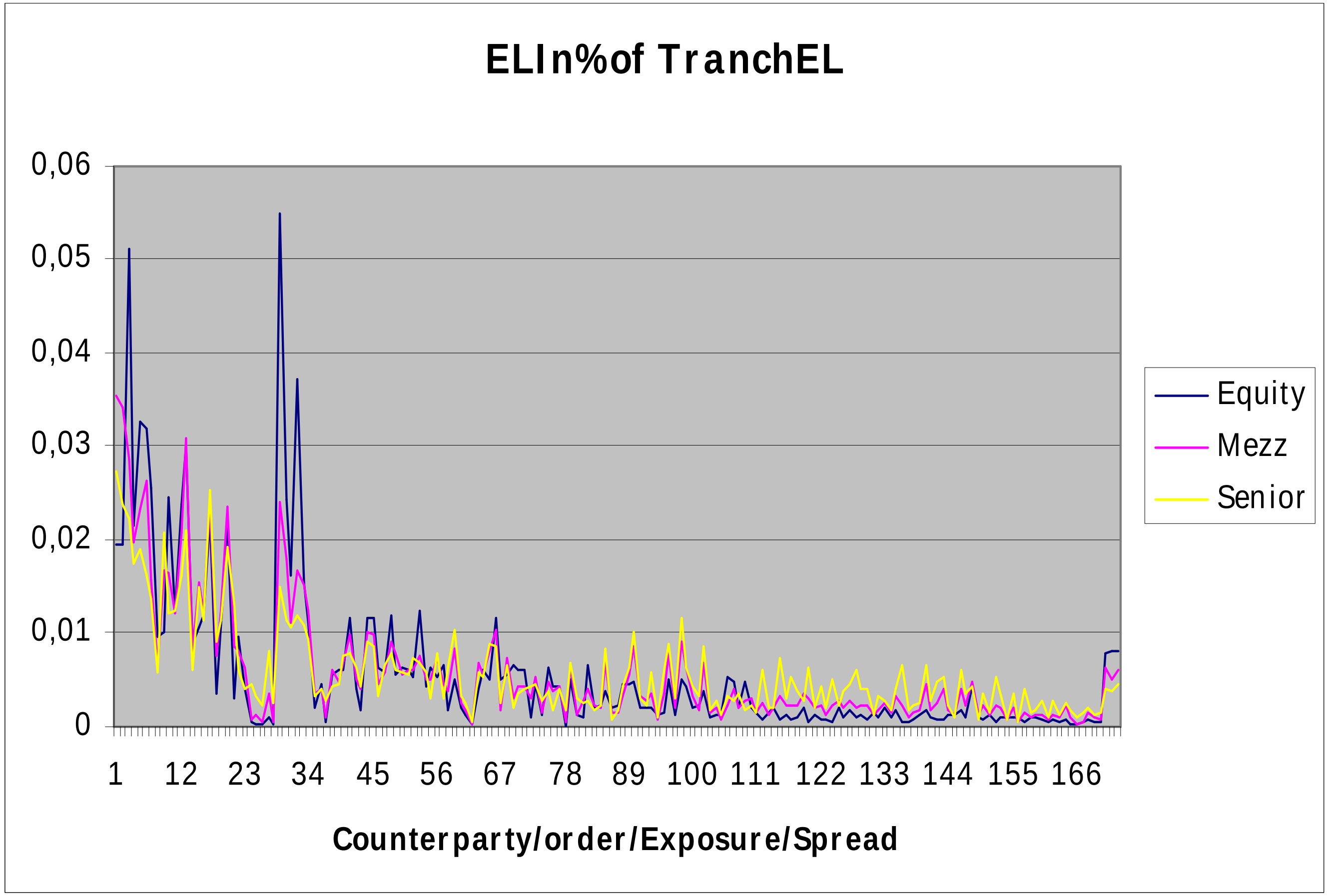
$$\sum_{i=1}^{N} V(i)(\omega) = L_t^{CDO}(\omega).$$

yielding the corresponding decomposition of the Expected Values.

