Corrected Random Approximations to Free Boundary Problems in Optimal Stopping: Theory and Applications

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Outline

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- Optimal Stopping Problem for Brownian Motion
- Free Boundary Approach
- Free Boundary Formulation Examples:
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Background and Motivation

- Financial options basics
- Real options formulations
- Spacecraft trajectory control
- Sequential project selection (bandit problem)
- Sequential testing of statistical hypotheses

Option Pricing Basics:
Definitions

- Option:

  Contract giving its holder the right to buy or sell

  - a specific asset (the *underlying*)
  - at a pre–arranged price K

    (the *strike* or *exercise price*)
  - up to, or at, a pre–arranged date T

    (the *expiration* or *maturity date*)

- Call option: right to buy

  Put option: right to sell

- American option:

  can exercise right any time up to expiration date

  European option:

  can only exercise right at expiration date

- Option pricing problem:

  Value of contract (and optimal strategy for American option)

Option Pricing Basics:
Standard Options Examples

- Stock option
  - current IBM price: \$92
  - July ’04 call options currently trading with strike at \$95 are of interest to investors concerned about IBM price rising significantly over the next three months
  - July ’04 put options currently trading with strike at \$85 are of interest to investors concerned about IBM price dropping significantly over the next three months
  - American–style options can be exercised any time up to the end of July ’04
  - European–style options can only be exercised at the end of July ’04

Option Pricing Basics:

Standard Options Examples (Cont'd)

- Currency option
  - U.S. manufacturer forecasts the need for \$10M worth of printed circuit boards to be imported from Japan
  - payment is to be made in Japanese Yens in three months
  - current exchange rate: 105 JPY = 1 USD
  - the company can insure to pay no more that \$10M in three months by purchasing a put option to sell USD at the current rate
  - contract can be can exercised European–style
  - manufacturer may want to take advantage of better rates during the next three months by entering into an American–style put contract

Option Pricing Basics:

European–Style Exercise

- Payoff: IBM stock put example

  Strike $K = \$90$

  Expiration date T : Friday, July 30, 2004

- On expiration date, option exercised only if stock price $S_T < \$90$.

- Upon exercise, option holder can purchase IBM stock at S_T (\$88 say) and sell it to gain (payoff) $K - S_T = 90 - 88 = \$2$ per share.

- For call option exercise payoff is $S_T - K$

  (buy at K and resell immediately at S_T)

Option Pricing Basics:

American–Style Exercise

- Can exercise option at any time, up to maturity
- At every time before maturity ponder question:

  if I exercise now, do I get the best payoff possible?
- Exercise decision based on:

  (i) history of price up to now;

  (ii) expected stock behavior (thus expected payoffs) between now and expiration date
- Put example:

  from payoff diagram, higher payoff with lower stock price
- Maximum payoff reached when $S = 0$,

  which is unattainable (we are assuming a solid company)
- Therefore every time we consider exercising put option, we should consider the likelihood that the stock price will go further down in the time remaining till expiration.

American–Style/Early Exercise Features in Financial Engineering

- Almost all exchange–traded options offer early exercise
- Financial engineering:

  Financial institutions create financial arrangements (instruments) tailored to their clients risks
- Many of these instruments allow early exercise
- Examples:
  - Knock–out option:

    contract no longer needed (or annulled) if a barrier is breached (e.g. interest rates or currency exchange rates fall below a certain level)
  - Interest rate swaption:

    option which when exercised gives its holder (usually a company) the right to swap interest rates (e.g. fixed vs. floating).

American–Style/Early Exercise Features in Real Options

- Real option approach:

  Methodology to evaluate risky endeavors in the widest sense.

- Examples:

  - Airline lease of aircraft with option to purchase should passenger demand pick up
  - Research and development:

    value of deferring development stage by allowing more research effort

    [“ *When you make an initial investment in a research project, you are paying an entry fee for a right, but you are not obligated to continue that research at a later stage.*” Judy Lewent, CFO, Merck.]
  - Dual–fuel industrial steam boiler:

    when to switch between fuel sources in the presence of tradeable emission rights capped by government regulation.
  - Global sourcing and manufacturing flexibility.

