American-Asian Options: Optimal Exercise Policies and Simulation-based Valuation

Where are the WMDs?

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OUTLINE

- Background on Options Pricing
- Motivation and Related Work
  Simple Examples for Gradient Estimation
- Problem Setting
- Structure of Optimal Exercise Policy
- Perturbation Analysis Estimators
- Parameterization of Exercise Boundary
  and Simplification of PA Estimators
- Numerical Results

REVIEW: What is an Option?

special type of DERIVATIVE SECURITY:

value *derived* from underlying (primary) security

- strike price K
- expiration date T

- **call option**: right to *purchase* the underlying asset at the strike price.
- **put option**: the right to *sell* at the strike price.

- **European option**:
  can only be exercised on the expiration date.
- **American option**:
  right can be exercised at any time up to and including the expiration date.
- In either case, the option does not have to be exercised at all on the expiration date if the stock price is below the strike price ("out of the money" in finance terminology).

REVIEW: Derivatives Pricing

MAJOR FOCUS of FINANCIAL ENGINEERING:
pricing and hedging of **DERIVATIVE SECURITIES**

EXAMPLE: call option on IBM 80 Jan
Price is given in the newspaper, so what's the problem?

What is hedging?

How are financial services providers like bookies?

Modeling *DYNAMICS* of underlying asset:
described by stochastic differential equation,
diffusion processes and/or jump processes.
ASSUMED GIVEN, NOT FOCUS OF OUR WORK

REVIEW: Derivatives Pricing

BIG BREAKTHROUGHS:

- Black-Scholes Model:
  1. constant proportional drift term plus a (geometric) Brownian motion term (Wiener process)
     (can be written as stochastic DE)
  2. leads to celebrated PDE
  3. can be solved in closed form:
     stock has lognormal distribution

- Risk-Neutral Valuation:
  use risk-neutral (equivalent **martingale**) measure,
  NOT **statistical** measure

  fundamental theorems of derivatives pricing:
  - existence: no arbitrage
  - uniqueness: complete markets

REVIEW: Derivatives Pricing

CHIEF TECHNIQUES

- Analytical Solutions/Approximations
- Binomial Method:
  replicating asset dynamics via probabilistic tree/lattice
- Finite Differences:
  solving PDE numerically
- Monte Carlo Simulation:
  stochastic simulation of asset dynamics (sample paths)
- Laplace/Fourier Transform Inversion

Motivation for our Work

Background on Simulation and Finance:

- "...the Monte Carlo method should prove most valuable in situations where it is difficult if not impossible to proceed using a more accurate approach." (Boyle 1977)
- "To obtain option values corresponding to different current stock prices a set of simulation trials has to be carried out for each starting stock price." (Boyle 1977)

Gauntlet laid down:

- "Monte Carlo simulation can only be used for European-style options." (Hull 1993, p.363)
- "the procedure is essentially only useful for European-style options..." (Luenberger 1998, p.363).

Related Work

Sampling of responses:

- Tilley (1993): bundling method.
- Fu & Hu (1995): PA & SA (to be described); see Fu, Wu, Gürkan, & Demir (2000) for corrections.
- Grant, Vora, & Weeks (1996, 1997): backwards induction simulation for exercise boundary; forward simulation to estimate price.
- Broadie & Glasserman (1996, 1997): simulated trees, stochastic mesh.
- Barraquand & Martineau (1997): consider only payoff.
- Patsis (1998): ordinal optimization.
- Carriere (1996), Longstaff & Schwartz (2001), Tsitsiklis & Van Roy (2001): approximate simulation-based DP; regression to approximate holding value function at each early exercise opportunity.
- Fu, Laprise, Madan, Su, and Wu (2001): compared most of these approaches computationally.

