Congestion Toll Pricing of Traffic Networks[1]

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Abstract. This paper concerns tolling methodologies for traffic networks which ensure that the resultant equilibrium flows are system optimal. A nonnegative vector β is defined to be a *valid toll vector*, if the set of tolled user equilibrium solutions is a subset of the set of untolled system optimal solutions. The problem of characterizing the *toll set* $\mathcal{T}$, which is the set of all valid toll vectors, is studied. Descriptions and characterizations of $\mathcal{T}$ are given for the cases when either the cost map is strictly monotonic or is affine monotonic. In the latter case, the cost map is of the form $Qv + c$, where Q is a not necessarily symmetric matrix and $Q + Q^T$ is positive semidefinite. The results are illustrated with several examples.

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1 Introduction

1.1 Motivation

As traffic congestion increasingly becomes a problem facing modern cities, new life is being given to an idea long proposed by transportation economists: the tolling of congested urban streets, especially during peak traffic periods. Arnott and Small, in a recent excellent popular article [2], have discussed the pros and cons of congestion pricing, noting, in particular, that modern technology makes the electronic collection of tolls possible.

Primary questions in the analysis of congestion tolling include which streets to toll, for how much, and with what objective. As noted in [2] the predominant idea is to set tolls, on possibly every street, according to the policy of *marginal social cost pricing,* with the objective of obtaining a traffic pattern which optimizes the collective use of the network.

The purpose of this paper is to define and characterize mathematically the **set** of all tolls which will cause drivers to make optimal use of a traffic network. In this setting, marginal social costs prices constitute just one vector in the set of possible tolls. Given algebraic descriptions of the toll set, it then becomes possible to define optimization models which choose a toll vector according to specific criteria, e.g., minimize the total amount of tolls collected from the drivers, minimize the number of tolled streets, etc.

After introducing notation and the familiar models of user and system optimal traffic assignment, we describe congestion toll pricing in more detail and give general results on toll sets. It is then shown that, under the assumptions often used in traffic assignment, the toll set is a polyhedron, i.e., it is defined by a system of linear equalities and inequalties. With this result, we illustrate the method of choosing a toll vector of minimum total cost using linear programming. Detailed treatment of several special cases is given, including the various paradox examples from [2].

1.2 Notation

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ be a network with $\mathcal{N}$ being the node set and $\mathcal{A}$ being the arc set, and denote by $\hat{A}$ the incidence matrix of $\mathcal{G}$. Suppose that $\mathcal{K}$ is a set of commodities that "flow" in the network. Corresponding to each commodity k, there is an associated demand vector $b(k)$, which is a vector of size $n = |\mathcal{N}|$, whose ith entry (for $i \in \mathcal{N}$) denotes the demand for that commodity at node i, and the entries of $b(k)$ sum to zero. The kth *commodity flow* (variable) vector is denoted by $x(k)$ and the sum of all the commodity flow vectors is the *aggregate flow* vector v. The system defining feasible flows is given by:

$$\begin{aligned} v &= \textstyle\sum_k x(k) \\ \hat{A}x(k) &= b(k) \ \forall \ k \in \mathcal{K} \\ x(k) &\geq 0 \ \forall \ k. \end{aligned}$$

This is the *arc-node formulation* of feasible flows in traffic assignment. The *Traffic Assignment Problem (TAP)* generically refers to various optimization and equilibrium problems involving the above type of constraints. The reader is referred to [6] for background on traffic assignment problems.

Often employed is the *arc-route formulation* in which one has a more restrictive demand pattern: each commodity is of a single-origin-single-destination type, i.e., if (o, d) is the origin-destination pair with demand t_k, then the demand vector $b(k) = t_k(e_d - e_o)$, where e_j denotes the jth standard unit vector. Let P_k denote the set of paths running from origin to destination for commodity k, and let $\mathcal{P}$ be the union of all the P_ks. The variables are v and h, where v is the aggregate flow as before, but h is the path flow vector indexed on $\mathcal{P}$. The feasible flows in this case are given by the system

$$\begin{aligned} v &= \Delta h \\ \Gamma h &= t \\ h &\geq 0, \end{aligned}$$

where Δ is the arc-path incidence matrix and Γ is the O-D pair-path incidence matrix.

For notational simplicity, the following format for the definition of feasible flows is employed in this paper:

$$\begin{aligned} v &= Zx \\ Ax &= b \\ x &\geq 0. \end{aligned}$$

It is clear that through an appropriate choice of the matrices A, Z and vector b, both arc-node and arc-path formulations can be recast in this format. Here, x will be called the vector of **individual flows**, and as before v will be called **aggregate flows**. Define the sets:

$$\begin{aligned} F &= \{(v, x) | v = Zx, Ax = b, x \geq 0\} \\ V &= \{v | \text{ there exists } x \text{ such that } (v, x) \in F\}. \end{aligned}$$

Here, F is the set of feasible flows, V the set of feasible aggregate flows.

It is assumed that a **cost map** $s : \mathcal{A} \to \mathcal{A}$ is given. The interpretation is that, when the aggregate flow in the network is v, then the cost incurred by a user on arc a is given by $s_a(v)$. Such a cost may be of several types, but for our purposes here, one may consider this to be the time taken to traverse the arc at the flow level of v. It will be assumed throughout that this cost map is continuously differentiable, and the Jacobian of s will be denoted by ∇s. Listed below are some special classes of maps that are of interest:

Monotonic Maps: s is said to be *monotonic* if $(s(v_1) - s(v_2))^T(v_1 - v_2) \geq 0, \forall v_1, v_2 \geq 0$. (Here, one may only consider maps that are monotonic on only V without a significant effect on the development to follow.) If this inequality is strict for every $v_1 \neq v_2$, then s is *strictly monotonic*.

S-Convex Maps: s is *S-Convex*, if $s(v)^T v$ is a convex function. If this function is strictly convex, then s will be said to be *SS-Convex*. As will be described in §1.4, the function $s(v)^T v$ is the *system objective function*.

Affine Maps: When $s(v) = Qv + c$, where Q is a not necessarily symmetric matrix and c is a vector, s is called and *affine*. When $Q + Q^T$ is positive semidefinite (i.e., its eigenvalues are nonnegative), then the map $s = Qv + c$ will be called *affine monotonic*. Note that an affine monotonic map is both monotonic and S-convex.

Separable Maps: If for each $a \in \mathcal{A}$, $s_a(v)$ is a function of v_a alone, then s is said to be *separable*.

