1 SOLVING CONGESTION TOLL PRICING MODELS

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Abstract:

Recently a methodology for traffic networks has been developed which extracts congestion toll sets such that the tolled user equilibrium is system optimal. Properties of toll sets, such as convexity, are investigated, as well as relationships with other problems. For a given toll set, various objectives can be defined and optimized with respect to the tolls. Examples include minimizing the total tolls collected, minimizing the number of toll booths and constraining net tolls collected to be zero. We illustrate with an example and report on our computational experience with the Stockholm network.

1.1 INTRODUCTION

Congestion toll pricing addresses the classic traffic assignment problem for which Wardrop enunciated two principles of traffic flow: user-optimal behavioral hypothesis and the notion and system-optimality. (See [7] for a recent review of the traffic assignment problem and [12] for a recent volume of papers on road pricing.) The traditional objective of congestion pricing has been to determine link tolls which will cause the solution of the tolled user-optimal problem to be optimal for the untolled system problem [1]. In most of the literature, the one choice given has been the vector of *marginal social cost pricing* tolls.

Recently, in [3], we have introduced the notion of a *toll set,* i.e., the set of all tolls which accomplish this objective for a given problem instance. A primary result concerns the situation where both problems have unique solutions; in this case the toll set has been shown to be a polyhedron defined in terms of the system optimal solution. The marginal social cost pricing vector is just one element of this set.

In this paper toll sets are defined in a more general way than in [3]. Specifically, we allow link tolls to be either positive or negative, with a negative toll corresponding to a subsidy for the associated link. Various properties of toll sets, such as convexity, are discussed as well as relations to other problems such as MPEC (mathematical programs with equilibrium constraints [13]) and bounded flow equilibrium problems [2, 11]. Further, an example problem is used to illustrate a Toll Pricing Framework for determining alternative tolls in a given problem by optimizing a secondary objective over the toll set. The alternatives include i) minimizing the total tolls collected, ii) minimizing the maximum toll on any link, iii) minimizing the number of toll booths, iv) constraining net tolls collected to be zero, and v) combining iii) and iv).

Finally, we report our computational experience with traffic assignment data from the Stockholm network.

1.2 THEORY OF CONGESTION TOLL PRICING

1.2.1 Notation

We employ the notation from [3]: $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ is a network with $\mathcal{N}$ being the node set and $\mathcal{A}$ being the arc set. $\hat{A}$ is the incidence matrix of $\mathcal{G}$ and $\mathcal{K}$ is a set of commodities that flow in the network. Corresponding to each commodity k, there is an associated demand vector $b(k)$ of size $n = |\mathcal{N}|$, whose ith entry denotes the demand for that commodity at node i. The entries of $b(k)$ sum to zero. The kth *commodity flow* (variable) vector is denoted by $x(k)$ and the sum of all commodity flow vectors is the *aggregate flow* vector v. The system defining feasible flows is given by:

$$\begin{array}{rcl} v & = & \sum_k x(k) \\ \hat{A}x(k) & = & b(k) \ \forall \ k \in \mathcal{K} \\ x(k) & \geq & 0 \ \forall \ k. \end{array}$$

This is known as the *arc-node formulation* of feasible flows in traffic assignment (see [7]). It is clear that the above system can be cast in the following more general format:

$$\begin{aligned} v &= Zx \\ Ax &= b \\ x &\geq 0. \end{aligned} \tag{1.1}$$

The *arc-route formulation*, often employed in Traffic Assignment literature, can also be expressed in the above form, by choosing Z to be the arc-path incidence matrix and A the O-D pair-path incidence matrix.

In (1.1), x is called the vector of **individual flows**, and v is the vector of **aggregate flows**. Now define the sets:

$$\begin{aligned} F &= \{(v,x) | v = Zx, Ax = b, x \geq 0\} \\ V &= \{v | \text{ there exists } x \text{ such that } (v,x) \in F\}. \end{aligned}$$

Here, F is the set of feasible flows, V the set of feasible aggregate flows.

A continuously differentiable **cost map** $s : \mathcal{A} \to \mathcal{A}$ is assumed given. Its components have the interpretation that, for feasible aggregated flows v, the cost (or time) incurred by a user on arc a is given by $s_a(v)$. The Jacobian of s is denoted ∇s. The traffic assignment problem refers to various optimization and equilibrium problems involving the set V, the cost map s, and certain functions derived from this map.