Monte Carlo Simulation Pros & Cons

STRENGTHS

- high-dimensional problems
  - convergence rate independent of dimensions
  - others generally exponential growth in computation
- most flexible modeling
  - path dependence
  - multiple assets
  - multiple sources of uncertainty
    (e.g., interest rates, volatility)
- easy to implement
- measure of precision (e.g., confidence intervals)

Monte Carlo Simulation Pros & Cons

POTENTIAL LIMITATIONS

- computationally intensive
  - convergence rate $1/\sqrt{n}$ considered “slow”
- forward-based technique
  - difficulties in handling early exercise features
- statistical estimation

Notation

$$
\begin{aligned}
S_t &= \text{stock price at time } t \ (S_0 \text{ initial}), \\
r &= \text{riskless interest rate}, \\
\mu, \sigma &= \text{drift, volatility of stock}, \\
K &= \text{strike}, \\
T &= \text{expiration date}, \\
J_T &= \text{net present value return at } T.
\end{aligned}
$$

European call option payoff:

$$J_T = (S_T - K)^+ e^{-rT}, \tag{1}$$

value

$$E^Q[J_T] = E[(S_T - K)^+ e^{-rT}],$$

Q is risk-neutral measure (e.g., Black-Scholes, $\mu = r$)

FIRST GOAL: estimate $\partial E[J_T]/\partial\theta$,
θ is the parameter of interest (e.g., $S_0, r, \sigma, \mu, K, T$)
"Greeks" used in hedging

IPA Estimators for the "Greeks"

$$\frac{\partial E[J_T]}{\partial \theta} = e^{-rT}\left[\frac{\partial E[(S_T-K)^+]}{\partial \theta} - E[(S_T-K)^+]\frac{\partial (rT)}{\partial \theta}\right].$$

IPA ESTIMATOR: differentiate (1)

$$\begin{aligned}\frac{\partial J_T}{\partial \theta} &= e^{-rT}\begin{cases}\frac{\partial (S_T-K)}{\partial \theta} - (S_T-K)\frac{\partial (rT)}{\partial \theta} & \text{for } S_T > K \\ 0 & \text{otherwise}\end{cases} \\ &= e^{-rT}\left[\frac{\partial (S_T-K)}{\partial \theta} - (S_T-K)\frac{\partial (rT)}{\partial \theta}\right]\mathbf{1}\{S_T > K\}.\end{aligned}$$

IPA Derivative Estimators for the European Option

θ	$\frac{\partial J_T}{\partial \theta}$ for $S_T > K$ (0 otherwise)
K	$-e^{-rT}$
S_0	$e^{-rT}\frac{\partial S_T}{\partial S_0}$
r	$e^{-rT}[T(K-S_T) + \frac{\partial S_T}{\partial r}]$
σ	$e^{-rT}\frac{\partial S_T}{\partial \sigma}$
T	$e^{-rT}[r(K-S_T) + \frac{\partial S_T}{\partial T}]$

Pricing American-Style Options via Simulation

Example: American call option value

$$\max_{\tau} E^Q[J_\tau] = E[(S_\tau - K)^+ e^{-r\tau}],$$

τ is a stopping time (cannot depend on future)

- stochastic control problem (continuous time);
  stochastic dynamic programming (discrete time)
- optimal stopping problem:
  exercise (stop) vs. hold (continue)
  immediate exercise value vs. expected value of holding
  (difficulty in handling latter, SAME PROBLEM again!)
- solution:
  PDE (Bellman equation) in continuous time
  backwards induction in discrete time,
  BUT simulation is forward technique
  - Example: American call option

$$E^Q[\max_{\tau} J_\tau] \neq \max_{\tau} E^Q[J_\tau]$$

American Call Option with Discrete Dividends Application of SPA

$$D_j = j\text{th dividend } D_j,$$

$$t_j = \text{epoch } j\text{th dividend distributed},$$

$$\eta(T) = \text{number of dividends.}$$

- STANDARD MODEL:
  after the dividend, the stock price drops by the amount of the dividend, i.e.,
  $$S_{t_j^+} = S_{t_j^-} - D_j.$$
- KEY RESULT:
  possibly exercise only right before a dividend date
  (or at expiration).
- CONSEQUENCE:
  Assume threshold exercise policy:
  Exercise at t_j if $S_{t_j^-} > s_j$,
  $s_j (\geq K)$ threshold parameters to determine.
  (NOTE: $s_j = \infty \quad \forall j$ corresponds to European.)

