Stochastic optimization with stochastic dominance constraints

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Utility Models

Expected Utility Models (von Neumann and Morgenstern, 1947)

$$\max_{z \in Z} \mathbb{E}\left[u(G(z))\right]$$

Z - subset of an appropriate vector space $\mathcal{Z}$

$(\Omega, \mathcal{F}, P)$ - probability space

$G : \mathcal{Z} \to \mathcal{L}_1(\Omega, \mathcal{F}, P; \mathbb{R})$ - random outcome

$u(\cdot)$ - utility function (non-decreasing)

Rank Dependent Utility (Quiggin 1982, Schmeidler 1986/89, Yaari 1987)

Inverse distribution function $F_{(-1)}(X;p) = \inf\,\{\eta : F_1(X;\eta) \geq p\} \quad 0 < p < 1$

Rank dependent utility $w : [0,1] \to \mathbb{R}$ nondecreasing continuous function such that $X \succeq Y$ if and only if

$$\int_0^1 F_{(-1)}(X;p)\, dw(p) \geq \int_0^1 F_{(-1)}(Y;p)\, dw(p)$$

Stochastic Dominance Constraints

$$\begin{aligned} \max\ & \mathbb{E}\Big[H(z)\Big] \\ \text{subject to}\ & G(z) \succeq_{(k)} Y \\ & z \in Z \end{aligned}$$

Y - reference random outcome $k \geq 1$

Z - convex subset of a separable locally convex Hausdorff vector space $\mathcal{Z}$

G and H − continuous operators from $\mathcal{Z}$ to the space $\mathcal{L}_1(\Omega, \mathcal{F}, P; \mathbb{R})$

The Dominance Constraint for $k = 2$ reflects risk aversion:

$G(z)$ is preferred over Y by all risk-averse decision makers

Stochastic Dominance

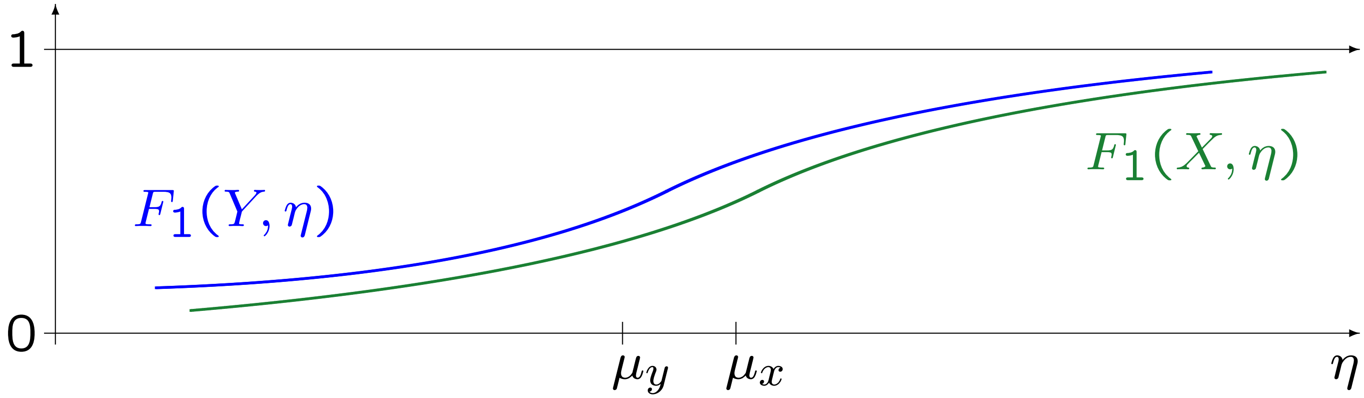
Distribution Functions

$$\begin{aligned} F_1(X,\eta) &= \int_{-\infty}^{\eta} P_X(d\xi) = \mathbb{P}\{X \le \eta\} \quad \text{for } \eta \in \mathbb{R} \\ F_k(X,\eta) &= \int_{-\infty}^{\eta} F_{k-1}(X,\xi)\, d\xi \quad \text{for } \eta \in \mathbb{R}, \quad k = 2, 3, \ldots \end{aligned}$$

kth degree Stochastic Dominance (kSD)

$$X \succeq_{(k)} Y \quad \Leftrightarrow \quad F_k(X,\eta) \le F_k(Y,\eta) \quad \text{for all } \eta \in \mathbb{R}$$