1.3 Wardrop's Principles

British traffic engineer Wardrop stated two principles that tend to model the nature of traffic behavior [6]. The first principle postulates the nature from a user's point of view, and the second principle in some sense prescribes the desirable behavior from the system designer's standpoint.

Wardrop's first principle: The travel times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

Wardrop's second principle: The average journey time is a minimum. This amounts to minimizing $s(v)^T v$ over feasible aggregate flows v.

1.4 User Optimum and Equilibrium

This notion emerges from Wardrop's first principle, which states essentially that the user, in an attempt to minimize his or her individual cost, will try to choose the least cost path as he or she sees from a snap-shot of the existing traffic flow. In other words, a *user equilibrium flow* (also called *user optimal flow*) will ensue when no individual user has an incentive for deviating from the currently chosen path.

This can equivalently be stated as a variational inequality problem (see [6]). A given feasible aggregate flow vector $\bar{v}$ is a user equilibrium solution if and only if

$$s(\bar{v})^T (v - \bar{v}) \geq 0, \forall v \in V. \qquad \text{(UOPT)}$$

By using LP duality, one can readily characterize user optimality of a given aggregate flow vector $\bar{v} \in V$.

Lemma 1. *Let $\bar{v} \in V$. Then the following are equivalent:*

1. *$\bar{v}$ is user optimal, i.e., $s(\bar{v})^T (v - \bar{v}) \geq 0, \forall v \in V$.*
2. *$\bar{v} \in argmin\{s(\bar{v})^T v | v \in V\}$, i.e., $\min\{s(\bar{v})^T v | v \in V\} = s(\bar{v})^T \bar{v}$.*
3. *There exists ρ such that*

$$Z^T s(\bar{v}) \geq A^T \rho$$
$$s(\bar{v})^T \bar{v} = b^T \rho.$$

Proof: The flow vector is user optimal if and only if $s(\bar{v})^T v \geq s(\bar{v})^T \bar{v}, \forall v \in V$, and since $\bar{v} \in V$, the equivalence of 1 and 2 follows. By using $\bar{v} \in V$ once again, *2* may be rewritten as

$$\min\{s(\bar{v})^T Zx | Ax = b, x \geq 0\} \geq s(\bar{v})^T \bar{v},$$

which by LP duality is the same as

$$\max\{b^T \rho | A^T \rho \leq Z^T s(\bar{v})\} \geq s(\bar{v})^T \bar{v}.$$

The above holds if and only if there exists ρ such that

$$\begin{aligned} A^T \rho &\leq Z^T s(\bar{v}) \\ b^T \rho &\geq s(\bar{v})^T \bar{v}. \end{aligned}$$

Letting $\bar{x} \geq 0$ be such that $\bar{v} = Z\bar{x}, A\bar{x} = b$, one sees that

$$b^T \rho \geq \bar{v}^T s(\bar{v}) = \bar{x}^T (Z^T s(\bar{v})) \geq \bar{x}^T A^T \rho = b^T \rho.$$

The lemma now follows. □

It is well known that when the cost map s is strictly monotonic, then the user equilibrium problem has at most one solution [6].

In this paper, equilibrium strategies resulting from perturbations of the cost map s will be considered. More specifically, these perturbations will involve the addition of a *toll vector* to the cost map, so that the net cost as experienced by users is

$$s_\beta(v) := s(v) + \beta.$$

The user equilibrium problem under the perturbed cost map is given by:

$$(s(\bar{v}) + \beta)^T (v - \bar{v}) \geq 0, \forall v \in V. \tag{UOPT-β}$$

1.5 System Optimum

The total system cost is given by $\sum_{a \in A} s_a(v) v_a$, or more compactly, $s(v)^T v$. From the point of view of a traffic system designer, it is therefore desirable to minimize this function; this is tantamount to Wardrop's second principle. The system optimum problem is stated as:

$$\begin{aligned} \min : \; & s(v)^T v \\ \text{s.t.} \quad & v = Zx \\ & Ax = b \\ & x \geq 0. \end{aligned} \tag{SOPT}$$

From the KKT necessary conditions for the above problem, after some straightforward manipulations, it can be concluded that for a given $\hat{v} \in V$, there exists a local minimum $(\hat{v}, \hat{x})$ only if *there exists* ρ *such that*

$$\begin{aligned} Z^T (s(\hat{v}) + \nabla s(\hat{v})\hat{v}) &\geq A^T \rho \\ \hat{v}^T (s(\hat{v}) + \nabla s(\hat{v})\hat{v}) &= b^T \rho. \end{aligned} \tag{KKT-SOPT}$$

A priori, it is neither the case that a system optimal solution is a user equilibrium solution nor the converse. This fact is illustrated using the well-known *Braess Paradox.*

Consider the Braess network depicted in Figure 1, and the associated cost map

$$s(v) = (10v_1, v_2 + 50, v_3 + 50, 10v_4, v_5 + 10)^T.$$

There is only one O-D pair, (1,2), and its demand is 6. The system optimal flows are 3 units on the path (1,3,2) and 3 units on the path (1,4,2), each of which has a cost of 83 units and the system optimal aggregate flow vector is $(3, 3, 3, 3, 0)^T$ with total cost $= \sum_{a \in A} v_a^* s_a(v^*) = 6 \times 83 = 498$. The user equilibrium flows are 2 units on each of the paths of the network, i.e., the new equilibrium aggregate flow vector is $(4, 2, 2, 4, 2)^T$. Since each path cost is 92, the system objective function has a value of 552 at user equilibrium.

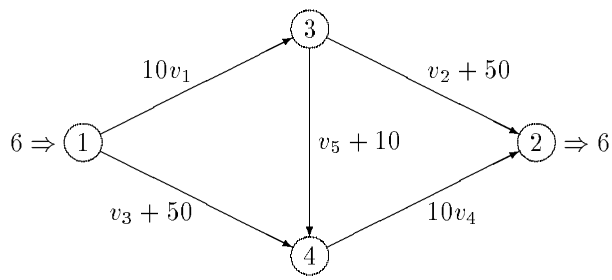


Figure 1: Braess Network

This example will be recalled in some sections to follow.

2 Congestion Toll Pricing

2.1 The Notion of Toll Pricing

As seen above, in general, the user optimal solutions can be different from the system optimal solutions. Since the latter, in a certain sense, is the desirable state of traffic flow, one would like to somehow induce these flows in the network. One possibility is to charge user tolls, and thus alter the effective cost map as seen by the users of the network. Of course, this is likely to change the equilibrium solutions. The objective of congestion toll pricing is to charge tolls in such a fashion that the following hold:

1. There exists a user equilibrium solution with the modified cost map.
2. Every (tolled) equilibrium solution is system optimal.