ANALYSIS

SAMPLE PERFORMANCE:

$$J_T = e^{-rT}\left(\sum_{i=1}^{\eta(T)}\left[\prod_{j=1}^{i-1}\mathbf{1}\{S_{t_j^-}\le s_j\}\right]\mathbf{1}\{S_{t_i^-}>s_i\}\left(S_{t_i^-}-K\right)e^{r(T-t_i)} + \prod_{j=1}^{\eta(T)}\mathbf{1}\{S_{t_j^-}\le s_j\}(S_T-K)^+\right),$$

GOAL: ESTIMATE

$$\partial E[J_T]/\partial\theta.$$

DIFFICULTY:
J_t no longer a.s. continuous due to jumps at dividend point.

MODEL:
Aside from dividends, stock price changes continuously.

Valuation of American Call Options

Let $\quad g(s) = \frac{\partial E[J_T(s)]}{\partial s}$,
gradient w.r.t. s, the early exercise threshold parameter.

MAIN IDEA:
Option value is point at which $E[J_T(s)]$ max w.r.t. s.

STOCHASTIC APPROXIMATION ALGORITHM:
solves $g(s) = 0$ iteratively

$$s^{(n+1)} = \Pi_\Theta\left(s^{(n)} + a_n \hat{g}_n\right),$$

$$\hat{g}_n = e^{-rT} \frac{\partial h^{-1}(y^*)}{\partial \theta} f(h^{-1}(y^*)) \Big(E[J_T | S_{t_1^-} = s^-] - E[J_T | S_{t_1^-} = s^+] \Big)$$

2nd term is simply $s - K$, $\quad y^* = s - D$,
h describes continuous dynamics, i.e., $S_{t+\Delta t} = h(S_t; \Delta t, ..., Z)$, $\quad Z \sim f$

Ex: GBM $h(S; \Delta t, r, \sigma, Z) = S e^{(r-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} Z}$, $\quad Z \sim N(0,1)$

NUMERICAL RESULTS

- constraint set: $\Theta = [K, 5K]$,
- step sizes: accelerated harmonic series with $a = 100$.
- starting point: $\theta_0 = K$,
- fixed observation lengths: 10 and 100.

PARAMETER VALUES

- $K = S_0 = 50, \qquad r = 0.10, \qquad \sigma = 0.30, \qquad (\tau_1, \tau_2) = (60, 30)$,
- three values for dividend: $D = 0.5, 1.0, 2.0$,
- lognormal distribution.

CONVERGENCE RESULTS

- within a penny within 1000 simulations for 5 out of 6 cases ;
- within a penny within 7000 simulations for other case;

- less effort than needed to simply *estimate* option payoff to within a penny

PROBLEM SETTING: Asian-American Call Option

- "Asian" background: sold mostly OTC; less volatile.

- arithmetic discrete average, finite horizon T.
- finite exercise opportunities (see Figure 1).
- asset price process $\{S_t\}$ Markovian.
- option price at time t:

$$\sup_{\tilde{\tau} \geq t} E^Q \left[e^{-r(\tilde{\tau}-t)} (\overline{S}_{\tilde{\tau}} - K)^+ | \overline{S}_t, S_t \right],$$

$$\text{where} \qquad \overline{S}_t = \frac{S_{t_0} + S_{t_0+\tau} + ... + S_t}{n_t}.$$

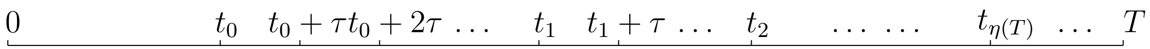

Figure 1: Time horizon: averaging dates and exercise opportunities

Structure of Optimal Exercise Policy

Related literature:
Grant, Vora, Weeks (1997) – heuristic guess, but no proof.
Karatzas and Shreve (1998) – "vanilla" American options.
Ben Ameur, Breton, and L'Ecuyer (2002) –
similar results found in parallel; DP algorithm for pricing, not simulation.

Write $\psi(x) = (x - K)^+$.
Continuation value given by:

$$c(\overline{S}_t, S_t, t) = \sup_{\tilde{\tau} > t} E[e^{-r(\tilde{\tau}-t)}(\overline{S}_{\tilde{\tau}} - K)^+ | \overline{S}_t, S_t]$$

$\Longrightarrow$ optimal exercise region R^* characterized by

$$R^* = \left\{(t, \overline{S}_t, S_t) | c(\overline{S}_t, S_t, t) \le \psi(\overline{S}_t)\right\}.$$

Structure of Optimal Exercise Policy (cont.)