Option Pricing Basics:

Black–Scholes–Merton Pricing Approach

- No–arbitrage argument (law of one price):

  Option price determined by value of a replicating portfolio

- Underlying (e.g. stock) assumed to follow geometric Brownian motion (lognormal process)

  $$dS_t = S_t(\mu - q)dt + S_t \sigma dW_t,$$

  where μ is the annual expected stock return, σ its corresponding standard deviation (known as the volatility), q is the annual dividend rate received from holding stock, and W_t a standard Brownian motion.

- No–arbitrage and continuous trading implication:

  Risk–neutral valuation (change of probability measure property)

  $$dS_t = S_t(r - q)dt + S_t \sigma dB_t,$$

  where r is the annual rate of return of a "riskless" asset (e.g. a savings bank account or more commonly a U.S. government short–term security) and B_t a standard Brownian motion under the new equivalent probability measure.

Option Pricing Basics:

Black–Scholes–Merton Pricing Approach (Cont'd)

- European option price:

$$U(S,t) = E\{e^{-r(T-t)} f(S_T) | S_t = S\},$$

where $f(S_T)$ is the option payoff at time T:

$$f(S_T) = \max(K - S_T, 0), \quad \text{for a put},$$

$$f(S_T) = \max(S_T - K, 0), \quad \text{for a call},$$

and E is expectation operator with respect to risk–neutral probability measure.

- Black–Scholes–Merton formulas:

$$\begin{aligned} U_{\mathrm{CALL}}(S,t) &= Se^{-q(T-t)}\Phi(d_1(S,t)) - Ke^{-r(T-t)}\Phi(d_2(S,t)), \\ U_{\mathrm{PUT}}(S,t) &= Ke^{-r(T-t)}\Phi(-d_2(S,t)) - Se^{-q(T-t)}\Phi(-d_1(S,t)), \end{aligned}$$

where

$$\begin{aligned} d_1(S,t) &= \frac{\ln(S/K) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2(S,t) &= d_1(S,t) - \sigma\sqrt{T-t}, \end{aligned}$$

and Φ is the cumulative standard normal distribution function.

Optimal Stopping Problem
for
Brownian Motion

- State Process:

  Standard Brownian motion X_t

- Actions: Stop or Continue

- Reward/Payoff function:

  Upon stopping in state z at time τ collect reward $g(z, \tau)$

- Horizon: T

  Finite or Infinite

- Objective:

  Find

  $$V(x, t) = \sup E\left\{g(X_\tau, \tau) | X_t = x\right\}$$

  over all stopping times $\tau \leq T$ (a.s.)

Free Boundary Approach

Let $\mathcal{C} = \{(x,t) : V(x,t) > g(x,t)\}$.

Then $V(x,t)$ solves

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial x^2} &= 0, && (x,t) \in \mathcal{C}, \\
V(x,t) &= g(x,t), && (x,t) \in \partial\mathcal{C}, \\
\frac{\partial V}{\partial x}(x,t) &= \frac{\partial V}{\partial x}(x,t), && (x,t) \in \partial\mathcal{C}
\end{aligned}$$

- $\mathcal{C}$ and V determined simultaneously.
- In our applications $\mathcal{C}$ is of the form

  $\mathcal{C} = \{(x,t) : x < b_1(t) \text{ and } x > b_2(t)\}$,

  where b_1 and b_2 are allowed to be infinite (not simultaneously for interesting cases) .
- Thus free boundary problem amounts to solving for

  V, $b_1(\cdot)$ and $b_2(\cdot)$.

Free Boundary Formulation:

Examples

- American option pricing
- Satellite control problem

American Option Pricing
in Black–Scholes–Merton Valuation Framework

$$U(S,t) = \sup_{\tau \in \mathcal{T}_{t,T}} E\{e^{-r(\tau - t)} f(S_\tau) | S_t = S\},$$

where $\mathcal{T}_{t,T}$ = set of all stopping times in $[t,T]$.