First Order Dominance



Relation to utility functions:

$$X \succeq_{(1)} Y \;\Leftrightarrow\; \mathbb{E}\,u(X) \geq \mathbb{E}\,u(Y) \quad \forall \textit{ nondecreasing } u(\cdot)$$

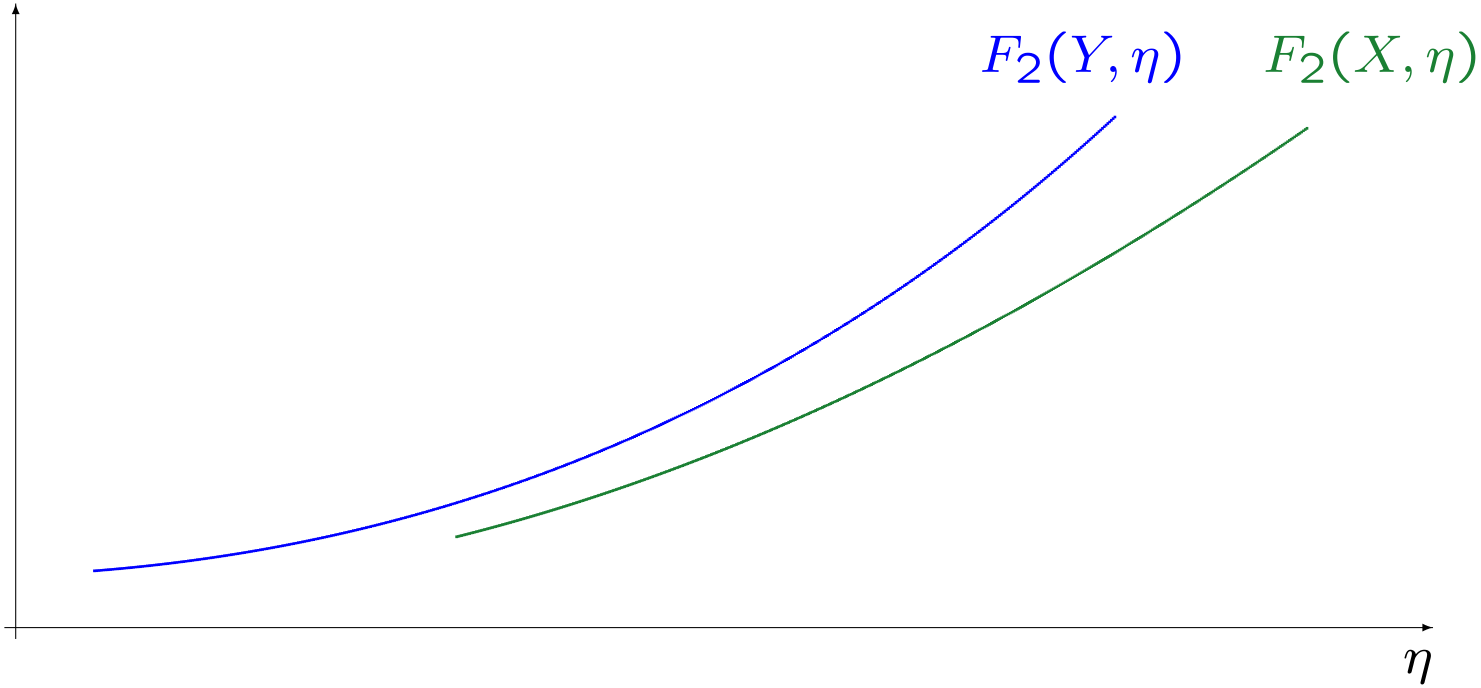
Quirk and Saposnik (1962)

First order dominance $\equiv$ Continuum of probabilistic constraints

$$X \succeq_{(1)} Y \quad \Leftrightarrow \quad F_{(-1)}(X;p) \geq F_{(-1)}(Y;p) \quad \text{for all} \quad 0 < p < 1.$$