Theorem 1 *<u>THRESHOLD</u> policy optimal, i.e.,*
$\exists \quad F_t^*(S_t)$ *s.t. optimal to exercise whenever*

$$\overline{S}_t \geq F_t^*(S_t) = \inf \left\{ \overline{S}_t | c(\overline{S}_t, S_t, t) \leq \psi(\overline{S}_t) \right\}.$$

Assumption 1 *If $S_{t,0}$ and $S_{t,1}$ are two stock prices at time t with $S_{t,0} > S_{t,1}$, then $c(\overline{S}_t, S_{t,0}, t) \geq c(\overline{S}_t, S_{t,1}, t)$.*

The higher current asset price, the higher call option value, e.g., multiplicative & additive models. ($S_{t+\Delta t} = S_t \times X_{\Delta t}$, GBM; $S_{t+\Delta t} = S_t + X_{\Delta t}$, BM.)

Theorem 2 *If Assumption 1 holds, $F_t^*(S_t)$ is <u>nondecreasing</u> in S_t.*

Theorem 3 For any $t \in \mathcal{E}(t \neq T)$, *$F_t^*(S_t)$ is <u>unbounded</u>, i.e., if $S_t \to \infty$, then $F_t^*(S_t) \to \infty$.*

Theorem 4 *If the stock price model is multiplicative or additive, then $F_t^*(.)$ is <u>convex</u>.*

Derivation of PA estimators

Formulate optimal stopping problem as following optimization problem:

$$\max_{\theta \in \Theta} E[\mathcal{L}(\theta, \omega)], \tag{2}$$

and apply iterative SA search scheme:

$$\theta_{n+1} = \Pi_{\Theta}(\theta_n + a_n \hat{g}_n), \tag{3}$$

PA in general superior to finite differences.

Related work:
Fu and Hu (1995), Fu, Wu, Gürkan and Demir (2000),
Fu et al. (2001) SPSA.

Assume asset price dynamics follows

$$S_t = h(Z; S_0, t, r, \sigma)$$

Derivation of PA estimators (cont.)

Lemma 2 *If $F_i(.)$ is convex and*

$$\{S_{t_i} | \overline{S}_{t_i} \geq F_i(S_{t_i})\} \neq \emptyset,$$

then we can always find $L_i(.)$ and $U_i(.)$ such that

$$\overline{S}_{t_i} \geq F_i(S_{t_i}) \Longleftrightarrow L_i(\overline{S}_{t_i-\tau}) \leq S_{t_i} \leq U_i(\overline{S}_{t_i-\tau}).$$

Payoff function:

$$\sum_{i=1}^{\eta(T)} \mathbf{1}\left\{\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i} \geq F_i(S_{t_i})\right\} \left(\overline{S}_{t_i} - K\right) e^{-rt_i} + \mathbf{1}\left\{\bigcap_{j=1}^{\eta(T)} \overline{S}_{t_j} < F_j(S_{t_j})\right\} \left(\overline{S}_T - K\right)^+ e^{-rT}.$$

PA estimator for general $\eta(T)$:

