# A Toll Pricing Framework for Traffic Assignment Problems with Elastic Demand

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Donald W. Hearn
hearn@ise.ufl.edu

Mehmet B. Yildirim
ybayram@cao.ise.ufl.edu

Center for Applied Optimization
Industrial and Systems Engineering Department
University of Florida

**Abstract:** This paper extends the notion of toll pricing and the toll pricing framework previously developed for fixed demand traffic assignment [4, 13] to the problem with elastic demand. The system problem maximizes net benefit to the network users [9, 20] and the user problem is the usual one of finding equilibrium with elastic demand. We define and characterize $\mathcal{T}$, the set of all tolls for the user problem that achieve the system optimal solution. When solutions to the two problems are unique, $\mathcal{T}$ is a polyhedron defined by the optimal solution of the system problem, similar to the case in [4, 13]. The Toll Pricing Framework in [13] is also extended to allow optimization of secondary criteria over $\mathcal{T}$. Examples include minimizing the number of toll booths and minimizing the maximum toll on any link. A numerical example illustrates the results.

**Keywords:** Congestion Toll Pricing, Road Pricing, Elastic Demand Traffic Assignment

## 1 Introduction

Motivation for congestion toll pricing stems from the fact that traffic jams and congestion on the roads are major problems that have direct economic impact on metropolitan areas in all countries. A recent study estimates that Bangkok loses one third of its potential output due to congestion [5]. In 1994, Arnott and Small [1] estimated that the total cost of delay in 39 metropolitan areas of United States is \$48 billion per year, an amount that translates to \$640 per driver.

Economists [1, 2] believe that roads are underpriced resources because users only experience their own traffic delays as their cost. To improve system utilization, the standard proposal is to charge (toll) individual users for the delays (externalities) they impose on other users. When developed in the mathematical context of traffic assignment (TA) models, it leads to a specific formula for what are known as *marginal social cost pricing* (MSCP) tolls. A simple derivation shows that when MSCP tolls are imposed, the equilibrium model for user behavior has a solution which agrees precisely with the solution of another (untolled) TA model that measures total impact on the system. The first of these models is known as the user optimal/equilibrium model and the second as the system optimal model. References on TA models include Sheffi [18], Patriksson [16] and Florian and Hearn [8].

[1]Date: January 3, 2002. *To appear in the edited volume "Current Trends in Transportation and Network Analysis — papers in honor of Michael Florian"*

It is now recognized that MSCP tolls are simply one possibility for defining tolls to achieve a system optimal solution. Bergendorff, Hearn and Ramana [4] introduce a notion of alternative tolls and formally characterize the *toll set* for fixed demand TA models. A primary result is that under typical uniqueness conditions for the fixed demand TA problem, the toll set is a polyhedron. When the user problem is perturbed by any toll vector in the toll set, the user optimal solution is the system optimal solution. Hearn and Ramana [13] further define a Toll Pricing Framework where the determination of specific tolls can be carried out by optimization methods such as linear or integer programming. The approach of defining alternative tolls to obtain system optimal flows is closely related to that of constraining user optimal flows to satisfy bound constraints. That approach appears in [10, 11, 3] and more recently in [14].

This paper extends results in [4, 13] to TA problems with elastic demand. The system problem is taken to be that of maximizing net benefit of the network users [9, 20] and the user equilibrium problem is the standard elastic demand TA problem. As before, when solutions to the two problems are unique, the toll set is a polyhedron defined by the optimal solution of the system problem. A new property, unique to the elastic demand case, is that all valid tolls generate the same total revenue. Thus MSCP tolls, in the aggregate, are no more expensive than other tolling schemes [14]. (This is very much not the situation in the fixed demand case, where the differences can be significant as shown in [4, 13].) However, MSCP tolls have other disadvantages, and we extend the Toll Pricing Framework [13] to allow optimization of secondary criteria over the toll set. Examples include minimizing the number of toll booths and minimizing the maximum toll on any link. A previously employed numerical example [13] is modified with linear demand functions to illustrate the results.

# 2 Traffic Assignment Models with Elastic Demand

This section introduces notation and then defines the system and user equilibrium models. The system model assumes that the goal of transportation planners is to maximize net user benefit [9, 20], while the equilibrium problem models the usual Wardrop principle [22] of users choosing routes from which there is no unilateral improvement available.