1.2.2 *Theory of Toll Pricing and Prior Results*

Suppose that every user of a given arc a on a traffic is charged an amount M_a of toll (if this number is negative, then the user receives a subsidy) in dollars. Assuming that R is the value (in $\$/min$, see [1]) of a unit of time to a given user, then the effective delay as perceived by that user on arc a is given by $s_a(v) + M_a/R$. Letting $\beta_a = M_a/R$, the effective time delay experienced by all users of the network is given by:

$$s_\beta(v) = s(v) + \beta,$$

where β is the vector whose components are β_a. A *tolled equilibrium* solution is a $\bar{v}$ such that

$$s_\beta(\bar{v})^T(v - \bar{v}) \geq 0 \ \forall \ v \in V.$$

The total delay in the network is given by $s(v)^T v$, and let S^* be the set of minimizers v^* of this function over V. Since it desirable that the total delay is minimized, a toll vector β should be such that every possible tolled equilibrium solution is in the set S^*. This leads to the following **Principle of Toll Pricing**:

> *The tolls imposed should be such that the resulting tolled user equilibrium problem has at least one solution and every such solution is an untolled system optimal solution.*

The mathematical formalism for the above is adopted from [3], and it goes as follows.

Consider the tolled user equilibrium problem:

$$(s(\bar{v})+\beta)^T(v-\bar{v}) \geq 0, \forall v \in V. \tag{UOPT-β}$$

Denote by U^*_β, the set of **Tolled Equilibrium Solutions**, i.e., those $\bar{v}$ that satisfy UOPT-β, and let

$$S^* := \operatorname{argmin}\{s(v)^T v | v \in V\}. \tag{SOPT}$$

Since we would like the toll vector to be such that the resulting user equilibrium problem has a solution, and *every* equilibrium solution is system optimal, we require

$$\emptyset \neq U^*_\beta \subseteq S^*.$$

Any β satisfying the above will be called a **Valid Toll Vector**. The Toll Set is defined by

$$\mathcal{T} := \{\beta | \emptyset \neq U^*_\beta \subseteq S^*\}.$$

The purpose of this section is to review some of the results obtained in [3], to develop some further results concerning the convexity of toll sets and to demonstrate the relations of the notion of toll pricing with certain problems pertaining to traffic assignment.

First, consider the following system:

$$\begin{array}{c} Z^T(s(\bar{v})+\beta) \geq A^T\rho \\ (s(\bar{v})+\beta)^T\bar{v} = b^T\rho. \end{array} \tag{2}$$

The lemma below is a consequence of LP duality and it can be found in [3].

Lemma 1 *Let β be any toll vector, and $\bar{v}$ be any feasible aggregate flow. Then, $\bar{v}$ is a tolled equilibrium solution with β being the toll vector (i.e., $\bar{v} \in U^*_\beta$) if and only if there exists a ρ such that system in (2) is satisfied.*

Similar to the definition in [3], let

$$W(\bar{v}) = \{\beta | \bar{v} \in U^*_\beta\},$$

where $\bar{v} \in V$, and this is the β part of the polyhedron (2). Further, it is not hard to show that

$$W(\bar{v}) = N(\bar{v}; V) - s(\bar{v}),$$

where

$$N(\bar{v}, V) = \{u | u^T(v-\bar{v}) \geq 0 \ \forall \ v \in V\}$$

denotes the normal cone to V at the point $\bar{v}$. Therefore, $W(\bar{v})$ is a shifted polyhedral cone.

The first result described below expresses the toll set as the difference of two sets, the first of which is the union of all $W(v^*)$ as v^* varies over the system optimal solution set S^*, while the second set is the union of the $W(\hat{v})$ as $\hat{v}$ varies over the complement $V \backslash S^*$.

Theorem 1 $\mathcal{T} = (\cup_{v^* \in S^*} W(v^*)) \backslash (\cup_{\hat{v} \notin S^*} W(\hat{v}))$

Proof: Let $\bar{W}$ denote the first union on the right hand side of the statement, and $\hat{W}$ the second. Then, $\beta \in \bar{W}$ if only if U^*_β is nonempty, and $\beta \notin \hat{W}$ if and only if $U^*_\beta \subset S^*$. $\square$

The next result specializes to the case when the cost map s is strictly monotonic:

Theorem 2 *If s is strictly monotonic, then*

$$\mathcal{T} = \cup_{v^* \in S^*} W(v^*).$$

Proof: By the hypothesis, the user-optimal solutions are unique. Hence, from the proof of Theorem 1, the sets $\bar{W}$ and $\hat{W}$ are disjoint. $\square$

(In [3] Theorem 2 was established differently, using an alternative expression: $\mathcal{T} = \cup_{v^* \in S^*} \{\beta \in W(v^*) | U^*_\beta \subseteq S^*\}$.)