We state the above as the **Principle of Toll Pricing**:

> *The tolls imposed should be such that the resulting tolled user equilibrium problem has at least one solution and every such solution is an untolled system optimal solution.*

Formally, let us define

$$s_\beta(v) := s(v) + \beta,$$

where $\beta \geq 0$ is a *toll vector*, and consider the perturbed user optimality problem:

$$(s(\bar{v}) + \beta)^T(v - \bar{v}) \geq 0, \forall v \in V. \qquad \text{(UOPT-}\beta\text{)}$$

Denote by U^*_β, the set of **Tolled Equilibrium Solutions**, i.e., those $\bar{v}$ that satisfy UOPT-β, and let S^* be the set of optimal solutions to the (untolled) system optimum problem SOPT:

$$S^* := \text{argmin}\{s(v)^T v | v \in V\}.$$

Since we would like the toll vector to be such that the resulting user equilibrium problem has a solution, and *every* equilibrium solution is system optimal, i.e.,

$$\emptyset \neq U^*_\beta \subseteq S^*.$$

Any $\beta \geq 0$ satisfying the above will be called a **Valid Toll Vector**. The set of all such vectors will be denoted by

$$\mathcal{T} := \{\beta \geq 0 | \emptyset \neq U^*_\beta \subseteq S^*\},$$

and this set is called the **Toll Set** for the Traffic Assignment Problem with cost map s over the network $\mathcal{G}$.

The general goal of this paper is to obtain descriptions of $\mathcal{T}$ or some of its subsets. For instance, when s is strictly monotone and $s(v)^T v$ (i.e., the system objective function) is strictly convex, then $\mathcal{T}$ turns out to be a polyhedron easily described using the system optimal flow vector.

For a given vector $\bar{v} \in V$ (typically chosen to be a system optimal solution, i.e., $\bar{v} \in S^*$), the following is precisely the set of all tolls that ensure that $\bar{v}$ is a solution of UOPT-β:

$$W(\bar{v}) = \{\beta \geq 0 | \bar{v} \in U^*_\beta\}.$$

When there is a unique system optimal solution, we will denote it by v^S, and similarly, if UOPT-β has a unique solution, then it will be denoted by v^U_β.

2.2 Marginal Social Cost Pricing

Informal Sketch The main traditional approach for congestion pricing is the concept of marginal social cost pricing (MSCP). This principle states that in order to achieve the most socially efficient usage of a road, one must force each additional user to pay not only the cost (in time and money) that the trip costs herself but also the cost of the delay that her decision to travel imposes on the other drivers. This cost has to be collected as a congestion toll. The toll will be the difference between the marginal private cost (the cost that the driver experiences) and the marginal social cost (the total cost that she and all the other drivers experience owing to her decision to travel). Early work in the area was done by Pigou [15], Knight [10], and Walters [18]. This scheme for road pricing is also discussed by Morrison [12].

The flow pattern that is most socially efficient (with respect only to the travel costs) will be the flow pattern that minimizes the total system cost $s(v)^T v$. If the link costs are separable it is easy to show that the tolls computed as $\frac{ds_a(v_a)}{dv_a} v_a$ will render the system optimal flow pattern. This is done by noting that the KKT conditions for the optimization formulation of the tolled user equilibrium problem (applicable when the cost map has a symmetric Jacobian) with the MSCP tolls are the same as the KKT conditions for the system optimization problem. Netter [14] emphasizes the importance of convexity of $s(v)^T v$ for achieving system optimum with MSCP tolls. Dafermos [5] and Smith [17] generalized this to the case when the link costs are not separable. Even in this case the tolls can be interpreted as the difference between the marginal social cost and the marginal private cost. Marginal social cost pricing makes travelers change routes or modes so that the resulting equilibrium is the (untolled) system optimal equilibrium.

Precise Description The mathematical description of the idea is as follows. Let $\bar{v}$ be a system optimal solution (i.e., $\bar{v} \in S^*$), and suppose that the following assumptions hold.

1. The cost map s is strictly monotonic.
2. The vector $\nabla s(\bar{v})\bar{v}$ is nonnegative.

The MSCP toll vector is defined to be:

$$\beta_{MSCP} := \nabla s(\bar{v})\bar{v}.$$

With $\beta = \beta_{MSCP}$, it is not hard to see (via an application of Lemma 1) that

$$\bar{v} \in U^*_\beta,$$

and further, by the strict monotonicity of s (and hence that of $s_\beta = s + \beta$), it follows that

$$U^*_\beta = \{\bar{v}\} \subseteq S^*.$$

Clearly, β_{MSCP} is a valid toll vector. However, there clearly may exist other valid toll vectors. This nonuniqueness can stem from two sources:

1. The choice of $\bar{v} \in S^*$ was arbitrary.
2. Even for a fixed choice of $\bar{v}$, there may well exist other tolls β which acheive the desired objective of $U^*_\beta = \{\bar{v}\}$. This is in essence the main observation behind the results of this paper.

MSCP Tolls for The Braess Paradox Consider the traffic assignment problem depicted in Figure 1. As mentioned there, the system optimal solution for this network is given by $v^S = [3\ 3\ 3\ 3\ 0]^T$ and the user optimal (equilibrium) solution is $v^U = [4\ 2\ 2\ 4\ 2]^T$. The MSCP tolls are calculated to be:

$$\beta_{MSCP} = [30\ 3\ 3\ 30\ 0]^T,$$

and as discussed, with these tolls being imposed, the resultant user equilibrium solution is v^*. However, the same can accomplished by using the toll vector $\tilde{\beta} = [0\ 0\ 0\ 0\ 13]^T$ (see §4.1). It is seen that the total toll (which is given by $\beta^T v^*$) imposed on the users in the latter case is zero (and hence the optimum), while that in the former is 198. Similar situations resurface for several other examples. The main point here is that there are usually many valid toll vectors; in the particular case of the Braess paradox, the toll set is a convex polyhedron.

3 Main Results

Consider once again the sets:

$$\begin{aligned} S^* &= \text{argmin}\{s(v)^T v | v \in V\} \\ U^*_\beta &= \{\bar{v} | (s(\bar{v}) + \beta)^T (v - \bar{v}) \geq 0, \forall v \in V\} \\ \mathcal{T} &= \{\beta \geq 0 | \emptyset \neq U^*_\beta \subseteq S^*\}. \end{aligned}$$

The objective is to obtain explicit descriptions of the toll set $\mathcal{T}$. The main object employed in the analysis is the following set

$$W(\bar{v}) = \{\beta \geq 0 | \bar{v} \in U^*_\beta\},$$

which turns out to be a polyhedron by Lemma 1.