## 2.1 Notation

Employing notation similar to [4, 13], let $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ be a directed network with $\mathcal{N}$ being the node set and $\mathcal{A}$ being the set of links, and denote the incidence matrix of $\mathcal{G}$ by $A$. Let $\mathcal{K}$ be a set indexing the origin-destination pairs $(o, d)$ where both $o \in \mathcal{N}$ and $d \in \mathcal{N}$. Then we may write $k = (o, d)$ for $k \in \mathcal{K}$ and refer to $k$ as a "commodity." The demand for travel from some origin $o$ to destination $d$ is expressed as $t_k(c_k)$ where $t_k$ is a nonnegative invertible demand function of $c_k$, the generalized cost of travel for commodity $k$. For brevity we write $t_k$ rather than $t_k(c_k)$, and denote the inverse of $t_k$ by $w_k(t_k)$. The $t_k$ are components of the vector $t$ and the inverse demand functions will be the components of the vector function $w(t)$. The $k$th *commodity flow* vector is denoted by the variable $x^k$, and the sum of all the commodity flow vectors is the *aggregate flow* vector $v$. We assume that a continuously differentiable cost map $s : \mathcal{A} \to \mathcal{A}$ is given. When the aggregate flow in the network is $v$, the cost incurred by a user on link $a$ is given by $s_a(v)$. The Jacobian of $s$ is denoted by $\nabla s$. Note that $\nabla_{x^k} s = \nabla_v s$ since $v = \sum_k x^k$.

The system defining the feasible flows and demands is given by:

$$
\begin{array}{rcll}
v & = & \sum_{k \in \mathcal{K}} x^k & \\
Ax^k & = & t_k E_k & \forall k \in \mathcal{K} \\
x^k & \geq & 0 & \forall k \in \mathcal{K} \\
t_k & \geq & 0 & \forall k \in \mathcal{K}.
\end{array}
$$

where $E_k = e_o - e_d$, and $e_o$ and $e_d$ are unit vectors. $E_k$ is thus a column incidence vector for commodity $k = (o, d)$ with $+1$ in position $o$ and $-1$ in position $d$. We define the set of aggregate feasible flows and demands as

$$V = \{(v, t) | \text{ there exists } x^k \text{ satisfying the system above } \}.$$

This is a *node-link formulation* of the constraints for the elastic demand TA models.

## 2.2 The System Problem

In the system problem, the traffic planner maximizes net user benefit [7, 9, 20], the difference between total user benefit and the system cost. From basic economic principles, the total network user benefit from travel is $\sum_{k \in \mathcal{K}} \int_0^{t_k} w_k(z)dz$ and the system cost is defined by $s(v)^T v$. Thus, using minimization in the objective, the elastic demand system problem (SOPT-ED) is

$$\begin{array}{ll} \min & s(v)^T v - \sum_{k \in \mathcal{K}} \int_0^{t_k} w_k(z)dz \\ \text{subject to} & (v, t) \in V. \end{array}$$

The following lemma characterizes the Karush-Kuhn-Tucker (KKT) points of SOPT-ED:

**Lemma 1** *A feasible point $(\bar{v}, \bar{t})$ is a KKT point for SOPT-ED if and only if there exists $\rho^k \;\; \forall k \in \mathcal{K}$ such that the following holds:*

$$\begin{array}{rcll} (s(\bar{v}) + \nabla s(\bar{v})\bar{v}) & \geq & A^T \rho^k & \forall k \in \mathcal{K} \\ w_k(\bar{t}_k) & \leq & E_k^T \rho^k & \forall k \in \mathcal{K} \\ (s(\bar{v}) + \nabla s(\bar{v})\bar{v})^T \bar{v} & = & w(\bar{t})^T \bar{t}. & \end{array}$$