If in addition to the strict monotonicity of s, we also have the strict convexity of the SOPT objective, then the toll set assumes a very simple form, which we will refer to as the **ideal case**:

Theorem 3 *Suppose that s is both strictly monotonic and $s(v)^T v$ is strictly convex. Then there exists a unique optimal solution v^* to the system optimum problem and thus the toll set $\mathcal{T}$ equals the polyhedron $W(v^*)$.*

Finally, in [3], the case when the cost map if of the form $s(v) = Qv + c$, where Q is a positive semidefinite (but not necessarily symmetric) matrix was considered. By the application of some previously known results regarding the solution sets of convex quadratic programs [14], an algebraic description of the toll set was derived:

Theorem 4 *Let $s = Qv + c$, where $Q + Q^T$ is positive semidefinite. Then β is a valid toll vector (i.e., $\beta \in \mathcal{T}$) if and only if there exist (scalars and vectors of appropriate dimensions) $x, v, \nu, \rho, \mu, u, z, w, \lambda$ such that the following multiquadratic system is satisfied:*

$$\begin{array}{rcl} v & = & Zx \\ Ax & = & b \\ x & \geq & 0 \\ Z^T((Q+Q^T)v+c) & \geq & A^T\nu \\ v^T((Q+Q^T)v+c) & \leq & b^T\nu \\ Z^T(Qv+c+\beta) & \geq & A^T\rho \\ v^T(Qv+c+\beta) & \leq & b^T\rho \\ \beta & \geq & 0 \\ \mu(c+\beta)+(Q+Q^T)u-Q^TZz+w & = & (Q+Q^T)v+c \\ Az & = & \mu b \\ A^T\lambda & \geq & Z^Tw \\ z^T & \geq & 0 \\ z^T(Z^T(Qv+c+\beta)-A^T\rho) & = & 0 \\ b^T\lambda & = & v^Tw. \end{array}$$

This description is rather interesting, for, as we will show in §2.3, even for the very special situation when $Q = 0$ (i.e., the map is constant) the toll set can be nonconvex.

1.2.3 Convexity of Toll Sets

This subsection addresses the natural question of whether toll sets are convex, motivated by the fact that they are central to toll pricing methodology. Of course, for the ideal case it has already been shown that $\mathcal{T} = W(v^*)$, which is a closed convex set. The two examples below show that when the map is linear, then the toll set may be either convex or nonconvex.

Example 1 A Convex Toll Set: *Consider a network with two nodes* $\mathcal{N} = \{1,2\}$ *and two arcs both of the type* $(1,2)$. *Suppose that one unit of a commodity flows from node* 1 *to node* 2. *If the cost map is given by* $s(v) = (1,1)^T$, *any feasible solution is system optimal and hence* $S^* = V = \{v \geq 0 | v_1 + v_2 = 1\}$. *In terms of Theorem 1, the toll set is the union of the three polyhedra:* $\{\beta | \beta_1 > \beta_2\}$, $\{\beta | \beta_1 < \beta_2\}$, *and* $\{\beta | \beta_1 = \beta_2\}$. *That is, all choices of* β *are valid tolls,* $\mathcal{T} = \Re^2$, *which is clearly convex.*

Example 2 A Nonconvex Toll Set: *Consider another network with two nodes* $\mathcal{N} = \{1,2\}$ *and three arcs, all of the type* $(1,2)$. *Again there is one unit of flow. Letting* $s(v) = (1,1,2)^T$, *we have*

$$\begin{array}{rcl} V & = & \{v | v_1 + v_2 + v_3 = 1\} \\ S^* & = & \{v \in V | v_3 = 0\}. \end{array}$$

For a given toll vector $\beta \in \Re^3$*, the tolled cost map is* $s_\beta = \{1+\beta_1, 1+\beta_2, 2+\beta_3\}$*. A vector* β *is valid if and only if there is no tolled user equilibrium solution with* $v_3 > 0$*, which happens if and only if*

$$\text{either } 1+\beta_1 < 2+\beta_3 \text{ or } 1+\beta_2 < 2+\beta_3,$$

and hence $\mathcal{T} = \{\beta | \beta_3 > \beta_1 - 1 \text{ or } \beta_3 > \beta_2 - 1\}$*, which is both nonconvex and open. The nonconvexity can be seen by considering* $\bar{\beta} = (0,4,0)^T$ *and* $\hat{\beta} = (4,0,0)^T$ *both of which are valid tolls, whereas* $\beta = \frac{1}{2}\bar{\beta} + \frac{1}{2}\hat{\beta}$ *is not.*

Now we prove that the toll set is convex when the cost map is monotonic with a symmetric Jacobian (in other words, it is the gradient map of a convex function) and the system optimum solution is unique.