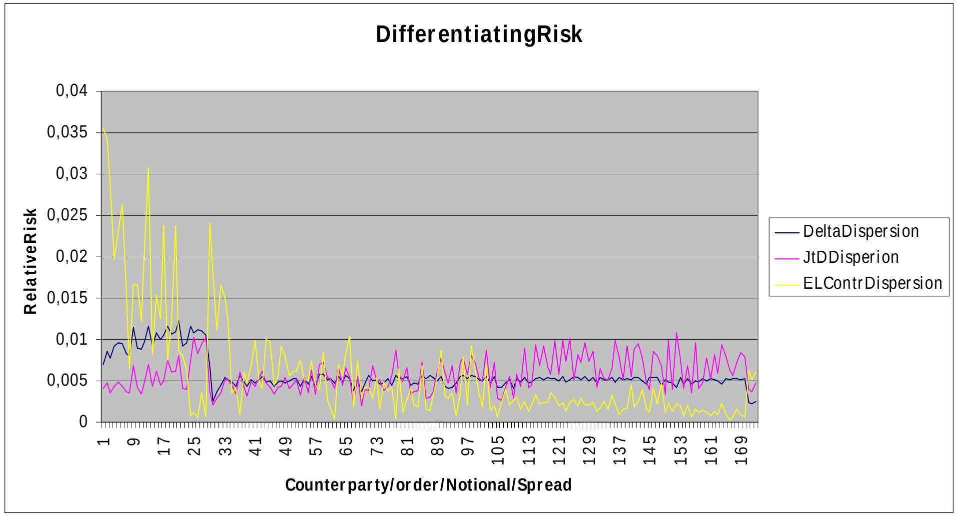
There is a symmetric version of this decompositions, in which the loss in a tranch is distributed to all defaulting obligors in the default path leading to a loss. The distribution of the tranch loss is proportional to all the exposures of these defaulting counterparties. The contribution in each scenario is therefore

$$\mathbf{1}_{\{\text{Tranch is hit}\}} \cdot \text{Tranch Loss} \cdot \frac{l_i \mathbf{1}_{D_i}}{\sum_j l_j \mathbf{1}_{D_j}}$$

ELcontribution



Comparison of stand-alone measures



Most differentiating one is EL contribution. Might be useful for an exposure equivalent.

Expected Loss contribution: Theory I

Derivatives and Risk Measures: Let L be the portfolio loss variable, and $l_1, .., l_k$ be exposure parameter. It is known that for optimization it is useful if the risk measure

$$\rho(L) = \rho(L(l_1, .., l_k))$$

is 1-homogenous with respect to the vector $\bar{l}$, i.e. for $a > 0$

$$a\rho(L) = \rho(L(al_1, .., al_k)).$$

Then we have the Euler differentiation property

$$\rho(l_1, .., l_m) = \sum_{i=1}^{k} l_i \frac{\partial \rho}{\partial l_i}(l_1, .., l_k),$$

which actually implies again 1-homogeneity.

Theory II

Even non-coherent measures satisfy this property - like Value-at-Risk -, which gives some portfolio optimization properties (cf. Tasche 1999). The Euler property remains valid under mixtures. Therefore all spectral risk measures (see portfolio risk measures) inherit it from quantiles.

How does it work if the risk functional, or even the loss variable L is not linear in the exposures?

Example

$$\rho(L) = E_Q[(L - K)^+]$$

the value of a supersenior tranche.

Theory III

One has to take a ”moving differential” (cf. Taras 2005, Presentation at Global Risk Management, Monte Carlo 2005). Better to understand, if the function $\rho(l_1,..,l_m)$ is slightly modified. If we concentrate on super-senior tranches and consider the attachement point in percentage k of the total exposure $\sum_{i=1}^m l_i$:

$$\begin{aligned}\tilde{\rho}(l_1,..,l_m) &= E\left[\left(\sum_{i=1}^m l_i \mathbf{1}_{D_i} - k\sum_{i=1}^m l_i\right)^+\right] \\ &= E\left[\mathbf{1}_{\left\{\sum_{i=1}^m l_i \mathbf{1}_{D_i} - k\sum_{i=1}^m l_i > 0\right\}} \sum_{i=1}^m l_i(\mathbf{1}_{D_i} - k)\right]\end{aligned}$$