**Proof:** The first two conditions follow from stationarity of the Lagrangian function for SOPT-ED, and the equality constraint is the aggregated complementarity condition. To prove, we use the disaggregated complementarity conditions:

$$(\bar{x}^k)^T \gamma^k = 0 = (\bar{x}^k)^T (-A^T \rho^k + \nabla s(\bar{v})\bar{v} + s(\bar{v}))$$

and

$$\bar{t}_k \delta_k = 0 = \bar{t}_k(-w_k(\bar{t}_k) + E_k^T \rho^k)$$

where the vector $\gamma^k$ and scalar $\delta_k$ are KKT multipliers for the constraints $-x^k \leq 0$ and $-t_k \leq 0$, respectively. Summing these gives

$$\begin{array}{rcl} 0 & = & \sum_k (\bar{x}^k)^T (-A^T \rho^k + \nabla s(\bar{v})\bar{v} + s(\bar{v})) + \sum_k \bar{t}_k(-w_k(\bar{t}_k) + E_k^T \rho^k) \\ & = & \sum_k (\bar{x}^k)^T (-A^T \rho^k) + \sum_k (\bar{x}^k)^T (\nabla s(\bar{v})\bar{v} + s(\bar{v})) + \sum_k \bar{t}_k E_k^T \rho^k - (\bar{t})^T w(\bar{t}) \\ & = & \bar{v}^T (\nabla s(\bar{v})\bar{v} + s(\bar{v})) - (\bar{t})^T w(\bar{t}) + \sum_k (\bar{x}^k)^T (-A^T \rho^k) + \sum_k (A\bar{x}^k)^T \rho^k \end{array}$$

Therefore $(s(\bar{v}) + \nabla s(\bar{v})\bar{v})^T \bar{v} = w(\bar{t})^T \bar{t}$. □

### 2.3 The User Problem

Wardrop's first principle states that the cost of travel (which is usually measured by travel time) on all routes actually used are equal, and they are less than or equal to the cost which would be experienced by a single vehicle on any unused route. In other words, if a feasible flow pattern has the property that there is no incentive for any user to deviate from the currently chosen route, then it is a *user equilibrium flow*. Mathematically, the user problem is stated as a variational inequality [8]:

Find $(\bar{v}, \bar{t}) \in V$ such that

$$s(\bar{v})^T(v - \bar{v}) - w(\bar{t})^T(t - \bar{t}) \geq 0 \quad \forall (v, t) \in V. \qquad \text{(UOPT-ED)}$$

ignoretrue The UOPT-ED solutions are characterized via the following lemma.

**Lemma 2** *$(\bar{v}, \bar{t}) \in V$ solves UOPT-ED if and only if $s(\bar{v})^T\bar{v} = w(\bar{t})^T\bar{t}$ and there exists $\rho^k \;\; \forall k \in \mathcal{K}$ such that*

$$\begin{aligned} s(\bar{v}) &\geq A^T\rho^k \;\; \forall k \in \mathcal{K} \\ w_k(\bar{t}_k) &\leq E_k^T\rho^k \;\; \forall k \in \mathcal{K}. \end{aligned}$$

**Proof:** The vector $(\bar{v}, \bar{t})$ is user optimal if and only if (UOPT-ED) holds. Equivalently,

$$s(\bar{v})^T v - w(\bar{t})^T t \geq s(\bar{v})^T\bar{v} - w(\bar{t})^T\bar{t} \quad \forall (v, t) \in V$$

Thus $(\bar{v}, \bar{t})$ is user optimal if and only if it solves the linear program

$$\min\{s(\bar{v})^T v - w(\bar{t})^T t | v = \sum_k x^k \text{ and } Ax^k = t_k E_k, x^k \geq 0, t_k \geq 0 \;\; \forall k \in \mathcal{K}\}$$

whose dual is

$$\max\{\sum_k 0^T\rho^k | A^T\rho^k \leq s(\bar{v}), w_k(\bar{t}_k) \leq E_k^T\rho^k \;\; \forall k \in \mathcal{K}\}.$$

Therefore, the optimality conditions ensure that there exist $\rho^k \;\; \forall\, k \in \mathcal{K}$ such that

$$\begin{aligned} s(\bar{v}) &\geq A^T\rho^k \;\; \forall k \in \mathcal{K} \\ w_k(\bar{t}_k) &\leq E_k^T\rho^k \;\; \forall k \in \mathcal{K}. \end{aligned}$$

Since the dual linear programs have the same optimal objective value, $s(\bar{v})^T\bar{v} - w(\bar{t})^T\bar{t} = 0.\Box$

In the next section, these two lemmas are used to extend the theory of toll pricing to elastic demand traffic assignment.