Recall

$$dS_t = S_t(r-q)dt + S_t \sigma dB_t,$$

Thus

$$S_t = S_0 \exp\{(r - q - \sigma^2/2)t + \sigma B_t\}$$

American Option Pricing

Optimal Stopping for Brownian Motion Formulation

Let

$$\begin{aligned} s &= t - T, \\ y &= \frac{1}{\sigma}\left[\ln S/K - \left(r - q - \frac{1}{2}\sigma^2\right) s\right], \\ V(y,s) &= Ke^{-rs}U(t(s), S(s,y)). \end{aligned}$$

Then

$$V(y,s) = \sup_{\tau \in \mathcal{T}_{s,0}} E\{g(B_\tau, \tau) | B_s = y\},$$

where

$$\begin{aligned} g(z,\tau) &= e^{-r\tau} \max(e^{(r-q-\sigma^2/2)\tau + \sigma z} - 1, 0) \text{ for a call (with } q > 0) \\ g(z,\tau) &= e^{-r\tau} \max(1 - e^{(r-q-\sigma^2/2)\tau + \sigma z}, 0) \text{ for a put} \end{aligned}$$

Bounded Variation Control Problem: Satellite Control Example

Objective: Control satellite deviation from target
in particular dimension (e.g. altitude)

- Assume target position is 0.

  Satellite position at time t: X_t, with $X_0 = x$ known.

- Perturbations modelled as standard Brownian motion $\{W_t : t \geq 0\}$

- Observation set $\mathcal{F}_t = \sigma(W_s : 0 \leq s \leq t)$

- Cost from deviation: $h(x)$ even, non-negative and strictly convex (plus additional technical conditions)

- Control process $\{\xi_t\}$ adapted to $\{\mathcal{F}_t\}$ leading to state:

$$X_t = x + W_t + \xi_t.$$

Bounded Variation Control Problem (Cont'd)

Given a horizon T, find $\{\xi_t : 0 \leq t \leq T\}$ of bounded variation that minimizes expected cost

$$E\left[\check{\xi}_T + \int_0^T h(x + W_t + \xi_t) d_t\right],$$

where, under canonical representation

$$\begin{aligned} \xi_t &= \xi_t^+ - \xi_t^-, \;\; \xi^\pm \geq 0 \;\text{ and non-decreasing,} \\ \check{\xi}_t &= \xi_t^+ + \xi_t^- \;\text{ is total variation.} \end{aligned}$$

Interpretation:

ξ_t^+: cumulative push in positive direction,

ξ_t^-: cumulative push in negative direction,

$\check{\xi}_T$: total cost of effort.

Bounded Variation Control Problem:
Free Boundary Formulation

Let

$$U(x,\tau) = \text{ minimal expected cost when current position is } x$$
$$\text{and } \tau \text{ units of time are left until } T$$

Application of DP principle and stochastic calculus lead to free–boundary problem:

Find (smooth enough) function $U(x,\tau)$ even in x and (smooth enough) positive function $b(\tau)$ such that

$$\begin{aligned}
\tfrac{1}{2}U_{xx}(x,\tau) + h(x) - U_\tau(x,\tau) &= 0, \text{ in } \overline{D}_T \\
&> 0, \text{ for } 0 < \tau \leq T,\ x > b(\tau) \\
U_\tau(x,\tau) &= 0, \text{ for } 0 \leq \tau \leq T, \\
U_x(x,\tau) &= 1, \text{ in } (\mathbb{R}^+ \times (0,T]) \backslash D_T, \\
U_{xx}(x,\tau) &> 0, \text{ in } D_T, \\
U(0,x) &= 0, \ x \in \mathbb{R}
\end{aligned}$$

where $D_T = \{(x,\tau) : 0 < \tau \leq T, 0 \leq x < b(\tau)\}$, and $\overline{D}_T = \{(x,\tau) : 0 < \tau \leq T, 0 \leq x \leq b(\tau)\}$.

The set $\{b(\tau) : 0 \leq \tau \leq T\}$ is called the (one–sided) reflecting boundary.

Bounded Variation Control Problem:

A Related Free Boundary Problem

Find (smooth enough) function $V(x,\tau)$ and (smooth enough) positive function $b(\tau)$ such that

$$\begin{aligned}
\tfrac{1}{2}V_{xx}(x,\tau)+h^{'}(x)-V_{\tau}(x,\tau) &= 0, \ \text{ in } \overline{D}_T \\
&\geq 0, \ \text{ for } 0<\tau\leq T, \ x>b(\tau) \\
V_{\tau}(x,\tau) &= 0, \ \text{ for } 0\leq\tau\leq T, \\
V_x(x,\tau) &= 1, \ \text{ in } (\mathbb{R}^{+}\times(0,T])\backslash D_T, \\
0<V_{xx}(x,\tau) &< 1, \ \text{ for } 0<\tau\leq T, \ 0<x<b(\tau), \\
V(0,x) &= 0, \ x\in\mathbb{R}
\end{aligned}$$