Second-Order Dominance

$$F_2(X,\eta) = \int_{-\infty}^{\eta} F_1(X,\xi)\,d\xi = \mathbb{E}(\eta - X)_+ \quad \text{for } \eta \in \mathbb{R}$$



Risk-Averse Consistency:

$$X \succeq_{(2)} Y \;\Leftrightarrow\; \mathbb{E}\,u(X) \geq \mathbb{E}\,u(Y) \quad \forall \text{ nondecreasing concave } u(\cdot)$$

Hadar and Russell (1969)

Inverse Dominance

Absolute Lorenz function $F_{(-2)}(X;\cdot) : \mathbb{R} \to \bar{\mathbb{R}}$:

$$F_{(-2)}(X;p) = \int_0^p F_{(-1)}(X;t)\,dt \quad \text{for} \quad 0 < p \leq 1,$$

$$F_{(-2)}(X;0) = 0 \quad \text{and} \quad F_{(-2)}(X;p) = +\infty \text{ for } p \notin [0,1]$$

$$X \succeq_{(2)} Y \quad \Leftrightarrow \quad F_{(-2)}(X;p) \geq F_{(-2)}(Y;p) \quad \text{for all} \quad 0 \leq p \leq 1.$$

The set $\mathcal{W}_0$ contains all continuous nondecreasing functions $w : [0,1] \to \mathbb{R}$ and $\mathcal{W}_1 \subset \mathcal{W}_0$ contains all concave subdifferentiable at 0 functions.

Theorem 1

(i) For any two random variables $X, Y \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ the relation $X \succeq_{(1)} Y$ holds true if and only if for all $w \in \mathcal{W}_0$

$$\int_0^1 F_{(-1)}(X;p)\,dw(p) \geq \int_0^1 F_{(-1)}(Y;p)\,dw(p) \tag{1}$$

(ii) $X \succeq_{(2)} Y$ holds true if and only if (1) is satisfied for all $w \in \mathcal{W}_1$.

Sets defined by dominance constraints

For all $k \geq 1$
Y - benchmark outcome in $\mathcal{L}_k(\Omega, \mathcal{F}, P)$
$[a, b] \subseteq \mathbb{R}$
we define the set

$$A_k(Y; [a, b]) = \{X \in \mathcal{L}_{k-1}(\Omega, \mathcal{F}, P) : X \succeq_{(k)} Y \text{ in } [a, b]\}$$

Proposition 1 The set $A_k(Y; [a, b])$ is convex and closed for all $[a, b]$, all Y, and $k \geq 2$. Its recession cone has the form

$$A_k^{\infty}(Y; [a, b]) = \{H \in \mathcal{L}_{k-1}(\Omega, \mathcal{F}, P) : H \geq 0 \quad \text{a.s. on } [a, b]\}$$

The set $A_1(Y; [a, b])$ is closed and

$$A_k(Y; [a, b]) \subseteq A_{k+1}(Y; [a, b]) \qquad \text{for all } k \geq 1.$$

$A_k(Y; [a, b])$ is a cone pointed at Y if and only if Y is a constant in $[a, b]$.

Inverse dominance constrained set

$$B(Y;[\alpha,\beta]) = \left\{X \in \mathcal{L}^1(\Omega,\mathcal{F},P) : F_{(-2)}(X;p) \geq F_{(-2)}(Y;p)\ \ \forall\ p \in [\alpha,\beta]\right\}.$$

Proposition 2 If $[a,b]$ contains all p-quantiles of Y for $p \in [\alpha,\beta]$, then

$$A(Y;[a,b]) \subset B(Y;[\alpha,\beta]).$$

Proposition 3 For every $p \in (0,1)$ the function $F_{(-2)}(\cdot;p)$ is concave and positive homogeneous on $\mathcal{L}^1(\Omega,\mathcal{F},P)$, and the set $B(Y;[\alpha,\beta])$ is convex and closed.