Theorem 5 *If* s *is a differentiable monotonic map with a symmetric Jacobian, and if* S^* *is a singleton* v^**, then the toll set* $\mathcal{T}$ *is convex.*

Proof: Let g be a convex function such that $s = \nabla g$. Then, it is clear that for $\beta \in \mathcal{T}$, the tolled user equilibrium solution is $U^*_\beta = \{v^*\}$, the unique minimizer of the function $g(v) + \beta^T v$. Let $\beta_1, \beta_2 \in \mathcal{T}, 0 \le \lambda \le 1$, and define $\beta = \lambda\beta_1 + (1-\lambda)\beta_2$. Since β_1, β_2 are valid toll vectors, we have for any $v \in V, v \ne v^*$,

$$g(v^*) + \beta_i^T v^* < g(v) + \beta_i^T v, i = 1,2$$

and hence

$$\begin{aligned} g(v^*) + \beta^T v^* &= \lambda(g(v^*) + \beta_1^T v^*) + (1-\lambda)(g(v^*) + \beta_2^T v^*) \\ &< \lambda(g(v) + \beta_1^T v) + (1-\lambda)(g(v) + \beta_2^T v) \\ &= g(v) + \beta^T v. \end{aligned}$$

Hence v^* is the unique minimizer of $g(v) + \beta^T v$, giving that $\beta \in \mathcal{T}$. □

Note that the conditions of the theorem are satisfied if s is separable, i.e., each of the $s_a(v_a)$ is a nondecreasing convex function of the flow on a alone. This proves for instance that the toll set for the Stockholm network discussed in §4 of this paper has a convex toll set. Does the theorem still hold if the assumption of symmetry and differentiability are dropped? We conjecture that it does.

Conjecture 1 *If* s *is a continuous monotonic map and* S^* *is a singleton, then the toll set* $\mathcal{T}$ *is convex.*

1.2.4 The Ideal Case

As mentioned, the ideal case is when both s is strictly monotonic and $s(v)^T v$ is strictly convex, and that is the assumption throughout this section. The SOPT problem has a unique solution, v^*, and the toll set is $W(v^*)$.

The next theorem reveals some interesting structure concerning the toll set for the ideal map case. It gives two valid toll vectors, which are readily available when v^* is known. The first of these is the well-known *marginal social cost pricing* (MSCP) toll vector, and the second we call *SCP* (abbreviation for *System Cost Pricing*), which simply involves giving the users on every arc a subsidy equal to the delay caused on that arc at system optimal flow.

Theorem 6 *If s is ideal and v^* is the system optimal solution, then the following toll vectors are valid:*

1. $\beta_{MSCP} = \nabla s(v^*) v^*$.
2. $\beta_{SCP} = -s(v^*)$.

Proof: Recall that $W(v^*)$ is the set of those β, for which there exists a ρ such that the following system is satisfied:

$$\begin{array}{rcl} Z^T(s(v^*)+\beta) & \geq & A^T\rho \\ (v^*)^T(s(v^*)+\beta) & = & b^T\rho. \end{array}$$

Since v^* is system optimal, the KKT conditions for SOPT guarantee the existence of a ρ^* such that optimality problem can be written as:

$$\begin{array}{rcl} Z^T(s(v^*)+\nabla s(v^*)v^*) & \geq & A^T\rho^* \\ (v^*)^T(s(v^*)+\nabla s(v^*)v^*) & = & b^T\rho^*, \end{array}$$

and it is then seen that the vector $\beta_{MSCP} := \nabla s(v^*)v^*$ is in the set $W(v^*)$.

If we set $\beta_{SCP} := -s(v^*)$, then choosing $\rho = 0$ will satisfy the system defining $W(v^*)$, and hence β_{SCP} is a valid toll vector. $\Box$.

While the SCP toll vector $-s(v^*)$ is, of course, impractical, it does have interesting consequences, owing mainly to the geometry of the toll set $\mathcal{T} = W(v^*)$, a shifted polyhedral cone with $-s(v^*)$ as its apex. It follows that every toll vector of the form

$$\beta = -s(v^*) + \lambda(\nabla s(v^*)v^* + s(v^*)), \lambda \geq 0$$

is valid. For the standard delay functions employed in the traffic assignment literature as well as practice, it is easily shown that $-s(v^*) < 0$ and $\nabla s(v^*)v^* > 0$, which gives us that for every θ in the range $[-s(v^*)^T v^*, \infty)$, one can readily calculate a valid toll vector β, such that the total toll collected is θ. This toll vector is given by

$$\beta = -s(v^*) + \lambda(\nabla s(v^*)v^* + s(v^*)),$$

where $\lambda = (\theta + s(v^*)^T v^*)/((v^*)^T \nabla s(v^*)v^* + s(v^*)^T v^*)$. This allows policy makers to set an appropriate target for the total toll to be collected, and then a valid toll that achieves this target is immediately calculated. In particular, the so-called *Robinhood* tolls (see §3), where the total toll collected is zero, can be calculated by setting $\theta = 0$ in the above formula:

$$\beta_{RH} = -s(v^*) + \lambda(\nabla s(v^*)v^* + s(v^*)),$$

where $\lambda = s(v^*)^T v^*/((v^*)^T \nabla s(v^*)v^* + s(v^*)^T v^*)$.