Bounded Variation Control Problem:

Connection to Optimal Stopping Problem

- (V, b) is a solution to an optimal stopping problem:

$$V(x,\tau) = \inf_{\sigma \in \mathcal{S}} E\left[\int_0^{S(x)\wedge\tau\wedge\sigma} h^{'}(x+W_t)dt + 1_{\{\sigma < S(x)\wedge\tau\}}\right],$$

where

$$\begin{aligned} S(x) &= \inf\{t \geq 0 : x + W_t = 0\} \\ &= +\infty, \text{ if } \{t \geq 0 : x + W_t = 0\} = \emptyset, \end{aligned}$$

and $\mathcal{S}$ is the set of all almost surely finite Brownian stopping times.

- A Solution (V, b) induces a solution (U, b) by defining U as follows:

$$\begin{aligned} U(x,\tau) = &\ \tfrac{1}{2}\textstyle\int_0^\tau V_x(0,s)ds + \int_0^x V(y,\tau)dy, && \text{in } D_T \\ = &\ U(\tau, b(\tau)) + x - b(\tau), && \text{on } \mathbb{R}_T^+ \backslash D_T \\ = &\ U(\tau, -x), && \text{for } 0 < \tau \leq T,\ x < 0 \\ = &\ 0, && \text{for } \tau = 0,\ x \in \mathbb{R}. \end{aligned}$$

- The single–sided reflecting boundary for the first problem is identical to the optimal stopping boundary for the second.

Solution Approaches
to
Optimal Stopping Problems

- Most optimal stopping problems have no closed–form solutions
- Traditional numerical solutions approaches:
  - finite differences
  - random walk approximations
- Recent developments in quasi–analytic methods:

  Ju (1998), AitSahlia and Lai (2001), AitSahlia, Imhof and Lai (2003, 2004)
- Error analysis with correction only known for random walk approximations for specific payoff functions.

Random Walk Approximations
and
Stochastic Dynamic Programming

- Recall problem is to find

$$V(x,t) = \sup E\left\{g(X_\tau,\tau)|X_t = x\right\}$$

  over all stopping times $\tau \le T$ (a.s.)

- Given interval $[0,T]$, consider subset of N possible stopping dates $0 = t_0 \le t_1 \le \cdots \le t_N = T$ such that
  $t_{i+1} - t_i = \delta (= T/N)$

- Approximate standard Brownian motion by random walk

$$S_n = \sum_1^n Z_i$$

  with i.i.d increments such that $EZ_i = 0 = EZ_i^3$, $EZ_i^2 = 1$.

- Backward induction of D.P.:

$$\begin{aligned} V_\delta(x,t_i) &= \max\Big\{g(x,t_i), E\{V_\delta(x+Z\sqrt{\delta},t_{i+1})\}\Big\}, \quad 1 \le i \le N-1 \\ V_\delta(x,t_N) &= g(x,T) \end{aligned}$$

Random Walk Approximations
and
Stochastic Dynamic Programming (Cont'd)

- Extend $V_\delta(x,t)$ by linear interpolation to approximate $V(x,t)$ for $x \in (t_i, t_{i+1})$:

$$V_\delta(x,t) = \frac{t_{i+1}-t}{t_{i+1}-t_i} V_\delta(x,t_i) + \frac{t-t_i}{t_{i+1}-t_i} V_\delta(x,t_{i+1})$$

- Z usually taken to be lattice–valued to avoid numerical integration. In this case $V_\delta(x,t)$ is only obtained for $x \in L = \{\cdots, -\sqrt{\delta}, \sqrt{\delta}, 2\sqrt{\delta}, \cdots\}$.

  Can then extend $V_\delta(x,t)$ to $x \notin L$ by interpolation.

  If $x_j < x < x_{j+1}$, for some $x_j,\ x_{j+1} \in L$, then

$$V_\delta(x,t) = \frac{x_{j+1}-x}{x_{j+1}-x_j} V_\delta(x_j,t) + \frac{x-x_j}{x_{j+1}-x_j} V_\delta(x_{j+1},t)$$

Boundary Error Estimation of Chernoff and Petkau

- Consider special stopping payoff function (*associated stopping problem*)

$$g(y, s) = -s1_{\{s<0\}} + y^2 1_{\{s=0, y\leq 0\}}.$$

- Its optimal stopping boundary is $b(s) \equiv 0$.
- Let b_δ = stopping boundary obtained through random walk approximation.