Dominance Constrained Problems

Given Y_i - benchmark random outcomes $i = 1..m$

Primal Stochastic Dominance Constraints

$$\begin{aligned} &\max\ \mathbb{E}\Big[H(z)\Big] \\ &\text{subject to } F_{(2)}(G_i(z);\eta) \geq F_{(2)}(Y_i;\eta), \quad i = 1..m \\ &\qquad \forall\ \eta \in [a,b], \\ &\qquad z \in Z \end{aligned}$$

Inverse Stochastic Dominance Constraints

$$\begin{aligned} &\max\ \mathbb{E}\Big[H(z)\Big] \\ &\text{subject to } F_{(-2)}(G_i(z);p) \geq F_{(-2)}(Y_i;p), \quad i = 1..m \\ &\qquad \forall\ p \in [\alpha,\beta], \\ &\qquad z \in Z \end{aligned}$$

$[\alpha,\beta] \subset (0,1)$ and $[a,b] \subset \mathbb{R}$

Portfolio Example

Assets $j = 1, \ldots, n$ with random returns R_j

Decision variables z_j, $j = 1, \ldots, n$, Z -simplex

Portfolio return $H(z) = G(z) = \sum_{j=1}^{n} z_j R_j$

Reference return Y (e.g. index, existing portfolio, etc.)

$$\max \; \mathbb{E}\Big[\sum_{j=1}^{n} z_j R_j \Big]$$

$$\text{subject to } \sum_{j=1}^{n} z_j R_j \succeq_{(2)} Y, \quad i = 1..m$$

$$z \in Z$$

All statements are equivalent:

$$\sum_{j=1}^{n} z_j R_j \succeq_{(2)} Y$$

$$F_{(-2)}\Big(\sum_{j=1}^{n} z_j R_j ; p\Big) \geq F_{(-2)}(Y;p) \quad \text{for all} \quad p \in [0,1]$$

continuum of CVaR (integrated chance) constraints for risk levels $p \in [0,1]$

$$\mathbb{E}u\Big(\sum_{j=1}^{n} z_j R_j\Big) \geq \mathbb{E}u(Y)$$

for all concave nondecreasing functions u (Von Neuman-Morgenstern utility)

$$\int_0^1 F_{(-1)}\Big(\sum_{j=1}^{n} z_j R_j ; p\Big)\, dw(p) \geq \int_0^1 F_{(-1)}(Y;p)\, dw(p)$$

for all concave nondecreasing functions w (rank dependent utility)

Abstract Problem Formulation

$$\begin{aligned} &\max\ f(X) \\ \text{subject to } &X \succeq_{(k)} Y \text{ in } [a,b] \qquad\qquad (\mathbf{P_k}) \\ &X \in C \subseteq \mathcal{L}_{k-1}(\Omega, \mathcal{F}, P) \end{aligned}$$

Main Results

- Concave nondecreasing functions $u(\cdot)$ play the role of Lagrange multipliers associated with the dominance constraint
- The function

  $$L(X,u) = f(X) + \mathbb{E}\Big[u(X)\Big] - \mathbb{E}\Big[u(Y)\Big]$$

  plays the role of the Lagrangian

The set $\mathcal{U}_k$ contains all functions $u : \mathbb{R} \to \mathbb{R}$, for which there exists a nonnegative, nonincreasing, left-continuous and bounded function $\varphi : [a,b] \to \mathbb{R}$ such that

$$\begin{aligned} &u^{(k-1)}(t) = (-1)^k \varphi(t), \quad \text{for a.a. } t \in [a,b] \\ &u^{(k-1)}(t) = (-1)^k \varphi(a), \quad \text{for } t < a \\ &u(t) = 0, \quad \text{for } t \geq b \\ &u^{(i)}(b) = 0, \quad i = 1, \dots, k-2 \text{ if } k \geq 2. \end{aligned}$$

The Uniform k-Dominance Condition: there exists $\tilde{X} \in C$ such that

$$\inf_{\eta \in [a,b]} \left\{ F_k(Y;\eta) - F_k(\tilde{X};\eta) \right\} > 0$$

Theorem 2

Assume that the Uniform k-Dominance Condition is satisfied. If $\hat{X}$ is an optimal solution of problem $(\mathbf{P_k})$ then there exists a function $\hat{u} \in \mathcal{U}_{k-1}$ such that

$$L(\hat{X}, \hat{u}) = \max_{X \in C} L(X, \hat{u}) \tag{2}$$

$$\mathbb{E}\left[\hat{u}(\hat{X})\right] = \mathbb{E}\left[\hat{u}(Y)\right] \tag{3}$$

Conversely, if for some function $\hat{u} \in \mathcal{U}_{k-1}$ an optimal solution $\hat{X}$ of (2) satisfies the dominance constraint in $(\mathbf{P_k})$ and (3), then $\hat{X}$ is an optimal solution of problem $(\mathbf{P_k})$.