Lemma 2. *For $\bar{v} \in V$, $W(\bar{v})$ is the polyhedron given by the β part of the following linear inequality system in β and ρ variables:*

$$\begin{aligned} Z^T (s(\bar{v}) + \beta) &\geq A^T \rho \\ (s(\bar{v}) + \beta)^T \bar{v} &= b^T \rho. \end{aligned}$$

The reason $W(\bar{v})$ plays a significant role in the analysis of $\mathcal{T}$ is that, firstly, this set is a polyhedron, and hence computations involving it tend to be computationally feasible, and secondly, this set is readily described in terms of the feasible flow $\bar{v}$, which is typically chosen to be a system optimal solution. The following result gives a simple expression for $\mathcal{T}$ in terms of the polyhedra $W(\bar{v})$.

Theorem 1 $\mathcal{T} = \cup_{\bar{v} \in S^*} \{\beta \in W(\bar{v}) | U^*_\beta \subseteq S^*\}$.

Proof:

Let $\beta \in \mathcal{T}$. Since U^*_β is nonempty, there exists $\bar{v} \in U^*_\beta$. Then, $\beta \in W(\bar{v})$, and hence

$$\mathcal{T} \subseteq \cup_{\bar{v} \in S^*} \{\beta \in W(\bar{v}) | U^*_\beta \subseteq S^*\}.$$

The $\supseteq$ direction is evident. □

In the two subsections to follow, a complete description of the toll set is given for the following cases:

1. s is strictly monotonic.
2. s is affine monotonic.

3.1 Special Case: Strictly Monotonic Maps

If s is strictly monotonic, then U_β^* has at most one element. This implies that for every system optimal solution $\bar{v}$, every β in the polyhedron $W(\bar{v})$ is a valid toll vector. Since the system optimal solution may not be unique, as we vary $\bar{v}$ over S^*, a family of polyhedra is obtained. Their union is the toll set as described in the result below.

Theorem 2 *If s is strictly monotonic, then*

$$\mathcal{T} = \cup_{\bar{v} \in S^*} W(\bar{v}).$$

Proof: As already seen, for every $\bar{v} \in S^*$, $\mathcal{T} \supseteq W(\bar{v})$. Thus,

$$\mathcal{T} \supseteq \cup_{\bar{v} \in S^*} W(\bar{v}).$$

For the reverse inclusion, let $\beta \in \mathcal{T}$. Then, since $U_\beta^* \neq \emptyset$, and since s strictly monotonic, there exists $\bar{v} \in U_\beta^* \subseteq S^*$. Clearly, then

$$\beta \in W(\bar{v}),$$

and the proof is complete. □

The special case in which the above theorem gives a very precise description of the toll set is when S^* is a singleton, which is ensured, for instance, by the strict convexity of $s(v)^T v$.

Corollary 1 *Suppose that s is both strictly monotonic and SS-Convex (i.e., $s(v)^T v$ is strictly convex). Then there exists a unique optimal solution v^S to the system optimum problem and thus the toll set $\mathcal{T}$ equals the polyhedron $W(v^S)$.*

3.2 Special Case: Affine Monotonic Maps

Now suppose that the map s is of the form:

$$s(v) = Qv + c,$$

where Q is a not necessarily symmetric matrix whose symmetrization $Q + Q^T$ is positive semidefinite. (If this latter matrix is positive definite, then s is strictly monotonic and $s(v)^T v$ is strictly convex, and hence the toll set for this case is a polyhedron as described in Corollary 1.)

In this section, an algebraic characterization of the toll set $\mathcal{T}$ for general affine monotonic maps is developed. More specifically, a system of quadratic equations and inequalities (such systems are called *Multiquadratic Systems*) in variables β as well other auxiliary variables is derived, such that the β part of the solutions to that system is precisely $\mathcal{T}$. The following tools are used along with Theorem 1 to obtain this result:

1. Linear and (convex) quadratic programming duality.
2. A well known characterization of the solution set of a convex QP (see [1], [11]. Also Theorem 3.1.7 of [4]).

The latter result is stated below.

Proposition 1 *Let M be a symmetric positive semidefinite matrix and V be a closed convex set, and consider the problem:*

$$\text{minimize } (1/2)v^T M v + c^T v \text{ subject to } v \in V.$$

Let V^* be the set of optimal solutions.

1. *The gradient $Mv + c$ is constant for $v \in V^*$.*
2. *If $\bar{v}$ is any fixed solution in V^*, then V^* is described by:*

$$V^* = \{v \in V | M(v - \bar{v}) = 0, c^T(v - \bar{v}) = 0\}.$$

If g is gradient of the objective over V^*, then an alternate description of V^* is:

$$V^* = \{v \in V | M(v - \bar{v}) = 0, g^T(v - \bar{v}) = 0\}.$$

In particular, when V is a polyhedron, the solution set V^* is a polyhedron as well.

The main steps of the derivation are:

1. Let $\bar{v}$ be any optimal solution to the system optimizing quadratic program SOPT. Its optimality is described using the KKT conditions (or QP duality).
2. Let β be a vector in the polyhedron $W(\bar{v})$. The conditions that ensure this are given in Lemma 2. Let ρ be the associated auxiliary variables.
3. Since the above ensures that $\bar{v} \in U^*_\beta$, the tolled user equilibrium problem for β is equivalent to the QP:

$$\begin{aligned} \min: \quad & v^T(Qv + c + \beta) - b^T\rho \\ & Z^T(Qv + c + \beta) - A^T\rho \geq 0, \end{aligned}$$

   and $\bar{v}$ is an optimal solution of the same. This formulation appears to be closely related to the notion of gap function studied in [8].
4. By using Proposition 1 (part *2*), a description of the solution set of the above QP can be obtained. The v part of the this set is U^*_β.
5. By using part *1* of Proposition 1, a linear program is set up for which $\bar{v}$ is an optimal solution if and only if $U^*_\beta \subset S^*$. This latter condition is characterized using LP duality.

Algebraic execution of the above steps is given below.