1.2.5 Relation to Other Problems

In this subsection, we explore connections between toll pricing and the two topics given below:

- Mathematical programs with equilibrium constraints [13] (abbreviated as MPECs).
- Bounded flow traffic equilibrium problems [2, 9, 11].

Toll Pricing and MPEC. An MPEC is an optimization problem in which some or all of the constraints prescribe that certain equilibrium conditions be satisfied. The text [13] by Luo, Pang and Ralph gives a comprehensive presentation of this relatively recent subject. The general MPEC problem is rather broad, and so we restrict ourselves to that problem as it pertains to toll pricing.

Consider the problem

$$\begin{array}{rl} \min: & s(v)^T v \\ \text{s.t.} & v \in V \\ & (s(v)+\beta)^T(u-v) \geq 0 \ \forall \ u \in V \end{array} \qquad \text{(MPEC)}$$

This problem is to be considered to have both β and v as its variables. Then, the constraints simply ensure that v is a feasible aggregate flow and it is a tolled user equilibrium problem with respect to the toll vector β. Thus, this problem is an MPEC.

Theorem 7 *Suppose that s is ideal. Then, the set of optimal solutions to the MPEC problem given above is precisely*

$$\{(\beta, v^*) | \beta \in \mathcal{T}\}.$$

Proof: Since $v \in V$ is among the constraints for the MPEC problem, and v^* is system optimal, it follows that for any v feasible for MPEC, $s(v)^T v \geq s(v^*)^T v^*$. But, on the other hand, $v = v^*, \beta = -s(v^*)$ is clearly feasible for MPEC, and therefore the optimal objective function value of the MPEC problem is $s(v^*)^T v^*$, and the v part of any optimal solution to that problem must be v^*. But then, any feasible solution of the type (v^*, β) must satisfy, by Lemma 1, $\beta \in W(v^*)$. Since s is ideal, from Theorem 3, $W(v^*) = \mathcal{T}$, and the theorem follows. □

Although it is not clear that the above reformulation of toll pricing as an MPEC would be of any practical use at this point, it might prove useful for further extensions. Also, it is not clear how one would minimize a further objective over the toll set in the MPEC framework.

Bounded Flows and Toll Pricing. Consider the following toll pricing situation. We have the traffic assignment problem as considered throughout this paper, and in addition, we have a vector of upper bounds for the aggregate flows. The objective is to ensure that the user equilibrium flows satisfy these upper bounds. In [9], the first author proposed that one impose tolls in order to achieve this goal. In particular, it was shown there (for the separable case) that when the cost map is ideal, then taking the tolls to be the (nonnegative) multipliers appearing in the generalized KKT conditions for the bounded flow equilibrium would suffice. This result was later extended to broad problem classes as well as variants by Bergendorff in her Master's thesis [2].

A description of some of those results in our current setting is given below. Suppose that v_{ub} is a given nonnegative vector, and let $V_B = \{v \in V | v \leq v_{ub}\}$. It is assumed that V_B is nonempty. Consider the variational inequality problem:

$$s(\bar{v})^T (v - \bar{v}) \geq 0 \ \forall \ v \in V_B.$$

Any $\bar{v} \in V_B$ for which the above holds will be called a *bounded flow equilibrium* solution. Under our assumptions, there is a unique such solution. It can then be shown that (see the development in [2]) if β is chosen to be a multiplier vector for the bound constraints in the generalized KKT conditions for the VI problem, then the tolled user equilibrium solution is precisely the bounded flow equilibrium solution. In the context of system optimality, if v^* is the system optimal solution, then choosing $v_{ub} = v^*$ will result in a valid toll as defined earlier.

Finally, since s is assumed to be ideal, the results above also hold if V_B is redefined with equality constraints on v, rather than bound constraints, i.e., if $V_B = \{v \in V | v = v^*\}$.

1.3 A TOLL PRICING FRAMEWORK

In this section, we will present a framework for the computation of alternate toll vectors for any given traffic assignment problem. The underlying assumption is that the cost map s is ideal. After solving SOPT for the system optimal v^*, the two valid toll vectors

1. MSCP toll vector $\beta_{MSCP} = \nabla s(v^*)v^*$.
2. SCP toll vector $\beta_{SCP} = -s(v^*)$

are immediately available (see §2.4). Further, for every $\theta \in [-s(v^*)^T v^*, \infty)$ one can easily obtain a valid toll vector satisfying $\beta^T v^* = \theta$. However, situations such as the following may arise:

- Owing to political or practical reasons, it may not be feasible to implement a policy involving subsidies, thus requiring the toll vector to be nonnegative.
- Whether nonnegative tolls are permitted or not, any immediate solution computed above is likely to have an excessively large number of arcs with nonzero tolls, thus increasing the infrastructure expenditure involved in the implementation.