# 3 Toll Set for the Elastic Demand Case

The aim is to obtain system optimal flows by perturbing the costs in the user problem. Let $s_\beta(v) = s(v) + \beta$ be the perturbed cost map where $\beta$ is a toll vector. Then the perturbed user problem is:

Find $(\bar{v}, \bar{t}) \in V$ such that

$$(s(\bar{v}) + \beta)^T(v - \bar{v}) - w(\bar{t})^T(t - \bar{t}) \geq 0, \quad \forall (v, t) \in V. \qquad \text{(UOPT}_\beta\text{-ED)}$$

ignoretrue

Let $U^*_\beta$ be the set of **tolled equilibrium solutions**, i.e., those $(\bar{v}, \bar{t}) \in V$ that satisfy the above variational inequality, and let $S^*$ be the optimal solution set for SOPT-ED. Any $\beta$ satisfying $\emptyset \neq U^*_\beta \subseteq S^*$ will be called a **valid toll vector**. The set of all such vectors denoted by $\mathcal{T} := \{\beta | \emptyset \neq U^*_\beta \subseteq S^*\}$ is called the **toll set**. For a given vector $(\bar{v}, \bar{t}) \in V$, $W(\bar{v}, \bar{t}) = \{\beta | (\bar{v}, \bar{t}) \in U^*_\beta\}$ is the set of all tolls for which $(\bar{v}, \bar{t})$ is a solution of UOPT$_\beta$-ED. Using Lemma 2, $W(\bar{v}, \bar{t})$ is the polyhedron given by the $\beta$ part of the following linear inequality system in $\beta$ and $\rho^k$ variables:

$$
\begin{aligned}
(s(\bar{v}) + \beta) &\geq A^T \rho^k \quad \forall k \in \mathcal{K} \\
w_k(\bar{t}_k) &\leq E_k^T \rho^k \quad \forall k \in \mathcal{K} \\
(s(\bar{v}) + \beta)^T \bar{v} &= w(\bar{t})^T \bar{t}.
\end{aligned}
$$

With these definitions, the following theorem gives expressions for $\mathcal{T}$ in terms of the polyhedra $W(\bar{v}, \bar{t})$.

**Theorem 1** *The toll set can be characterized by*

$$\mathcal{T} = \cup_{(\bar{v},\bar{t}) \in S^*} \{\beta \in W(\bar{v}, \bar{t}) | U^*_\beta \subseteq S^*\}.$$

*If $s$ and $w$ are strictly monotone, then $\mathcal{T} = \cup_{(\bar{v},\bar{t}) \in S^*} W(\bar{v}, \bar{t})$.*

*If $S^*$ is the singleton $(v^*, t^*)$, then $\mathcal{T} = W(v^*, t^*)$.*

Proof of this theorem is omitted since the results follow from arguments similar to those in [13] where expressions for $\mathcal{T}$ are given in the fixed demand case.

For a given $(\bar{v}, \bar{t})$, a valid toll $\beta$ generates total toll revenue $\beta^T \bar{v}$. Interestingly, this quantity is *constant* for all $\beta \in \mathcal{T}$ as shown in the corollary below. Larsson and Patricksson [14] also make this observation. Examples in [13, 12] show this is clearly not true in the fixed demand case, and, in fact, one of the original motivations for that work was to determine tolls that are less expensive than MSCP tolls [3, 4].

**Corollary 1** *The total toll revenue is the constant $w(\bar{t})^T \bar{t} - s(\bar{v})^T \bar{v}$ for any valid toll $\beta \in \mathcal{T}$.*

**Proof:** Follows from Lemma 2, since $(s(\bar{v}) + \beta)^T \bar{v} = w(\bar{t})^T \bar{t}$. □

Valid tolls are also restricted in that total costs are the same on utilized routes, as the equilibrium principle dictates for the UOPT$_\beta$-ED problem. To state precisely, let $r_k$ be a route for commodity $k$ and $\gamma_{ar_k}$ be 1 if link $a$ is on route $r_k$ and zero otherwise.