– Step 1 can be stated as:

$$\begin{aligned} Z^T((Q + Q^T)\bar{v} + c) &\geq A^T\nu \\ \bar{v}^T((Q + Q^T)\bar{v} + c) &\leq b^T\nu. \end{aligned}$$

– The condition $\beta \in W(\bar{v})$ is given by:

$$\begin{aligned} Z^T(Q\bar{v} + c + \beta) &\geq A^T\bar{\rho} \\ \bar{v}^T(Q\bar{v} + c + \beta) &\leq b^T\bar{\rho} \\ \beta &\geq 0. \end{aligned}$$

– The set of optimal solutions of the convex QP in Step 3 can be written down via an application of Proposition 1:

$$\tilde{U} := \{(v,\rho) | (Q{+}Q^T)(v{-}\bar{v}) = 0, (c{+}\beta)(v{-}\bar{v}) = b^T(\rho{-}\bar{\rho}), Z^T(Qv{+}c{+}\beta) \geq A^T\rho, v \in V\}.$$

Note also that S^* is given by:

$$S^* = \{v \in V | (Q + Q^T)(v - \bar{v}) = 0, g^T(v - \bar{v}) = 0\},$$

where g is the gradient $(Q + Q^T)\bar{v} + c$ of the system objective function at $\bar{v}$.

– Since the v part of the above polyhedron is U^*_β, requiring that $U^*_\beta \subset S^*$ is the same as having $(v,\rho) \in \tilde{U}$ imply that $g^T(v - \bar{v}) = 0$. But, as $\bar{v}$ is a system optimal solution, $g^T(v - \bar{v}) \geq 0, \forall v \in V$. Therefore, $U^*_\beta \subset S^*$ if and only if $\bar{v}$ is a maximizer of g^Tv over U^*_β. Thus, $U^*_\beta \subset S^*$ is equivalent to:

$$\bar{v} \in \operatorname{argmax}\{g^Tv | (c{+}\beta)(v{-}\bar{v}) = b^T(\rho{-}\bar{\rho}), Z^T(Qv{+}c{+}\beta) \geq A^T\rho, v = Zx, Ax = b, x \geq 0\}.$$

Now, one can write the dual program of the above LP, and apply complementary slackness to arrive at: the condition $U^*_\beta \subset S^*$ holds if and only if there exist μ, u, z, w, λ such that

$$\begin{aligned} \mu(c + \beta) + (Q + Q^T)u - Q^TZz + w &= (Q + Q^T)\bar{v} + c \\ Az &= \mu b \\ A^T\lambda &\geq Z^Tw \\ z &\geq 0 \\ z^T(Z^T(Qv + c + \beta) - A^T\rho) &= 0 \\ b^T\lambda &= v^Tw. \end{aligned}$$

Putting all the pieces together readily yields a description of the toll set $\mathcal{T}$ (with $\bar{v}$ now being replaced by v).

Theorem 3 *Let $s = Qv + c$, where $Q + Q^T$ is positive semidefinite. Then β is a valid toll vector (i.e., $\beta \in \mathcal{T}$) if and only if there exist (scalars and vectors of appropriate dimensions) $x, v, \nu, \rho, \mu, u, z, w, \lambda$ such that the following multi-quadratic system is satisfied:*

$$
\begin{aligned}
v &= Zx \\
Ax &= b \\
x &\geq 0 \\
Z^T((Q+Q^T)v+c) &\geq A^T\nu \\
v^T((Q+Q^T)v+c) &\leq b^T\nu \\
Z^T(Qv+c+\beta) &\geq A^T\rho \\
v^T(Qv+c+\beta) &\leq b^T\rho \\
\beta &\geq 0 \\
\mu(c+\beta)+(Q+Q^T)u-Q^TZz+w &= (Q+Q^T)v+c \\
Az &= \mu b \\
A^T\lambda &\geq Z^Tw \\
z^T &\geq 0 \\
z^T(Z^T(Qv+c+\beta)-A^T\rho) &= 0 \\
b^T\lambda &= v^Tw.
\end{aligned}
$$

3.3 Computational Implications

The system optimum problem is an uncapacitated nonlinear multicommodity network flow problem. There are well studied algorithms for solving such problems which are also commercially available. Most such codes are based on the Frank-Wolfe technique, for which, in the case of the traffic assignment problem, the subproblems reduce to shortest path problems. Typically, one needs the convexity of the system objective function (i.e., S-convexity of s) for establishing convergence to global optimal solutions. Irrespective of the method employed, let v^* be a system optimum solution.

Then, assuming that s is strictly monotonic, it can be concluded that the polyhedron $W(v^*)$ gives valid tolls. If one wants to optimize a linear function of the toll vector, then a linear program will need to be solved. More general functions of β can also be considered for minimization over the polyhedron $W(v^*)$.

In [3], it was proposed that one minimize β^Tv^* over $W(v^*)$ so that, the *total toll* imposed on the users is minimized; this technique for computing the tolls was named the "MINSYS" approach. Specifically, the MINSYS approach is

Step 1: Solve the system optimum problem to obtain an optimal solution v^*.
Step 2: Minimize β^Tv^* over the polyhedron $W(v^*)$ defined by the inequalities:

$$
\begin{aligned}
Z^T(s(v^*)+\beta) &\geq A^T\rho \\
(v^*)^T(s(v^*)+\beta) &= b^T\rho \\
\beta &\geq 0
\end{aligned}
$$

to obtain optimal MINSYS toll vector β_{MINSYS} (this need not be unique).

It is clear that imposing the MINSYS tolls will make equilibrium flows system optimal, when s is strictly monotonic. The results of [3] as well as those of its predecessor [7] were developed in the context of bounded flow traffic assignment problems.

When the system optimum is not unique, one may perform the above computations for two or more alternate system optima to further reduce the total toll imposed, while still enforcing that the tolled user equilibrium solutions are system optimal.

A related aspect is the effect of the formulation used, i.e., whether arc-node or arc-path formulation is used. For simplicity, assume that s is both strictly monotonic and SS-convex, so that the toll set is given precisely by $W(v^S)$, v^S is the unique system optimal solution. It is evident that different descriptions of the toll set are obtained based on which formulation is used, although the toll set itself is invariant. This is explained by the fact that the description of $W(v^S)$ as given by Lemma 3 involves an additional (other than β) set of variables, namely ρ, which differ based on the formulation. Actually, the description of $\mathcal{T}$ under the arc-path formulation may be considered as being obtained from that under the other formulation via the elimination of some of the ρ variables of the latter.

The observation is that, for the arc-node formulation, the resulting description of the toll set has polynomially many variables and constraints, while the arc-path formulation could potentially have exponentially many constraints.