The MINSYS approach from [2, 3] addresses the first of these situations. It and and other strategies, such as MINTB, which addresses the second, are encompassed in a general toll pricing framework:

Step 1: Solve the system optimum problem to obtain an optimal solution v^*.

Step 2: Define the toll set. When the map s is ideal (§2.4), this will be the β part of the polyhedron $W(v^*)$:

$$\begin{array}{rcl} Z^T(s(v^*)+\beta) & \geq & A^T\rho \\ (v^*)^T(s(v^*)+\beta) & = & b^T\rho \end{array}$$

which may be intersected with $\beta \geq 0$ when tolls, but not subsidies, are allowed.

Step 3: Define and optimize an objective function over the toll set.

The system optimum problem is an uncapacitated nonlinear multicommodity network flow problem and can be addressed by familiar techniques such as the Frank-Wolfe method or restricted simplicial decomposition [10].

In Step 3 the problem to solve will depend on the choice of objective. The formulations which follow all assume that $W(v^*)$ defines the full set of possible tolls. The objective is linear in each case and thus the problem is either a linear program or a linear integer program.

MINSYS As mentioned above, the objective is to minimize the total tolls collected; tolls are nonnegative:

$$\begin{array}{rl} \min_{(\beta,\rho)} & \beta^T v^* \\ \text{s.t.} & (\beta,\rho) \in W(v^*) \\ & \beta \geq 0 \end{array}$$

MINMAX The goal is to minimize the largest nonnegative toll to be collected:

$$\begin{array}{rl} \min_{(z,\beta,\rho)} & z \\ \text{s.t.} & (\beta,\rho) \in W(v^*) \\ & z \geq \beta_a \quad \forall\ a \in \mathcal{A} \\ & \beta \geq 0 \end{array}$$

A variation would be to measure the maximum toll, which is in time units, relative to some link parameter, such as uncongested travel time.

ROBINHOOD(RH) By allowing negative tolls, network users collect a payment (credit) on some of the links and pay a toll on others. The total tolls collected can be constrained to zero. A simple derivation of RH tolls was given in §2.4, but to be included in the framework above, some objective function needs to be chosen, e.g., the MINSYS objective:

$$\begin{array}{rl} \min_{(\beta,\rho)} & \beta^T v^* \\ \text{s.t.} & (\beta,\rho) \in W(v^*) \\ & \beta^T v^* = 0 \end{array}$$

MINTB Toll booths, whether traditional or the more modern toll toll sensing stations, are expensive. In this integer program, the objective is to minimize the number. The formulation requires a positive constant M which exceeds the largest toll and a vector y of binary variables y_a:

$$\begin{array}{rl} \min_{(y,\beta,\rho)} & \sum_{a\in\mathcal{A}} y_a \\ \text{s.t.} & (\beta,\rho) \in W(v^*) \\ & \beta_a \leq M z_a \quad \forall\ a \in \mathcal{A} \\ & y_a \in \{0,1\} \\ & \beta \geq 0 \end{array}$$

MINTB/RH Combining the two preceding formulations leads to minimizing the number of toll booths while constraining the total tolls collected to zero:

$$\begin{array}{rl} \min_{(y,\beta,\rho)} & \sum_{a\in\mathcal{A}} y_a \\ \text{s.t.} & (\beta,\rho) \in W(v^*) \\ & \beta_a \le M z_a \quad \forall\, a \in \mathcal{A} \\ & y_a \in \{0,1\} \\ & \beta^T v^* = 0 \end{array}$$

Of course many other formulations and variations exist. For example, it might be desirable to have tolls only on certain links (bridges, tunnels, etc.) which enter a central business district. In this case the tolls on other links would be constrained to zero. Feasibility of the resulting formulation could then be determined by the software used in Step 3.

1.3.1 An Example

To provide a comparison of the formulations given above as well as comparison with MSCP tolls [3], we have again employed the nine node example from [11] which has data similar to large-scale traffic assignment problems. It has 18 links and all of the links have cost functions with the same structure:

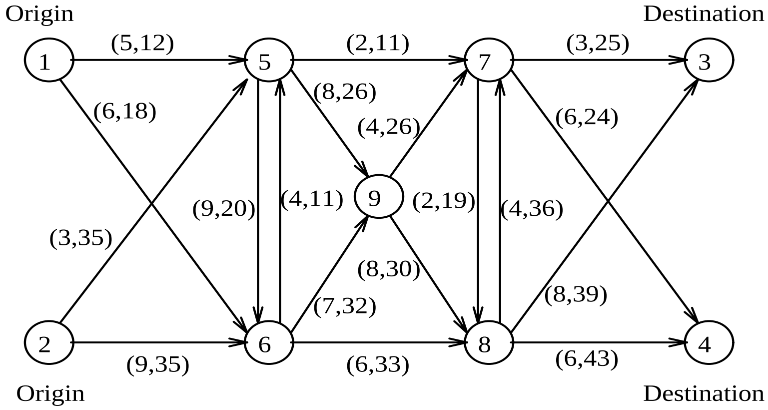
$$s_a(v) = s_a(v_a) = T_a(1 + 0.15(v_a/b_a)^4)$$

where T_a and b_a are constants. There are four OD-pairs: (1,3), (1,4), (2,3) and (2,4). The network is shown in Figure 1. The tuple near link a is (T_a, b_a).