Duality

The Lagrangian

$$L(X,u) = f(X) + \mathbb{E}[u(X)] - \mathbb{E}[u(Y)]$$

The dual functional

$$D(u) = \sup_{X \in C} L(X,u)$$

The dual problem

$$\min_{u \in \mathcal{U}_k} D(u)$$

Theorem 3

Assume that the uniform dominance condition is satisfied and problem $(\mathbf{P_k})$ has an optimal solution. Then the dual problem has an optimal solution and the optimal values of both problems coincide. Furthermore, the set of optimal solutions of the dual problem is the set of functions $\hat{u} \in \mathcal{U}_k$ satisfying (2)–(3) for an optimal solution $\hat{X}$ of problem $(\mathbf{P_k})$.

Problem with Inverse Stochastic Dominance Constraint

$$\begin{aligned} &\max\ f(X) \\ &\text{subject to } F_{(-2)}(X;p) \geq F_{(-2)}(Y;p) \quad \forall\ p \in [\alpha,\beta] \qquad\qquad \mathbf{PID} \\ &\qquad X \in C \subseteq \mathcal{L}_1(\Omega,\mathcal{F},P) \end{aligned}$$

$$[\alpha,\beta] \subset (0,1)$$

Main Results

- Concave nondecreasing functions $w : [0,1] \to \mathbb{R}$ play the role of Lagrange multipliers associated with the inverse dominance constraint
- The function $\Phi : C \times \mathcal{W}_1([\alpha,\beta]) \to \mathbb{R}$

$$\Phi(X,w) = f(X) + \int_0^1 F_{(-1)}(X;p)\,dw(p) - \int_0^1 F_{(-1)}(Y;p)\,dw(p)$$

  plays the role of the Lagrangian

The set $\mathcal{W}_1([\alpha,\beta])$ contains functions $w : [0,1] \to \mathbb{R}$ such that:

$w(\cdot)$ is concave and nondecreasing;
$w(p) = 0$ for all $p \in [\beta,1]$;
$w(p) = w(\alpha) + c(p-\alpha)$, with some $c > 0$, for all $p \in [0,\alpha]$.

Problem (PID) satisfies the *uniform inverse dominance condition* if there exists a point $\tilde{X} \in C$ such that

$$\inf_{p \in [\alpha, \beta]} \left\{ F_{(-2)}(\tilde{X}; p) - F_{(-2)}(Y; p) \right\} > 0$$

Theorem 4 Assume the uniform inverse dominance condition.
If $\hat{X}$ is an optimal solution of problem (PID), then there exists a function $\hat{w} \in \mathcal{W}_1([\alpha, \beta])$ such that

$$\Phi(\hat{X}, \hat{w}) = \max_{X \in C} \Phi(X, \hat{w}) \tag{4}$$

and

$$\int_0^1 F_{(-1)}(\hat{X}; p)\, d\hat{w}(p) = \int_0^1 F_{(-1)}(Y; p)\, d\hat{w}(p) \tag{5}$$

If for some function $\hat{w} \in \mathcal{W}_1([\alpha, \beta])$ an optimal solution $\hat{X}$ of (4) satisfies the inverse dominance constraints and (5), then $\hat{X}$ is an optimal solution of problem (PID).

Duality theory

For every function $w \in \mathcal{W}_1([\alpha,\beta])$ the problem

$$\max_{X \in C} \left\{ f(X) + \int_0^1 F_{(-1)}(X;p)\,dw(p) - \int_0^1 F_{(-1)}(Y;p)\,dw(p) \right\} \tag{6}$$

is a Lagrangian relaxation of problem (PID).