  For normal increments, Chernoff (1965) shows

$$b_\delta(s) = b(s) - .5824\sqrt{\delta} + o(\sqrt{\delta}).$$

  If increments are symmetric Bernoulli, then Chernoff–Petkau (1976) show

$$b_\delta(s) = b(s) - .5\sqrt{\delta} + o(\sqrt{\delta}).$$

Boundary Error Estimation of Chernoff and Petkau: Early Extension Attempts

- For more general increments with mean 0 and variance 1, Hogan (1986) uses renewal theory argument to show that

  $$b_\delta(s) = b(s) - \rho\sqrt{\delta} + o(\sqrt{\delta}) \tag{1}$$

  where

  $$\rho = \frac{E(S_{\tau_+}^2)}{2E(S_{\tau_+})},$$

  with $\tau_+ = \inf\{n : S_n > 0\}$, $S_n = \sum_1^n Z_i$.

- Using Taylor expansion of $V - g$ near boundary and optimality conditions at the boundary (equality of value function to payoff and smooth fit) Chernoff (1972) argues heuristically for the applicability of (1) to more general payoff function g.

Boundary Error Estimation of
Chernoff and Petkau:
Current Extension Approach

- Theorem (AitSahlia, Lai and Yao (2004)):

  $V_\delta(x,t) - V(x,t) = O(\delta)$ and $b_\delta(t) = b(t) - \rho\sqrt{\delta} + o(\sqrt{\delta}$

- proof proceeds by induction on time intervals (blocks), showing the relation to hold in each.

- First block is initialized by method other than random walk approximation (integral equation approach)

- For subsequent blocks define modifications of Chernoff's associated problem (called in our approach *canonical problem*)

- Payoff for Canonical Optimal Stopping Problem:

  $$\begin{aligned} g_\delta(y,s) = & -s + r_\delta(y,s) && \text{if } s < 0, \\ = & \; h_\delta(y) && \text{if } s = 0, \end{aligned}$$

  where for some $A > 0$,

  $$\begin{aligned} |r_\delta(y,s)| \leq & \; A\{\delta s^2 + \sqrt{\delta}|sy| + \sqrt{\delta}|y|^3\} && \text{for } y \leq -\delta^{-1/6}, \\ h_\delta(y) = & \; y^2 I\{y \leq 0\} + a_\delta(y) && \text{for } y \geq -\delta^{-1/6} \end{aligned}$$

  with $|a_\delta(y)| \leq A$.

Boundary Error Estimation of
Chernoff and Petkau:
Current Extension Approach (Cont'd)

Two main steps in proof:

Lemma.

Let $G(x,t) = (\partial/\partial t + \frac{1}{2}\partial^2/\partial x^2)g(x,t)$.

Fix $t^* < T$ and let $x^* = b(t^*)$.

Define the quadratic function

$$\begin{aligned}\pi_g(x,t) = g(x^*,t^*) + \frac{\partial g}{\partial x}(x^*,t^*)(x-x^*) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(x^*,t^*)(x-x^*)^2 \\ - \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(x^*,t^*)(t-t^*).\end{aligned}$$

Letting $y = \delta^{-1/2}(x-x^*)$, $s = \delta^{-1}(t-t^*)$, define

$$g_\delta(y,s) = \begin{cases} \delta^{-1}\{g(x,t) - \pi_g(x,t)\} & \text{if } t < t^*, \\ \delta^{-1}\{v(x,t) - \pi_g(x,t)\}I\{y \le 0\} + C_\delta(y) & \text{if } t = t^*, \end{cases}$$

where $|C_\delta(y)|$ is bounded by some positive constant A. Then $g_\delta/|G(x^*,t^*)|$ is a payoff for the canonical stopping problem.