$$\begin{aligned}
&\sum_{i=1}^{\eta(T)} \mathbf{1}\left\{\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j})\right\} \times \\
&\quad \left\{ \frac{\partial h^{-1}(L_i(\overline{S}_{t_i-\tau}); S_{t_i-\tau}, \tau)}{\partial \theta} f(h^{-1}(L_i(\overline{S}_{t_i-\tau}); S_{t_i-\tau}, \tau)) \times \left[E\left[\mathcal{L} \left| \bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\tau}, S_{t_i} = L_i(\overline{S}_{t_i-\tau}) \right.\right] - (\overline{S}_{t_i} - K)e^{-rt_i} \left| S_{t_i} = L_i(\overline{S}_{t_i-\tau}) \right.\right] \right. \\
&\qquad \left. - \frac{\partial h^{-1}(U_i(\overline{S}_{t_i-\tau}); S_{t_i-\tau}, \tau)}{\partial \theta} f(h^{-1}(U_i(\overline{S}_{t_i-\tau}); S_{t_i-\tau}, \tau)) \times \left[E\left[\mathcal{L} \left| \bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\tau}, S_{t_i} = U_i(\overline{S}_{t_i-\tau}) \right.\right] - (\overline{S}_{t_i} - K)e^{-rt_i} \left| S_{t_i} = U_i(\overline{S}_{t_i-\tau}) \right.\right] \right\} \\
&\qquad + \sum_{i=1}^{\eta(T)} \mathbf{1}\left\{\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i} \geq F_i(S_{t_i})\right\} \frac{\partial}{\partial \theta}\left[(\overline{S}_{t_i} - K)e^{-rt_i}\right] + \mathbf{1}\left\{\bigcap_{j=1}^{\eta(T)} \overline{S}_{t_j} < F_j(S_{t_j})\right\} \frac{\partial}{\partial \theta}\left[(\overline{S}_T - K)^+ e^{-rT}\right]. \qquad (4)
\end{aligned}$$

Parameterization of Exercise Boundary

Increasing and convexity properties:
approximate exercise boundary at t_i
by piecewise linear function (see Figure 2):

$$F_i(S_{t_i}) = \begin{cases} s_i & \text{if } S_{t_i} \leq s_i - v_i \\ S_{t_i} + v_i & \text{if } S_{t_i} > s_i - v_i \end{cases}$$

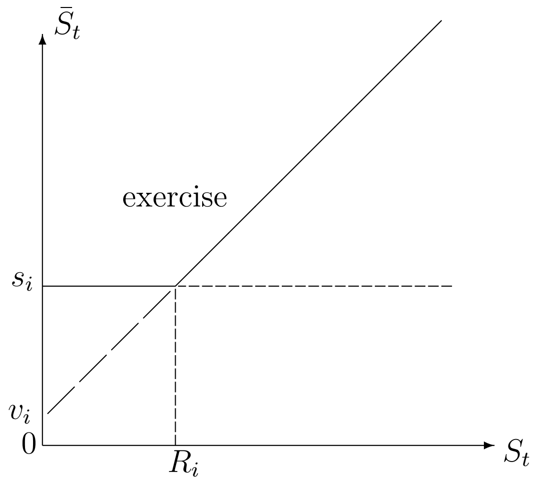

Figure 2: Early exercise boundary for Asian option.

Simplification of PA Estimator

$$\sum_{i=1}^{\eta(T)} \mathbf{1}\left\{\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\tau} > s_i + \frac{v_i}{n_i-1}\right\}$$
$$\left\{\mathbf{1}\left\{L_i(\overline{S}_{t_i-\tau}) > 0\right\} \frac{\partial h^{-1}(L_i(\overline{S}_{t_i-\tau}); S_{t_i-\tau}, \tau)}{\partial \theta} f(h^{-1}(L_i(\overline{S}_{t_i-\tau}); S_{t_i-\tau}, \tau)) \times\right.$$
$$\left[E\left[\mathcal{L}\left|\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\tau}, S_{t_i} = L_i(\overline{S}_{t_i-\tau})\right.\right] - (s_i - K)e^{-rt_i}\right]$$
$$- \frac{\partial h^{-1}(U_i(\overline{S}_{t_i-\tau}); S_{t_i-\tau}, \tau)}{\partial \theta} f(h^{-1}(U_i(\overline{S}_{t_i-\tau}); S_{t_i-\tau}, \tau)) \times$$
$$\left.\left[E\left[\mathcal{L}\left|\bigcap_{j=1}^{i-1} \overline{S}_{t_j} < F_j(S_{t_j}), \overline{S}_{t_i-\tau}, S_{t_i} = U_i(\overline{S}_{t_i-\tau})\right.\right] - (\overline{S}_{t_i-\tau} - \frac{v_i}{n_i-1} - K)e^{-rt_i}\right]\right\},$$

where $L_i(\overline{S}_{t_i-\tau}) = n_i s_i - (n_i - 1)\overline{S}_{t_i-\tau}$ and $U_i(\overline{S}_{t_i-\tau}) = \overline{S}_{t_i-\tau} - \frac{n_i v_i}{n_i-1}$.
The last two terms in (4) are zero, because the underlying asset price process is independent of the threshold parameters.