The dual functional $\Psi(w) = \sup_{X \in C} \Phi(X,w)$

The dual problem $\min_{w \in \mathcal{W}_1([\alpha,\beta])} \Psi(w)$

$\mathcal{W}_1([\alpha,\beta])$ is a closed convex cone and $\Psi(\cdot)$ is a convex functional

Theorem 5 Under the uniform inverse dominance condition if problem (PID) has an optimal solution, the dual problem has an optimal solution and the optimal values of both problems coincide. Furthermore, the set of optimal solutions of the dual problem is the set of functions $\widehat{w} \in \mathcal{W}_1([\alpha,\beta])$ satisfying (4)–(5) for an optimal solution $\hat{X}$ of the problem.

Nonlinear Second Order Dominance Constraints

Problem (SSD)

$$\begin{aligned} \text{max} \quad & \mathbb{E}[H(z)] \\ \text{subject to} \quad & G_i(z) \succeq_{(2)} Y_i \quad \text{in} \quad [a_i, b_i] \quad i = 1, \ldots, m \\ & z \in Z \end{aligned}$$

Z - convex subset of a separable locally convex Hausdorff vector space $\mathcal{Z}$

G_i and H – continuous operators from $\mathcal{Z}$ to the space $\mathcal{L}_1(\Omega, \mathcal{F}, P; \mathbb{R})$

G_i and H are concave in the following sense: for P-almost all $\omega \in \Omega$ the functions $[G_i(\cdot)](\omega)$, $i = 1, \ldots, m$, and $[H(\cdot)](\omega)$ are concave and continuous

Lagrangian

We define the set $\mathcal{U}_2([a,b])$ of functions $u(\cdot)$ such that:

$$\begin{aligned}&u(\cdot) \text{ is concave and nondecreasing;}\\&u(t)=0 \text{ for all } t\ge b;\\&u(t)=u(a)+c(t-a),\ \text{with some } c>0, \text{ for all } t\le a.\end{aligned}$$

We denote the convex cone

$$\mathcal{U}_2^m = \mathcal{U}_2([a_1,b_1]) \times \cdots \times \mathcal{U}_2([a_m,b_m])$$

The Lagrangian, $L : \mathcal{Z} \times \mathcal{U}_2^m \to \mathbb{R}$,
associated with problem (SSD) takes on the form:

$$L(z,u) := \mathbb{E}\Big[H(z) + \sum_{i=1}^{m} \Big(u_i(G_i(z)) - u_i(Y_i)\Big)\Big]$$

Optimality Conditions

Definition 1 Problem (SSD) satisfies the Uniform Dominance Condition if there exists a point $\tilde{z} \in Z$ such that

$$\inf_{\eta \in [a_i, b_i]} \Big\{ F_2(Y_i;\eta) - F_2(G_i(\tilde{z});\eta) \Big\} > 0, \quad i = 1, \ldots, m$$

Theorem 6

Assume that the uniform dominance condition is satisfied. If $\hat{z}$ is an optimal solution of problem (SSD) then there exist $\hat{u} \in \mathcal{U}_2^m$ such that

$$L(\hat{z}, \hat{u}) = \max_{z \in Z} L(z, \hat{u}) \tag{7}$$

$$\mathbb{E}\Big[\hat{u}_i(G_i(\hat{z})\Big] = \mathbb{E}\Big[\hat{u}_i(Y_i)\Big], \quad i = 1, \ldots, m \tag{8}$$

Conversely, if for some function $\hat{u} \in \mathcal{U}_2^m$ an optimal solution $\hat{z}$ of (7) is feasible for (SSD) and satisfies (8), then $\hat{z}$ is an optimal solution of (SSD).