Boundary Error Estimation of Chernoff and Petkau: Current Extension Approach (Cont'd)

Theorem. For $\epsilon > 0$,

$$V(x,t) = E_{x,t}V(B_{T-\epsilon}, T-\epsilon) - \int_t^{T-\epsilon} E_{x,t}[G(B_s,s)I\{B_s \geq b(s)\}]ds.$$

If g is left–continuous at T, then $\lim_{\epsilon \downarrow 0} v(x, T-\epsilon) = g(\cdot, T)$, so letting $\epsilon \downarrow 0$ yields

$$\begin{aligned} V(x,t) &= E_{x,t}g(B_T,T) \\ &\quad - \int_t^T E[G(x+\sqrt{s-t}Z, s)I\{x+\sqrt{s-t}Z \geq b(s)\}]ds, \quad (2) \end{aligned}$$

where Z is a standard normal random variable.

In particular, for American options, (2) corresponds to the decomposition formula due to Jacka (1991) and Carr, Jarrow and Myneni (1992):

$$\begin{aligned} \text{American option price} &= \text{European option price} \\ &\quad + \text{Early exercise premium}, \end{aligned}$$

in which the integrand in the integral defining the early exercise premium is an explicit function of the optimal stopping boundary.

Boundary Continuity Correction of Chernoff and Petkau

- Recall DP algorithm:

$$\begin{aligned} V_\delta(x,t_i) &= \max\left\{g(x,t_i), E\{V_\delta(x+Z\sqrt{\delta},t_{i+1})\}\right\}, \quad 1 \le i \le N-1 \\ V_\delta(x,t_N) &= g(x,T) \end{aligned}$$

- In order to retrieve continuous-time boundary from discrete–time boundary via error formula, one needs estimate of latter.

- When applied to lattice-based random walk, DP algorithm only identifies stopping points (where $V_\delta(x_j,t_i)=g(x_j,t_i)$) and continuation points (where $V_\delta(x_j,t_i)E\{V_\delta(x+Z\sqrt{\delta},t_{i+1})\} > g(x_j,t_i)$), not necessarily the actual optimal stopping boundary with discrete time steps.

- This boundary can be estimated via extrapolation approaches

- Chernoff and Petkau (1986) develop an approach that bypasses the need to estimate actual optimal stopping boundary with discrete time steps.

Boundary Continuity Correction of Chernoff and Petkau (Cont'd)

- For discrete time t, let $x^0(t)$, $x^1(t) \in \{0, \pm\sqrt{\delta}, \pm 2\sqrt{\delta}, \dots\}$ be, respectively, the first and second closest grid points in the continuation region to a stopping point on the grid.

- For $k = 0, 1$ define:

$$D_k(t) = V_\delta(x^k(t), t) - g(x^k, t)$$

- Then continuous-time boundary at t estimated as

$$b(t) = x^0(t) \pm \left| D_1(t) / \left(2D_1(t) - 4D_0(t)\right) \right| \sqrt{\delta},$$

  where the sign is determined so as to make the continuation region of the continous–time larger (e.g. - for a put and + for a call).

- This corrected boundary satisfies $b_\delta(t) = b(t) - 0.5\sqrt{\delta} + o(\sqrt{\delta})$ for Z such that $P(Z = 1) = 1 - P(Z = -1) = 1/2$.

Benchmarking the Correction Method: Monotone Follower Problem

Given a horizon T, find non-decreasing function $\{\xi_t : 0 \leq t \leq T\}$ (with $\xi_0 = 0$) adapted to Brownian motion filtration that minimizes expected cost

$$E\left[\check{\xi}_T + \int_0^T h(x + W_t - \xi_t) d_t\right].$$

- When $h(x) = x^2$ this problem has a known explicit solution (Benes, Shepp and Witsenhausen (1980))
- It is reduceable to an optimal stopping problem
- We apply random walk approximation to stopping problem and Chernoff–Petkau correction
- Then compare uncorrected discrete boundary and corrected boundary vs. closed–form solution.

Application of the Correction Method:
American Option Pricing

- Applied to standard and barrier–type American options
- Application first led to support fact that optimal exercise boundaries are well approximated by piecewise linear functions with only a few knots (typically 3 or 4) for overwhelming majority of parameters of practical interest.
- In turn, these approximations led to efficient and accurate pricing of American options (cf. AitSahlia and Lai (2001), AitSahlia, Imhof and Lai (2003), AitSahlia, Imhof and Lai (2004)).

Conclusions

- Optimal stopping problems are prevalent in many areas of practical interest
- Can be directly or indirectly formulated
- Most have not been solved in closed–form
- Provided here a generalization of error estimation for random–walk based approximations
- Applied corrected random walk approximations to price efficiently American options.