Let us turn our attention to the affine monotonic special case for which a multiquadratic description was derived in the previous subsection. Suppose that one wishes to minimize a linear or, more generally, a quadratic function of β over the toll set $\mathcal{T}$. The resulting optimization problem is a so-called **Multiquadratic Programming Problem(MQP)**. This problem was studied in the third author's Ph.D. thesis [16]. There, some theoretical and algorithmic results were developed for MQP. However, the general MQP is a highly intractable NP-hard problem, and hence, it remains to be studied whether the MQP resulting from our toll set description has certain special structure enabling one to solve certain optimization problems over $\mathcal{T}$ in polynomial time. It remains to be seen if one can reduce the description to a semidefinite programming problem, for which several efficient algorithms have been developed, see for instance [13].

4 Numerical Examples

4.1 The Toll Set for the Braess Paradox

A complete treatment of the valid toll set for the Braess paradox is given below. As mentioned before, the user equilibrium aggregate flows are $v^U = [4\ 2\ 2\ 4\ 2]^T$. However, the aim is to determine tolls that make the users choose routes so that a system optimizing link flow pattern of $v^S = [3\ 3\ 3\ 3\ 0]^T$ is achieved. The link cost at the system optimizing link flows are $[30\ 53\ 53\ 30\ 10]$.

The MINSYS approach is formulated as follows. First note that the cost map s strictly monotonic and the system objective function $s(v)^T v$ is strictly convex.

From the arc-path formulation, the matrix Z is given by

$$Z = \begin{bmatrix} 1\ 0\ 1 \\ 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 1\ 1 \\ 0\ 0\ 1 \end{bmatrix},$$

and $A = [1\ 1\ 1]$. The linear programming problem to be solved is thus:

$$\begin{aligned} \min: \quad & 3\beta_1 + 3\beta_2 + 3\beta_3 + 3\beta_4 \\ & \begin{bmatrix} \beta_1 + \beta_2 \\ \beta_3 + \beta_4 \\ \beta_1 + \beta_4 + \beta_5 \end{bmatrix} - \begin{bmatrix} \rho \\ \rho \\ \rho \end{bmatrix} \geq - \begin{bmatrix} 83 \\ 83 \\ 70 \end{bmatrix} \\ & 3\beta_1 + 3\beta_2 + 3\beta_3 + 3\beta_4 - 6\rho = -498 \\ & \beta \geq 0. \end{aligned}$$

The optimal solution to this problem is $\beta^* = [0\ 0\ 0\ 0\ 13]$ and $\rho = 83$. The tolls calculated by the marginal social cost pricing methodology (the toll on link a is $\frac{ds_a(v_a^S)}{dv_a} v_a^S$ since the link costs are separable) are $\beta_{MSCP} = [30\ 3\ 3\ 30\ 0]$. It is not difficult to see that these tolls are a feasible solution to the system above along with the multiplier $\rho = 116$. However, this solution is not optimal since its objective value is 198 while the optimal objective value is 0.

4.2 Two More Examples

Now we will discuss two paradoxes given in the article by Arnott and Small [2]. The network for both problem has two nodes A, B and two arcs as shown in Figure 2. The O-D pair (A, B) has a demand of 1000 travelers.

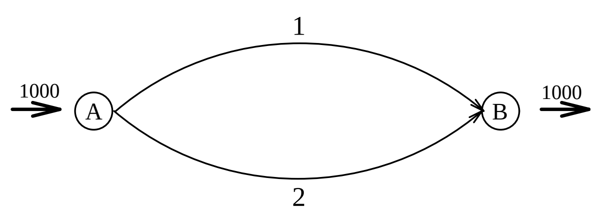


Figure 2: The Paradox Network

The cost maps for the two paradoxes are given as below, where C denotes the "capacity" of the bridge that is depicted as arc 1 in the figure. The capacity and the flows of the two problems will be scaled to be in units of 1000.

Pigou-Knight-Downs Paradox (PKDP) : $s(v) = [(10 + 10v_1/C), 15]^T$.

Downs-Thompson Paradox (DTP) : $s(v) = [(10 + 10v_1/C), 20 - 10v_2/3]^T$.

Note that the cost map is not monotonic for DTP, and weakly monotonic in the case of PKDP. The former fact is explained in [2] as follows: arc 2 is assumed to be a train service, and so if there is more traffic, then the frequency of the service would be higher and thus the waiting time is reduced. This nonmonotonicity has interesting consequences on the toll set obtained, as we will shortly see.

The toll sets of these two problems are given below, with the calculations given in the appendix.

Tolls for the Pigou-Knight-Downs Paradox The system optimal solution for PKDP is unique for every positive C and it is given by:

$$\begin{aligned} v^S &= [C/4, 1 - C/4]^T \text{ if } C \leq 4 \\ &= [1, 0]^T \qquad\qquad\quad \text{if } C \geq 4. \end{aligned}$$

The untolled user optimum is:

$$\begin{aligned} v^U &= [C/2, 1 - C/2]^T \text{ if } C \leq 2 \\ &= [1, 0]^T \qquad\qquad\quad \text{if } C \geq 2. \end{aligned}$$

The toll set $\mathcal{T}$ in this case turns out to be the same as $W(v^S)$, and is given by:

$$\begin{aligned} \mathcal{T} &= \{(\beta_1, \beta_2) \geq 0 | \beta_1 = 2.5 + \beta_2\} & \text{if } C < 4 \\ \mathcal{T} &= \{(\beta_1, \beta_2) \geq 0 | \beta_1 \leq (5 - 10/C) + \beta_2\} & \text{if } C \geq 4. \end{aligned}$$

It is interesting to observe that, the half line given by $\beta_1 = 2.5 + \beta_2$ and $\beta_1, \beta_2 \geq 0$ always gives valid toll vectors, irrespective of the capacity C. One particular choice on this line is $\beta_1 = 2.5, \beta_2 = 0$.

Downs-Thompson Paradox This is a more interesting example. Here, we will see that even though the system optimum is still unique, it turns out that the toll set is smaller than $W(v^S)$ when $C > 3$.