Duality

The dual functional $D : \mathcal{U}_2^m \to \overline{\mathbb{R}}$

$$\begin{aligned} D(u) &:= \sup_{z \in Z} L(z,u) \\ &= \sup_{z \in Z} \mathbb{E}\Big[H(z) + \sum_{i=1}^{m} \Big(u_i(G_i(z)) - u_i(Y_i) \Big) \Big] \end{aligned}$$

The dual problem

$$\min \; \{ D(u) : u \in \mathcal{U}_2^m \}$$

Theorem 7 Assume that the uniform dominance condition is satisfied. If problem (SSD) has an optimal solution, then the dual problem has an optimal solution and the optimal values of both problems coincide. Furthermore, for every solution $\hat{u}$ of the dual problem, any maximizer $\hat{z}$ of the Lagrangian, which is feasible for (SSD) and satisfies the complementarity conditions, is an optimal solution of the primal problem (SSD).

Discrete Distributions

$\Omega = \{\omega_1, \ldots, \omega_n\}$ with probabilities $p_j = P(\{\omega_j\})$, $j = 1, \ldots, n$
$J = \{1, \ldots, n\}$, $I = \{1, \ldots, m\}$

$$h_j(z) = H(z)(\omega_j),\ \ g_{ij}(z) = G_i(z)(\omega_j),\ \ y_{ij} = Y_i(\omega_j),\ \ x_{ij} = X_i(\omega_j)$$

Problem Formulation

$$\begin{aligned} \text{max} \quad & \sum_{j=1}^{n} p_j h_j(z) \\ \text{subject to} \quad & \sum_{j=1}^{n} p_j (y_{ik} - x_{ij})_+ \le \sum_{j=1}^{n} p_j (y_{ik} - y_{ij})_+, \quad i \in I, \quad k \in J \\ & x_{ik} \le g_{ik}(z), \quad i \in I, \quad k \in J \\ & z \in Z \end{aligned}$$

Optimality for Discrete Distributions

Slater Condition: there exist $\tilde{z} \in \operatorname{relint} Z$ and $\tilde{X}_i$, $i \in I$, such that

$$\tilde{x}_{ik} < g_{ik}(\tilde{z}), \quad i \in I, \quad k \in J$$

and the dominance constraints are satisfied.

The Lagrangian

$$L(z, X, u, \theta) = \sum_{j=1}^{n} p_j \Big[h_j(z) + \sum_{i=1}^{m} \theta_{ij} g_{ij}(z) \Big] + \sum_{i=1}^{m} \sum_{j=1}^{n} p_j \Big[u_i(x_{ij}) - u_i(y_{ij}) - \theta_{ij} x_{ij} \Big]$$

Optimality and duality conditions remain valid under the Slater Condition.

The utility functions corresponding to the ith group of dominance constraints are concave nondecreasing functions $u_i(\cdot)$ which are piecewise–linear with break points at y_{ik}, $k \in J$.

Remark 1 The inverse dominance constraint does not reduce to finitely many constraints except in the case of equal probabilities.

Application to Portfolio Optimization

Assets $j = 1, \ldots, N$ with random returns R_j
Decision variables z_j, $j = 1, \ldots, N$, Z -simplex
Portfolio return $G(z) = \sum_{j=1}^{N} z_j R_j$
Reference return Y (e.g. index, existing portfolio, etc)