Implementation and Results

- Stochastic approximation algorithm employed.
- Compared with other procedures in literature.
- Better parameter values means higher option value.
  2,000,000 simulations used to estimate option price once associated threshold parameters are derived in comparison with LS method.
- Conclusions:
  much faster convergence than most other methods,
  comparable (slightly faster) than LS method.

Related Work in Progress

- Pricing of American-style derivatives using
  piecewise **linear interpolation** of value function (Laprise)
  - express price as a portfolio of European options
  - in some cases: analytical results, upper/lower bounds
  - convergence with # interpolating points

  [paper presented at WSC01]

  weighted stochastic mesh (Xiong)

  [paper presented at 12th Derivatives Securities Conference]

- Estimation of Value-at-Risk (VaR):
  probability error bounds for quantile estimation

  [paper in Feb. 2003 issue of *Management Science* (Jin, Xiong)]

- Optimal importance sampling in securities pricing (Su)

  [paper *Journal of Computational Finance*, 2002]

- Pricing of mortgage-backed securities and credit derivatives (Chen)

  [paper presented at 11th Derivatives Securities Conference, WSC02 proceedings]

- Simulation of bank defaults for pricing deposit insurance (Jarrow et al.)

  [paper presented at 2003 FDIC conference on finance and banking]

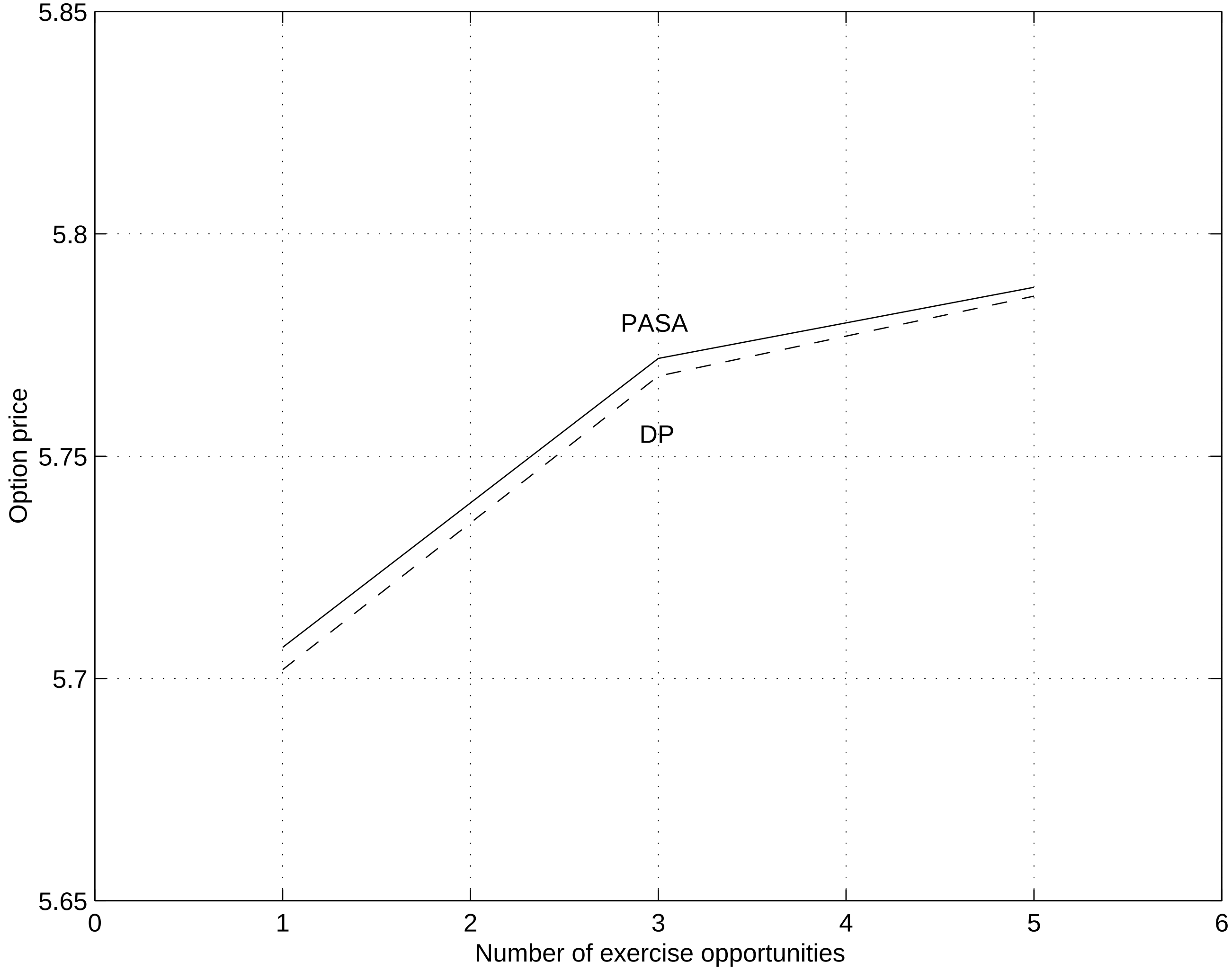

Figure 3: Comparison of option value for two methods.