**Corollary 2** *At the UOPT$_\beta$-ED solution $(\bar{v}, \bar{t})$ the total generalized cost on any route, for any commodity $k$, satisfies the inequality $\sum_{a \in A} \gamma_{ar_k} (s_a(\bar{v}_a) + \beta_a) \geq w_k(\bar{t}_k)$. For any route with positive flow, the inequality holds as an equality. Therefore the costs on utilized routes are constant for any commodity.*

**Proof:** For $a = (i, j)$ and $k = (o, d)$, in a given route $r_k$, $s_a(\bar{v}_a) + \beta_a \geq \rho^k_j - \rho^k_i$ and $w_k(\bar{t}_k) \leq \rho^k_o - \rho^k_d$. By summing over a route, the inequality in the corollary statement is obtained. Further, the complementarity conditions force the inequality to hold as an equality when there is flow on the route. The conclusion of the corollary then follows. □

This corollary also shows clearly that every commodity reaches an equilibrium exactly when the route costs, including tolls, equals the user benefit at the final demand level. MSCP tolls are easily computed by the formula $\beta_{MSCP} = \nabla s(v^*) v^*$ whenever $(v^*, t^*)$ solves SOPT-ED, hence $\mathcal{T}$ is nonempty. However, there are disadvantages to MSCP tolls. A primary disadvantage, recognized in the theory of *second-best* tolls [15, 19], is that some links may not be available for tolling. In fact, there is a positive component of $\beta_{MSCP}$ for any congested link with flow, implying the need for many toll booths. Also, MSCP tolls do not allow the possibility for *subsidies*, or negative tolls, on some links. The toll pricing framework of the following section is a prescription for defining $\mathcal{T}$ and then choosing a particular $\beta \in \mathcal{T}$ based on some secondary criteria.

# 4 The Toll Pricing Framework

The toll pricing framework from [13] extends readily given the results above. Assume uniqueness of the solutions for SOPT-ED and for UOPT-ED. Uniqueness will follow, for example, if both $s$ and $w$ are strictly monotone and $s(v)^T v$ is strictly convex. The framework then consists of the following steps:

**Step 1:** Solve SOPT-ED to obtain the system optimal solution $(v^*, t^*)$. The algorithm of Evans [6] can be used for this step.

**Step 2:** Define the toll set which is the $\beta$ part of the polyhedron $W(v^*, t^*)$:

$$\begin{array}{rcl} (s(v^*) + \beta) & \geq & A^T \rho^k \quad \forall k \in \mathcal{K} \\ w_k(t_k^*) & \leq & E_k^T \rho^k \quad \forall k \in \mathcal{K} \\ (s(v^*) + \beta)^T v^* & = & w(t^*)^T t^*. \end{array}$$

**Step 3:** Define and optimize an objective function over the toll set, possibly intersected with other constraints. (See Table 1 for examples.)

| TOLL | OBJECTIVE FUNCTION | EXTRA CONSTRAINTS |
|---|---|---|
| MINREV | $\min_{(\beta,\rho)} \beta^T v^*$ | |
| MINMAX | $\min_{(z,\beta,\rho)} z$ | $z \geq \beta_a, \quad \forall a \in \mathcal{A}, \beta \geq 0$ |
| MINTB | $\min_{(z,\beta,\rho)} \sum_{a \in \mathcal{A}} z_a$ | $\beta_a \leq M z_a \quad \forall a \in \mathcal{A}, z_a \in \{0, 1\}, \beta \geq 0$<br>$M$ a large constant |

Table 1: Alternative Optimization Formulations

As in the fixed demand case, various objectives in step 3 lead to linear programs or linear integer programs. For the elastic demand problem, two natural choices are the MINTB and MINMAX objectives in [13]. The first of these aims to minimize the number of toll booths and the second minimizes the maximum toll on any one link. The objective of minimizing total revenue from nonnegative tolls (MINSYS) is of less interest due to corollary 1. MINREV is similar to MINSYS, but tolls are free to be negative as well as positive. It is worth noting that the step 3 objectives can lead to problems with nonunique solutions. Hence further objectives might be possible in specific problems. Table 1 summarizes the choices suggested.

To compare the formulations given above as well as provide comparison with MSCP tolls, we have modified the nine node example from [11, 4, 13] which has cost data similar to large-scale traffic assignment problems. The nine node network has 18 links and all links have cost functions with the same structure: $s_a(v) = s_a(v_a) = T_a(1+0.15(v_a/C_a)^4)$ where $T_a$ is a measure of travel time when there is zero flow and $C_a$ is the practical capacity of link $a$. It is depicted in Figure 1. The tuple near link $a$ is $(T_a, C_a)$. There are four OD-pairs: (1,3), (1,4), (2,3) and (2,4). The demand functions between these OD-pairs are, $t_{(1,3)} = t_{(1,3)}(c_{(1,3)}) = 10-0.5c_{(1,3)}$, $t_{(1,4)} = t_{(1,4)}(c_{(1,4)}) = 20-0.5c_{(1,4)}$, $t_{(2,3)} = t_{(2,3)}(c_{(2,3)}) = 30 - 0.5c_{(2,3)}$ and $t_{(2,4)} = t_{(2,4)}(c_{(2,4)}) = 40 - 0.5c_{(2,4)}$.