Theory IV

Since

$$\left\{\sum_{i=1}^{m} l_i \mathbf{1}_{D_i} - k\sum_{i=1}^{m} l_i > 0\right\} = \left\{\sum_{i=1}^{m} a l_i \mathbf{1}_{D_i} - k\sum_{i=1}^{m} a l_i > 0\right\}$$

it follows that

$$\begin{aligned}
&\tilde{\rho}(al_1, .., al_m)) = a\tilde{\rho}(l_1, .., l_m)) \\
&\frac{\partial\tilde{\rho}}{\partial l_i}(l_1, .., l_m) = E\left[\mathbf{1}_{\left\{\sum_{i=1}^{m} l_i \mathbf{1}_{D_i} - k\sum_{i=1}^{m} l_i > 0\right\}}\left(\mathbf{1}_{D_i} - k\right)\right] = \\
&P[\text{Loss in Tranche}]\left(P\left[i \text{ defaults}|\text{ loss in tranche }\right] - k\right)
\end{aligned}$$

Theory vers. Practice

The symmetric version of the Expected Loss Contribution can be written as

$$P[\text{Loss in Tranche}]E\left[\mathbf{1}_{D_i}\frac{\text{Tranche Loss}}{\text{Portfolio loss}}|\text{ loss in tranche }\right]$$

In order to understand the difference in contribution we have to analyze the difference between

$$\mathbf{1}_{D_i}\frac{\text{Portfolio loss-K}}{\text{Portfolio loss}}$$

and

$$\mathbf{1}_{D_i} - k$$

under the conditional measure that there is a tranch loss. The first one equals

$$\mathbf{1}_{D_i}(1 - K/L).$$

Therefore the first practical contribution is proportional to the probability that i defaulted when the tranch was hit and $K/L < 1$ takes a fixed value, whereas the theoretical one is proportional to the deviation from this probability to the tranch thickness (independent and hence for each realized value of K/L). The relative order should be the same!?

Summary: Stand-alone Risk Measures

The risk management of CDOs should be based on several stand-alone risk measures. As examples we presented

- Delta w.r.t. change in default probabilities
- Gap and Jump risk
- Expected loss contribution

- All three are very sensitive to the correlation assumption
- In the normal copula framework the first two can be calculated analytically
- Expected loss contribution is the most risk sensitive
- Expected loss contribution seems reasonable as the basis for an exposure equivalent when working in a credit risk management framework

Portfolio Risk Measures

1. Spectral Risk Measures Given a weight function $w : [0,1] \to [0,\infty)$, increasing with integral 1,

   $$\begin{aligned} ES_w(L) &= \int_0^1 w(\alpha) q_\alpha(L) d\alpha \\ &= \int_0^1 w(\alpha) F_L^{-1}(\alpha) d\alpha \end{aligned}$$

   The function w can be viewed as a risk aversion function.

2. Expected Shortfall distribution

   $$w(x) = w_\alpha(x) = \frac{1}{1-\alpha}\mathbf{1}_{(\alpha,1]}(x)$$

3. Value-at-Risk. Point mass, not an increasing weight function. Not coherent

   $$w(x) = \delta_\alpha$$

A generalized scenario

Theorem 1 *The density of the scenario associated with the risk measure ES_w equals*

$$L_w \ := \ g_w(L) := \int_0^1 g_\alpha(L)(1-\alpha)\mu_w(d\alpha) \tag{4}$$

with μ_w is the measure associated with the increasing function w, $w(b) = \mu_w([0,b])$ and

$$g_\alpha(Y) := (1-\alpha)^{-1}(\mathbf{1}_{\{Y>\mathrm{q}_\alpha(Y)\}} + \beta_Y \mathbf{1}_{\{Y=\mathrm{q}_\alpha(Y)\}}), \tag{5}$$

where

$$\beta_Y := \frac{\mathbb{P}(Y \le \mathrm{q}_\alpha(Y)) - \alpha}{\mathbb{P}(Y = \mathrm{q}_\alpha(Y))} \quad \textit{if} \quad \mathbb{P}(Y = \mathrm{q}_\alpha(Y)) > 0.$$

In particular

$$ES_w(L) \ = \ E[LL_w] \tag{6}$$

It follows from general allocation rules (cf. Kalkbrener, Lotter, O. and Kalkbrener) that

$$ESC_w(L_i, L) = E[L_i L_w].$$

Since CDOs are investments in specific quantiles of the loss distribution the choice of the risk measure becomes even more important. Purely tail-risk measures seem not to be useful. All parts of the loss distribution should be considered. The question how to specify a risk aversion weight function is still open, from a conceptual as well from a financial/risk point of view.