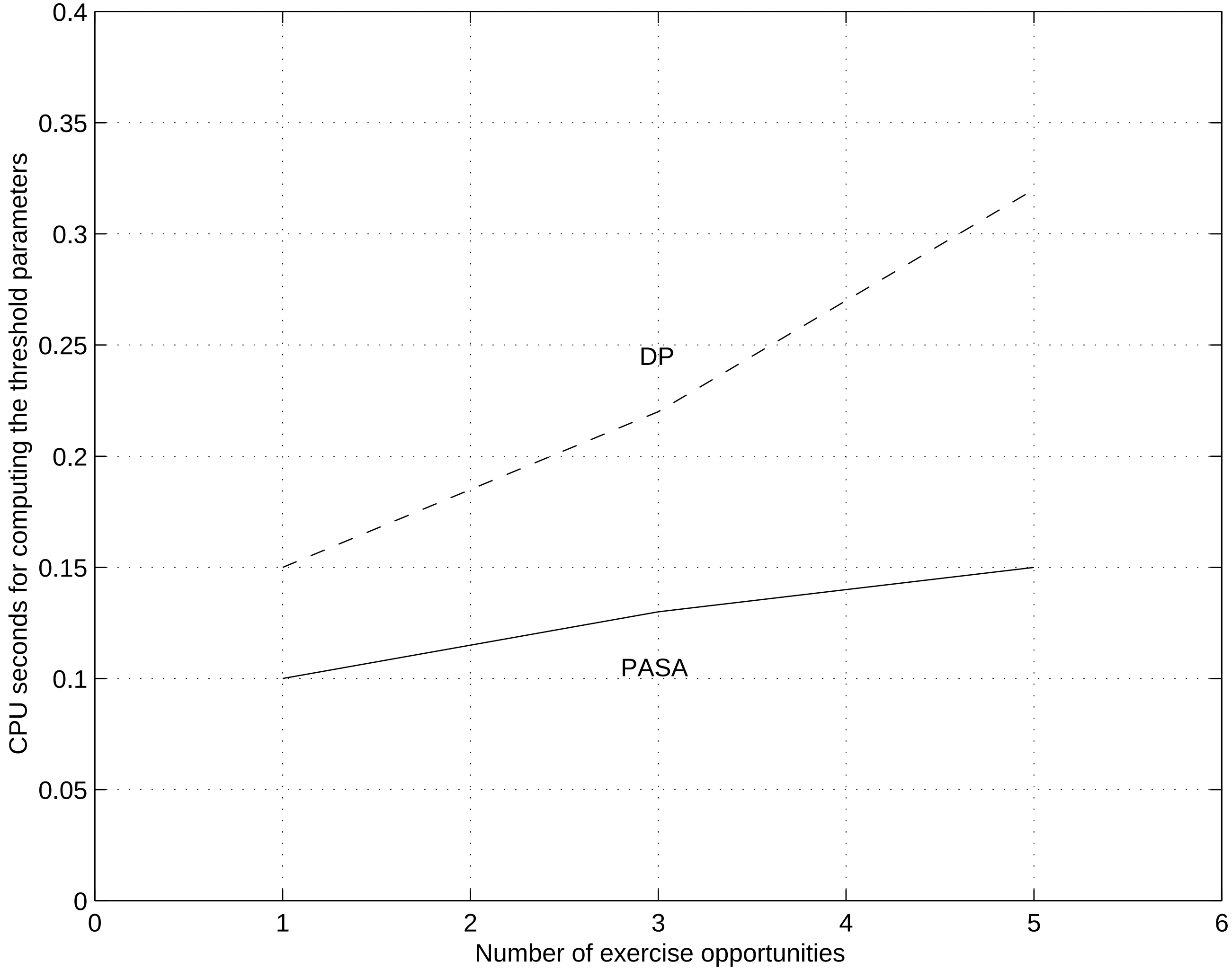

Figure 4: Comparison of CPU time for two methods.

	t_i=105, 120			t_i=105, 110, 115, 120			t_i=105, 108, ..., 120		
	Price	StdErr	CPU	Price	StdErr	CPU	Price	StdErr	CPU
				$K = 90$					
PASA	13.091	0.007	0.12	13.179	0.007	0.17	13.197	0.007	0.18
DP	13.078	0.007	0.15	13.169	0.007	0.22	13.189	0.007	0.32
DIFF	0.013			0.010			0.008		
				$K = 95$					
PASA	9.019	0.006	0.11	9.108	0.006	0.17	9.123	0.006	0.18
DP	9.021	0.006	0.15	9.101	0.006	0.21	9.122	0.006	0.33
DIFF	-0.002			0.007			0.001		
				$K = 100$					
PASA	5.707	0.005	0.10	5.772	0.005	0.13	5.788	0.005	0.15
DP	5.702	0.005	0.15	5.768	0.005	0.22	5.786	0.005	0.32
DIFF	0.005			0.004			0.002		
				$K = 105$					
PASA	3.287	0.004	0.07	3.329	0.004	0.10	3.338	0.004	0.12
DP	3.280	0.004	0.15	3.329	0.004	0.22	3.337	0.004	0.33
DIFF	0.007			0.000			0.001		
				$K = 110$					
PASA	1.720	0.003	0.06	1.748	0.003	0.08	1.747	0.003	0.10
DP	1.716	0.003	0.15	1.745	0.003	0.22	1.751	0.003	0.32
DIFF	0.004			0.003			-0.004		

Table 1: σ =0.2, r=0.09, S_0=100.