Note first, that SOPT-ED and UOPT-ED have different demands (Table 2). The total demand in the system problem is 57.411 and it is 60.753 in the user problem, so there is a 5.5% difference. This table also contains the generalized costs at the optimal demand levels.

Table 3 provides optimal flows for the SOPT-ED and UOPT-ED problems. Although the UOPT-ED solution has higher user benefit than the SOPT-ED solution, it also has a higher user cost

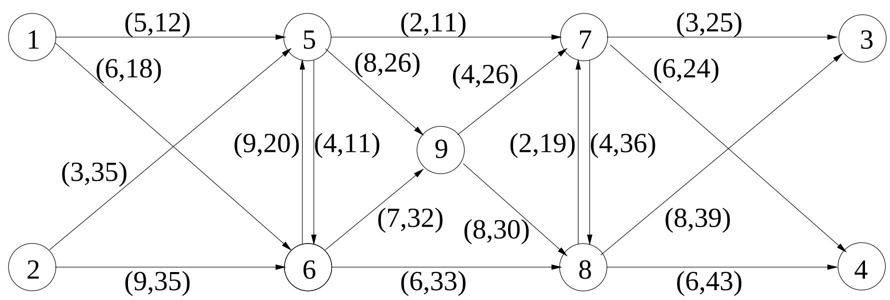


Figure 1: The Nine Node Network

| OD-PAIR | SOPT-ED SOLUTION | UOPT-ED SOLUTION |
|---|---|---|
| (1,3) | (0.000, 20.000) | (0.151, 19.698) |
| (1,4) | (9.696, 20.607) | (10.698, 18.605) |
| (2,3) | (19.476, 21.047) | (20.672, 18.656) |
| (2,4) | (28.239, 23.523) | (29.232, 21.537) |

Table 2: Demand and Generalized Cost $(t^*, w(t^*))$ for the Nine Node ED Problem

due to increased traffic on the network. Thus the net user benefit (NUB) in the system problem is greater. Examining the individual link flow values, notice that they are within 10% of each other on almost all links of the network, but not on links (5,7), (5,9) and (9,7). The *total* flow between nodes 5 and 7 for both user and system problems is within 10%, but the link flows differ substantially. Relative to the system solution, in the user problem (5,7) is over utilized while (5,9) and (9, 7) are under utilized. Therefore, systems efficiency is increased by diverting traffic on (5,7) to the route 5-9-7. This can be done by making (5,7) less attractive, i.e., increasing the cost by tolling (5, 7).

Table 4 contains alternative toll vectors for the nine node problem obtained with an optimization modeling package (GAMS/CPLEX) implementation of the Toll Pricing Framework. (Tolls are expressed in time units; Arnott and Small [1] discuss the conversion to dollars based on studies in the U.S.) For comparison, the MSCP tolls are also listed. As expected, tolls on the link (5,7) are high for the tolling schemes with positive tolls, namely, MSCP, MINMAX and MINTB. However, MINREV, which allows negative tolls, rewards travel on the 5-9-7 route with subsidies in order to achieve the SOPT-ED solution. By corollary 1, all tolling schemes produce the same toll revenue. For this example the revenue is 268.519, which is 17.44% of net user benefit. MSCP and MINREV tolls require 10 toll booths. MINMAX has an objective value of 8.00 with eight toll booths. It happens that MINTB obtains the same result (i.e., the maximum toll is 8.00) with only five toll booths. Thus it could be argued that the MINTB solution is best in that it achieves the SOPT-ED solution and is cheapest to implement.