Although small, this example illustrates that a variety of alternative tolling patterns can be found by use of the formulations above. It should also be pointed out that multiple solutions can be the result for several of these formulations. We will summarize a few observations based on the particular solutions in Table 1:

- A comparison between MINSYS and MSCP shows that the tolling pattern and toll amounts are quite different. Further, the total toll system cost (total system cost + total toll cost) in the MSCP case is equal to 3747 (2254 + 1493) and in the MINSYS case equal to 3142 (2254 + 888). So with the MSCP principle the users of the nine-node network pay 68% more in tolls than with the MINSYS pricing principle. The MINSYS solution also happened to coincide with the MINTB solution, so it also gives the minimum number of toll booths, 5 versus the 14 of MSCP. (This points out a disadvantage of MSCP evident from the formula

$$\beta_{MSCP} := \nabla s(\bar{v})\bar{v}.$$



O-D Pair:	[1,3]	[1,4]	[2,3]	[2,4]
Demand:	10	20	30	40

Figure 1: The Nine Node Network

Every link with a nonlinear travel time function has an MSCP toll if it has positive flow at the SOPT solution $\bar{v}$.)

- The maximum toll on any link is 16.88 for MSCP and 11.2 for MINSYS (link 5-7 in both cases). When this maximum is minimized (MINMAX) the largest toll reduces to 8.00. MINMAX also provides another set of nonnegative tolls which significantly reduce the total tolls when compared with MSCP; the toll total is 28% higher in the latter case.
- The high toll costs of MSCP relative to the total travel time raises an important question: By how much would the system optimal objective increase if there were no tolls of any kind, i.e., if the flow were user-optimal? For this example, the SOPT objective value at the untolled UOPT solution is 2455.84, an increase of just 8.95% which is significantly less than the increases of MSCP, MINSYS and MINMAX, the strategies with nonnegative tolls. The viewpoint taken by economists is that the extra cost of of the UOPT solution represents real resource consumption whereas the collection of tolls does not. To quote Arnott and Small [1],

  > "If one regards people only in their roles as travelers, everyone is made worse off by being forced to pay a toll that raises the cost of using the road, even with a reduction in congestion....But travelers are also citizens, so one must consider what happens to the toll revenues. Paying a toll, after all, does not use up resources; it is

> only a paper transaction — or, more likely, an electronic one. If the toll revenues are used to benefit citizens generally, the gains people receive as citizens more than offset their losses as travelers. In fact, the more formal statement of "efficiency" is precisely this: There is some way of redistributing the toll revenues that leaves everyone as well, or better, off."

In our example, ROBINHOOD and MINTB/RH, by design, have zero net toll costs. These could be considered as two tolling strategies which leave travelers "as well off" without the need for an additional mechanism to accomplish the redistribution.

- ROBINHOOD and MINTB/RH introduce the idea of negative tolls, but it is clear that this concept requires further examination. As mentioned earlier, $\beta = -s(v^*)$, where v^* is system-optimal provides a negative toll for every link, implying that the users are totally reimbursed for their time on the network. It is difficult to imagine that such a policy would ever be put in place. However, the selective use of negative tolls luring users to certain links might have some appeal (provided negative *cycles* are not induced in the network!).

All of the data in Table 1 was produced by a single GAMS [8] program which first solves the SOPT problem using the nonlinear code MINOS and then each of the toll optimization problems using the CPLEX solver.

Link	SOPT SOLUTION			Alternative Tolls β for Nine Node Problem				
	v_a	$s_a(v_a)$	$v_a s_a(v_a)$	MSCP	MINSYS & MINTB	MINMAX	RH	MINTB/RH
1	9.411	5.284	49.728	1.135				
2	20.589	7.541	155.262	6.162				
3	38.334	3.648	139.842	2.590	4.000	8.000	4.000	
4	31.666	9.905	313.652	3.618		4.000		-4.000
5	.000	9.000						
6	21.303	6.220	132.505	16.880	11.200	8.000	2.877	8.000
7	26.442	9.284	245.487	5.135				
8	0.000	4.000						
9	39.474	7.843	309.595	7.370	7.200	7.200	- 0.816	
10	12.781	7.027	89.812	0.107				
11	29.608	3.885	115.027	3.541	4.000	7.200	2.618	
12	20.757	6.504	135.004	2.014		3.200		-2.126
13	0.000	2.000				1.079		
14	10.392	8.006	83.198	0.024			-1.689	
15	39.243	6.624	259.946	2.497			-0.307	1.874
16	0.000	4.000						0.121
17	29.062	4.937	143.479	3.746	3.200		-5.123	
18	10.162	8.016	81.459	0.063			-8.016	-7.200
Total Time =			2253.918					
Total Tolls = $\beta^T v^S$				1493.458	887.574	1167.572	0.000	0.000
Total Tolls/Total Time (%)				66.38	39.38	51.80	0.00	0.00
Toll Booths				14	5	7	8	6