CPU seconds are for computing the threshold parameters only.

	t_i=105, 120			t_i=105, 110, 115, 120			t_i=105, 108, ..., 120		
	Price	StdErr	CPU	Price	StdErr	CPU	Price	StdErr	CPU
$K = 90$									
PASA	14.376	0.010	0.11	14.502	0.010	0.13	14.534	0.010	0.15
DP	14.368	0.010	0.15	14.490	0.010	0.21	14.526	0.010	0.33
DIFF	0.008			0.012			0.008		
$K = 95$									
PASA	10.815	0.009	0.09	10.913	0.009	0.14	10.944	0.009	0.16
DP	10.797	0.009	0.14	10.911	0.009	0.20	10.942	0.009	0.33
DIFF	0.018			0.002			0.002		
$K = 100$									
PASA	7.820	0.008	0.08	7.920	0.008	0.11	7.933	0.008	0.12
DP	7.814	0.008	0.15	7.916	0.008	0.21	7.935	0.008	0.32
DIFF	0.006			0.004			-0.002		
$K = 105$									
PASA	5.450	0.007	0.07	5.526	0.007	0.09	5.544	0.007	0.10
DP	5.447	0.007	0.15	5.526	0.007	0.21	5.538	0.007	0.32
DIFF	0.003			0.000			0.006		
$K = 110$									
PASA	3.665	0.006	0.06	3.723	0.006	0.07	3.730	0.006	0.09
DP	3.661	0.006	0.15	3.725	0.006	0.20	3.738	0.006	0.31
DIFF	0.004			-0.002			-0.008		

Table 2: σ =0.3, r=0.09, S_0=100.
CPU seconds are for computing the threshold parameters only.