$$\max \mathbb{E}\Big[\sum_{j=1}^{n} z_j R_j\Big]$$

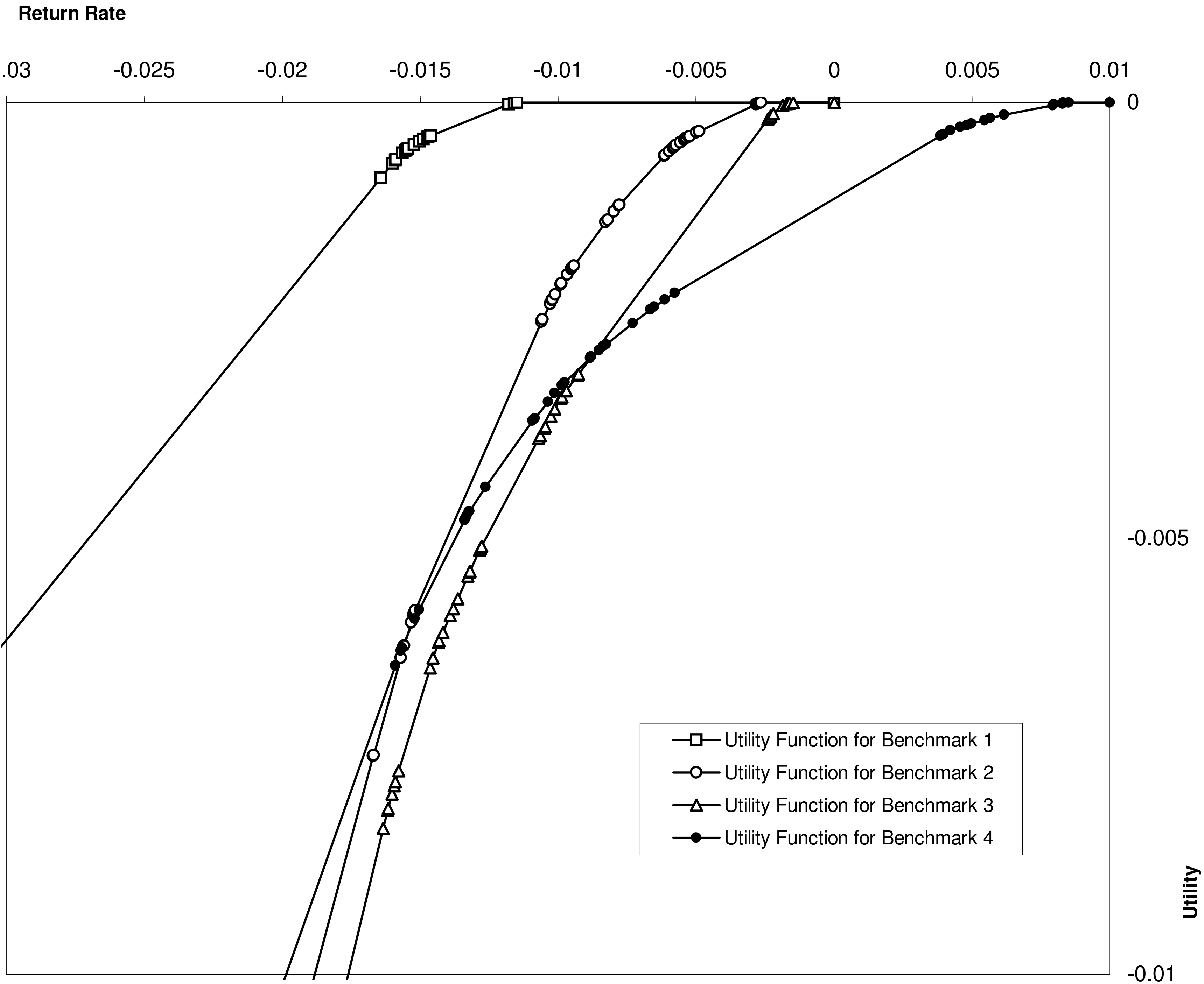
$$\text{subject to } \sum_{j=1}^{n} z_j R_j \succeq_{(2)} Y, \quad i = 1..m$$

$$z \in Z$$

719 real-world assets, 616 possible realizations of their joint returns.

Benchmarck Portfolios:

equally weighted indexes composed of N of our assets
Portfolio 1 corresponds to $N = 26$,
Portfolio 2 corresponds to $N = 54$,
Portfolio 3 corresponds to $N = 82$,
and Portfolio 4 corresponds to $N = 200$.


Return Rate
-0.03
-0.025
-0.02
-0.015
-0.01
-0.005
0
0.005
0.01
0
-0.005
-0.01
Utility
Utility Function for Benchmark 1
Utility Function for Benchmark 2
Utility Function for Benchmark 3
Utility Function for Benchmark 4

Conclusion

- Von Neuman-Morgenstern Utility Optimization and Rand Dependent Utility Optimization are dual problems to the Problem with Stochastic Dominance Constraints.

- When the stochastic dominance is expressed in primal form via integrated distribution functions, the Lagrange multiplies to the dominance constraints are von Neuman-Morgenstern utility functions.

- When the dominance constrains is expressed via the Lorenz curve then the Lagrange multiplies to the dominance constraints are rank dependent utility functions.

- In finite probability space the stochastic dominance constraints in primal form reduces to finitely many constraints.