Table 1: The Nine Node Problem - Alternative Tolls

1.4 MINSYS TOLLS FOR THE STOCKHOLM NETWORK

To gain experience with a larger network, we have applied the above procedure and obtained MINSYS tolls for a particular version of the Stockholm network which has 417 nodes and 963 links. The origin-destination matrix contains demands between 46 of the nodes during a one-hour morning rush period. Total demand is 272,873 trips.

This problem was too large for our GAMS code, so the SOPT problem of Step 1 was solved using code RSDTA, which applies the restricted decomposition methodology of [10] to traffic assignment problems. Solution time on a Sun Ultrasparc was 108 seconds to obtain a relative duality gap of 0.0025.

The linear system of Step 2 consisted of 20,145 β and ρ variables restricted by 44,298 inequalities and one equality constraint. A direct attempt at solving the MINSYS formulation in Step 3 encountered numerical difficulties, primarily in obtaining a feasible solution within the default tolerances of the the CPLEX [4] linear program solver. To circumvent this difficulty, we dualized the equality constraint, adding it to the objective function with a penalty parameter $\mu = 1000$. This penalized problem had the form:

$$\begin{array}{rcl} \min_{(\beta,\rho)} \beta^T v^* + \mu((v^*)^T(s(v^*)+\beta) & - & b^T\rho) \\ \text{s.t. } Z^T(s(v^*)+\beta) & \geq & A^T\rho \\ \beta & \geq & 0 \end{array}$$

An approximate solution was then obtained in about 20 minutes on the Sun Ultrasparc.

While the solutions to Step 1 and Step 3 were not obtained to high accuracy, they provide an comparison between MINSYS and MSCP with even more pronounced differences than in the small example:

SOPT Average Travel Time: 42.96 minutes

MSCP Average Toll: 128.53 minutes

MINSYS Average Toll: 9.40 minutes

Measuring in time units, the MSCP tolls cost the average user three times as much as their travel time. The MINSYS tolls are significantly less, but still add about 22% to the travel time. Converting to monetary units using the figures in [1], the MSCP tolls translate to \$13.67 (88.86 SK) and the MINSYS tolls are \$1.25 (8.125 SK). MINSYS tolled only 185 links versus 830 for MSCP.

We also calculated the untolled user-optimal flows and the average travel time was 43.77 minutes, a nearly negligible increase of 0.81 minutes over the SOPT average. Of course, 0.81 minutes per user is 2.2×10^5 total vehicle minutes, so there is an aggregate savings which might justify small tolls or RH tolls, which net to zero.

While the average times are in close agreement, it should be noted that the individual link flows of the UOPT solution varied significantly from the SOPT link flows, on the order of 26% for the nonlinear cost links, 24% for all links. This emphasizes an obvious fact: tolls are more important in controlling individual link flows rather than the total system time.

1.5 FUTURE DIRECTIONS

In this paper and its predecessor paper [3], we have introduced and analyzed a notion of toll sets for the standard traffic assignment problem with fixed demands, and shown its computational feasibility. However, one should note that for the moderately sized Stockholm network, the times taken to calculate the system optimum and that to calculate optimal tolls were 0.0025 seconds and 20 minutes, respectively. It would certainly be desirable to be able to make the latter step more efficient so that it takes a time proportionate with the former step. To this end, we have recently devised a Bender's decomposition type approach for attacking toll optimization problems. The advantage of that decomposition is that, the size of the linear programs (or integer programs) that need to be solved is the same as the number of arcs, and hence there is good hope that one can handle much larger toll optimization problems. This method and its computational behavior will be reported on in a forthcoming article.

Another important avenue of future research is that of extending the notion of toll sets to other models of traffic equilibrium. First consider the case of elastic demand. For this scenario, no satisfactory notion of system equilibrium has been given in the literature, to our knowledge. Interestingly, as will be reported in another upcoming article of ours, by juxtaposing the notion of valid tolls with that of system optimality, one can define a coherent model which when specialized to the fixed demand case would yield the correct system optimum and valid tolls. It should also be interesting to devise a meaning notion of valid tolls (and toll sets) for the case of transit equilibrium (see [15] and [17]) as well as multimode transporation networks ([5], [6] and [16]).

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