To illustrate Corollary 2, consider the routes 2-6-8-4 and 2-5-7-4. The delay functions on the first route give a total cost of 21.505, while on the second this total is only 13.504. However, the tolls on the first route are 2.018 and they are 10.018 on the second. Thus the total route cost is 23.523, in agreement with the total generalized cost in Table 2. There is no incentive for additional 2-4 trip demand. Any increase would result in lower user benefit and higher costs since the demand and cost functions are strictly monotone.

| Link | SOPT-ED SOLUTION $v_a^*$ | $s_a(v_a^*)$ | $v_a^* s_a(v_a^*)$ | UOPT-ED SOLUTION $v_a^U$ | $s_a(v_a^U)$ | $v_a^U s_a(v_a^U)$ |
|---|---|---|---|---|---|---|
| 1-5 | | 5.000 | | | 5.000 | 20.000 |
| 1-6 | 9.696 | 6.076 | 58.914 | 10.849 | 6.119 | 66.380 |
| 2-5 | 31.715 | 3.303 | 104.769 | 34.458 | 3.423 | 117.940 |
| 2-6 | 15.999 | 9.059 | 144.938 | 15.446 | 9.051 | 139.805 |
| 5-6 | | 9.000 | | | 9.000 | |
| 5-7 | 17.978 | 4.140 | 74.433 | 26.442 | 12.016 | 317.730 |
| 5-9 | 13.738 | 8.094 | 111.188 | 8.016 | 8.011 | 64.215 |
| 6-5 | | 4.000 | | | 4.000 | |
| 6-8 | 25.696 | 6.331 | 162.677 | 26.295 | 6.363 | 167.308 |
| 6-9 | | 7.000 | | | 7.000 | |
| 7-3 | 19.476 | 3.166 | 61.657 | 20.823 | 3.217 | 66.979 |
| 7-4 | 12.239 | 6.061 | 74.180 | 13.785 | 6.098 | 84.063 |
| 7-8 | | 2.000 | | | 2.000 | |
| 8-3 | | 8.000 | | | 8.000 | |
| 8-4 | 25.696 | 6.115 | 157.125 | 26.144 | 6.123 | 160.078 |
| 8-7 | | 4.000 | | 0.151 | 4.000 | 0.604 |
| 9-7 | 13.738 | 4.047 | 55.594 | 8.016 | 4.005 | 32.108 |
| 9-8 | | 8.000 | | | 8.000 | |
| **User Benefit** | | | 2544.75 | | | 2613.50 |
| **System Cost** | | | 1005.474 | | | 1217.21 |
| **Net User Benefit** | | | 1539.284 | | | 1396.285 |

Table 3: The Nine Node Problem-SOPT-ED and UOPT-ED Solutions

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| **Link** | **MSCP** | **MINREV** | **MINMAX** | **MINTB** |
|---|---|---|---|---|
| 1-5 | | | 8.000 | |
| 1-6 | 0.303 | 2.085 | 2.085 | 0.067 |
| 2-5 | 1.214 | -7.444 | | 2.018 |
| 2-6 | 0.236 | 2.018 | 2.018 | |
| 5-6 | | 6.218 | | |
| 5-7 | 8.561 | | 8.000 | 8.000 |
| 5-9 | 0.374 | -5.953 | | |
| 6-5 | | | | |
| 6-8 | 1.323 | | | |
| 6-9 | | | | |
| 7-3 | 0.663 | 17.882 | 2.438 | 0.420 |
| 7-4 | 0.243 | 17.462 | 2.018 | |
| 7-8 | | 15.408 | | |
| 8-3 | | | 2.320 | |
| 8-4 | 0.459 | | | 2.018 |
| 8-7 | | | 0.716 | |
| 9-7 | 0.187 | -4.047 | | |
| 9-8 | | 9.408 | | |
| $\beta^T v^*$ | 268.519 | 268.519 | 268.519 | 268.519 |
| $\beta^T v^*$/**(NUB) (%)** | 17.44 | 17.44 | 17.44 | 17.44 |
| **Toll Booths** | 10 | 10 | 8 | 5 |

Table 4: The Nine Node Problem-Alternative Tolls

## 5 Summary

The toll pricing methodology previously introduced extends readily to the elastic demand case when the system problem maximizes net user benefit. It offers clear advantages to marginal social cost pricing and heuristic tolling schemes. Further, it is anticipated that standard optimization software will be capable of solving modest sized urban networks. This expectation is based on prior experience with real data reported in [13, 12]. Alternative tolls were calculated for a Stockholm model with 417 nodes, 963 links and fixed demand of 272,873 rush hour trips using the GAMS/CPLEX combination, and the methodology of this paper is only marginally more complicated. Implementations and customized network algorithms for larger problems are important computational issues to be addressed in future research.

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