As was the case for PKDP, the system optimal solution for DTP is unique for every positive C and is given by:

$$\begin{aligned} v^S &= (1/2(3 - C))[C, 3(2 - C)]^T & \text{if } C \leq 2 \\ &= [1, 0]^T & \text{if } C \geq 2. \end{aligned}$$

The untolled user optimum is:

$$\begin{aligned} v^U &= (1/(3 - C))[2C, 3 - 3C]^T & \text{if } C \leq 1 \\ &= [1, 0]^T & \text{if } C \geq 1. \end{aligned}$$

To describe the toll set for this problem three separate intervals for C need to be considered, namely, $(0, 2), [2, 3)$ and $[3, \infty)$. First, the following gives expressions for the set $W(v^S)$:

$$\begin{aligned} W(v^S) &= \{(\beta_1, \beta_2) \geq 0 | \beta_1 = 5 + \beta_2\} & \text{if } C < 2 \\ W(v^S) &= \{(\beta_1, \beta_2) \geq 0 | \beta_1 \leq 10(1 - 1/C) + \beta_2\} & \text{if } C \geq 2. \end{aligned}$$

In the interval $(0,3)$ the toll set $\mathcal{T}$ turns out to be the same as $W(v^S)$:

$$\begin{array}{ll} \mathcal{T} = \{(\beta_1,\beta_2) \geq 0 | \beta_1 = 5 + \beta_2\} & \text{if } C < 2 \\ \mathcal{T} = \{(\beta_1,\beta_2) \geq 0 | \beta_1 \leq 10(1-1/C) + \beta_2\} & \text{if } 2 \leq C < 3, \end{array}$$

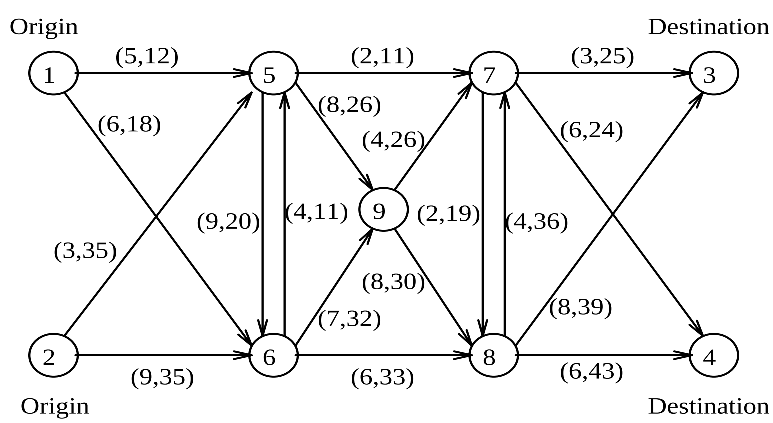
whereas, in the interval $[3, \infty)$, in can be shown (see Appendix) that

$$\mathcal{T} = \{(\beta_1,\beta_2) \geq 0 | \beta_1 < \tfrac{20}{3} + \beta_2\} \text{ if } C \geq 3.$$

Also, similar to the PKDP, the line $\beta_1 = 5 + \beta_2, \beta_1, \beta_2 \geq 0$ gives valid tolls for every positive capacity C.

4.3 The Nine-node Problem

To provide a comparison of MINSYS versus MSCP tolls on an example problem with data similar to large-scale traffic assignment problems, we have employed the *nine-node network* from [9]. The network consists of 9 nodes and 18 links and all of the links have cost functions with the same structure, $s_a(v) = s_a(v_a) = T_a(1 + 0.15(v_a/b_a)^4)$, where T_a and b_a are constants. There are four OD-pairs: (1,3), (1,4), (2,3) and (2,4). The network is shown in Figure 3 wherein the tuple near arc a is (T_a, b_a).



O-D Pair:	[1,3]	[1,4]	[2,3]	[2,4]
Demand:	10	20	30	40

Figure 3: The Nine-node Network

The MINSYS tolls are quite different from the tolls calculated by the MSCP principle as shown in Table 1. The total toll system cost (total system cost + total toll cost) in the MSCP case is equal to 3747 (2254 + 1493) and in the MINSYS case equal to 3142 (2254 + 888). So with the MSCP principle the users of the nine-node network pay 68 % more tolls than with the MINSYS pricing principle.

Arc	v_a	$s_a(v_a)$	$v_a s_a(v_a)$	MSCP Tolls	Toll Cost	MINSYS tolls	Toll Cost
1-5	9.411	5.284	49.728	1.135	10.681	0	0
1-6	20.589	7.541	155.262	6.162	126.869	0	0
2-5	38.334	3.648	139.842	2.590	99.285	4.000	153.336
2-6	31.666	9.905	313.652	3.618	114.566	0	0
5-6	0.000	9.000	0	0	0	0	0
5-7	21.303	6.220	132.505	16.880	359.595	11.200	238.594
5-9	26.442	9.284	245.487	5.135	135.780	0	0
6-5	0.000	4.000	0	0	0	0	0
6-8	39.474	7.843	309.595	7.370	290.923	7.200	284.213
6-9	12.781	7.027	89.812	0.107	1.368	0	0
7-3	29.608	3.885	115.027	3.541	104.841	4.000	118.432
7-4	20.757	6.504	135.004	2.014	41.805	0	0
7-8	0.000	2.000	0	0	0	0	0
8-3	10.392	8.006	83.198	0.024	0.249	0	0
8-4	39.243	6.624	259.946	2.497	97.990	0	0
8-7	0.000	4.000	0	0	0	0	0
9-7	29.062	4.937	143.479	3.746	108.866	3.2	92.998
9-8	10.162	8.016	81.459	0.063	0.640	0	0
Total			2253.918		1493.458		887.574

Table 1: The Nine-node Problem - MSCP/MINSYS Tolls

5 Concluding Remarks

In this paper, a notion of *valid tolls* was introduced and investigated. For the case when the cost map is strictly monotonic and the system objective function $s(v)^T v$ is strictly convex, the set of valid tolls is an easily described polyhedron. Hence, in this case, approaches such as MINSYS for computing least cost tolls can be easily incorporated into existing software packages by simply adjoining them with a linear programming solver.

An interesting observation one can make from Table 1 is that, the total numbers of arcs with nonzero tolls for MSCP tolls is 14, while that for the MINSYS tolls is 5. Since a nonzero toll implies installing a toll booth or perhaps electronic hardware, it is clearly desirable to have as few nonzero tolls as possible. To take this issue to the extreme, one can minimize the total number of toll booths by solving an integer program built from the MINSYS constraint set.

Some directions for future research are the investigation of theoretical properties of the toll set $\mathcal{T}$ for various classes of maps and the development of specialized (perhaps combinatorial) algorithms for MINSYS and related problems.

Appendix: Derivations for the problems PKDP and DTP

Note that, for these two problems, $v \in V$ if and only if $v_1 + v_2 = 1, v_1, v_2 \geq 0$. Let us characterize the condition $v \in U_\beta^*$ for $v \in V$. We have $v \in U_\beta^*$ if and only if there exists ρ such that

$$\begin{aligned} s_1(v) + \beta_1 &\geq \rho \\ s_2(v) + \beta_2 &\geq \rho\,, \\ v_1(s_1(v) + \beta_1) + v_2(s_2(v) + \beta_2) &= \rho \end{aligned}$$

which holds if and only if

$$s_1(v) + \beta_1, s_2(v) + \beta_2 \geq v_1(s_1(v) + \beta_1) + v_2(s_2(v) + \beta_2),$$

which is the same as

$$\begin{aligned} v_2(s_1(v) - s_2(v) + \beta_1 - \beta_2) &\geq 0 \\ v_1(s_1(v) - s_2(v) + \beta_1 - \beta_2) &\leq 0 \end{aligned},$$

and this last condition is equivalent to one of the following being valid:

$$\begin{array}{lll} \text{Case A:} & v_1 = 0, v_2 = 1 & \text{and} \quad s_1(v) - s_2(v) + \beta_1 - \beta_2 \geq 0 \\ \text{Case B:} & v_1 = 1, v_2 = 0 & \text{and} \quad s_1(v) - s_2(v) + \beta_1 - \beta_2 \leq 0 \\ \text{Case C:} & 0 < v_1, v_2 < 1 & \text{and} \quad s_1(v) - s_2(v) + \beta_1 - \beta_2 = 0. \end{array}$$

5.1 Analysis for PKDP

First, let us derive expressions for $W(v^S)$. Note that for PKDP $s_1(v) - s_2(v) = 10v_1/C - 5$. When $0 < C < 4$, $0 < v_1^S, v_2^S < 1$ (see §4.2.1), and hence Case C applies here:

$$W(v^S) = \{\beta \geq 0 | (10/C)(C/4) - 5 + \beta_1 - \beta_2 = 0\} = \{\beta \geq 0 | \beta_1 - \beta_2 = 2.5\}.$$

On the other hand, when $C \geq 4$, then $v^S = [1, 0]^T$ and hence case B applies:

$$W(v^S) = \{\beta \geq 0 | ((10/C) - 5) + \beta_1 - \beta_2 \leq 0\} = \{\beta \geq 0 | \beta_1 - \beta_2 \leq 5 - 10/C\}.$$

Since the system optimum is unique in all the cases for the two problems, clearly $\mathcal{T} \subseteq W(v^S)$. To identify the toll set, we need to determine the set of those $\beta \in W(v^S)$ for which $U_\beta^* \subset S^* = \{v^S\}$. Since by definition $v^S \in U_\beta^*$ for every $\beta \in W(v^S)$, this is the same as requiring that U_β^* contain no *spurious* solutions, i.e., it contains only v^S. In most of the cases, we can employ the following idea to show that $\mathcal{T} = W(v^S)$.

First, note that the set of tolled used equilibrium solutions (whenever this set is nonempty) are precisely the v part of the optimal solutions for the problem:

$$\min\{v^T(s(v) + \beta) - b^T\rho | C^T(s(v) + \beta) \geq A^T\rho, v \in V\} \qquad (*)$$

owing to the fact that the constraints ensure that the objecive function is non-negative over the feasible region. Since there exists an equilibrium solution, the optimal objective value is zero, and the claim follows. Now we eliminate the variable v_2 from the above optimization problem. This is achieved by setting $v_2 = 1 - v_1$. This elimination turns the objective function

$$v_1(s_1(v) + \beta_1) + v_2(s_2(v) + \beta_2) - \rho$$

into the following:

$$(10/C)v_1^2 + v_1(10 + \beta_1 - 15 - \beta_2) + 15 + \beta_2 - \rho.$$

Since this is a strictly convex function of v_1, it follows that the tolled user equilibrium solution is unique for this problem. That is sufficient to conclude that for this problem $\mathcal{T} = W(v^S)$.

Analysis for DTP For this problem, $s_1(v) - s_2(v) = 10(v_1/C + v_2/3 - 1)$. When $0 < C < 2$, $0 < v_1^S, v_2^S < 1$ (see §4.2.2), and hence Case C applies here:

$$W(v^S) = \{\beta \geq 0 | (10/(2(3-C)))(1+(2-C)-2(3-C))+\beta_1-\beta_2 = 0\} = \{\beta \geq 0 | \beta_1-\beta_2 = 5\}.$$

When $C \geq 2$, the system optimum is $v^S = [1, 0]^T$ and hence Case B will apply, giving us:

$$W(v^S) = \{\beta \geq 0 | \beta_1 - \beta_2 \leq 10(1 - 1/C)\}.$$

Performing the elimination as for PKDP gives a quadratic objective function (for $(*)$ whose pure quadratic term is

$$10v_1^2(1/C - 1/3).$$

Thus, if $C < 3$, then the above conclusion of the uniqueness of tolled equilibrium solutions holds true once again. This leaves us to consider the situation when $C \geq 3$.

We claim that when $C \geq 3$ in DTP, the toll set is given by

$$\mathcal{T} = \{\beta \geq 0 | \beta_1 - \beta_2 < \frac{20}{3}\}.$$

To prove this, suppose that

$$20/3 \leq \beta_1 - \beta_2 \leq 10(1 - 1/C). \qquad (**)$$

It is claimed that the vector $[0, 1]^T$ is in U_β^*. So, substituting this vector in the expression for Case A, we get

$$20/3 \leq \beta_1 - \beta_2,$$

which holds. Therefore $[0, 1]^T$ is a tolled equilibrium solution. This is sufficient to prove the validity of the exression for the toll set. However, let us continue and determine what other solutions are tolled user equilibrium solutions for $\beta \in W(v^S) \backslash \mathcal{T}$.

First, if $C = 3$, then the optimization problem $(*)$ turns into a linear program. Since $[1, 0], [0, 1]$ are the extreme points of the feasible region, it follows that for this case every point in the simplex is a tolled equilibrium solution.

Now supose that $C > 3$. Then to see if Case C type solutions exist, we solve

$$v_1 + v_2 = 1, \beta_1 - \beta_2 + 10(v_1/C - v_2/3 - 1) = 0.$$

This gives us

$$v_1 = 3C/(10(C-3))(\beta_1 - \beta_2 - 20/3).$$

Since $\beta_1 - \beta_2 \leq 10(1-1/C)$, it can be shown that $v_1 \leq 1$. It is also seen that, as $\beta_1 - \beta_2$ varies from $20/3$ to $10(1-1/C)$, v_1 varies from 0 to 1. Thus, the tolled equilibrium solutions are two in number when at least one of the inequalities in $(**)$ holds as equality, and three if they are both strict.

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