Homework 6 Solution

1 Problem 9.2

To evaluate the incremental investment, we will compare the making investment decision versus not making investment decision.

$E[U(W-w+x)]$ = expected value of the utility function of wealth if the investment is made

$E[U(W)]$ = expected value of the utility function of wealth if the investment is not made.

To make the investor inclined to invest the expected value of the utility function in case of investment should be greater then the expected value of the utility function without an investment.

$$E[U(W-w+x)] > E[U(W)]$$
$$E[-e^{-a(W-w+x)}] > E[e^{-aW}]$$
$$E[-e^{-aW}]E[e^{-a(x-w)}] > E[-e^{-aW}]$$
$$E[e^{-a(x-w)}] < 1$$

Therefore, his evaluation is independent of W.

2 Problem 9.4

$$\mu = \frac{xU''(x)}{U'(x)}$$

- $U(x) = \ln(x)$: $U'(x) = \frac{1}{x}$, $U''(x) = -\frac{1}{x^2}$

$$\mu = \frac{x \times -\frac{1}{x^2}}{\frac{1}{x}} = -1 \text{ - constant}$$

- $U(x) = \gamma x^\gamma$: $U'(x) = \gamma^2 x^{\gamma-1}$, $U''(x) = \gamma^2(\gamma-1)x^{\gamma-2}$

$$\mu = \frac{x \times \gamma^2(\gamma-1)x^{\gamma-2}}{\gamma^2 x^{\gamma-1}} = \gamma - 1 \text{ - constant}$$

3 Problem 9.5

$$V(x) = aU(x) + b$$

Since U(x) is known on the interval [A,B], then the system of two equations need to be solved to find a and b:

$$\begin{cases} V(A') = aU(A') + b = A' \\ V(B') = aU(B') + b = B' \end{cases}$$

Subtracting one from another gives

$$a = \frac{A' - B'}{U(A') - U(B')}$$

and multiplying the first equation by $U(B')$ and the second equation by $U(A')$ and subtracting one from another gives

$$b = \frac{A'U(B') - B'U(A')}{U(B') - U(A')}$$

4 Problem 10.2

The first thing that should be noticed is that the formula consists an error. If there are no storage costs then the theoretical forward price should be $F = \frac{S}{d(0,M)}$. If there are storage costs then by formula 10.2 it can be concluded that the forward price should be greater than $\frac{S}{d(0,M)}$. But the problem asks you to prove that it is less than this value.

As suggested by the hint to this problem, consider a portfolio which has 1 unit of the asset at time zero and pays the storage costs by selling part of the asset. From the problem formulation it follows that every period we need to sell a fraction of the asset equal to q to cover storage costs. Therefore, after the first period there will be $(1 - q)$ units of the asset in our portfolio. Then, after the second period there will be $(1 - q)^2$ units of the asset in our portfolio, etc. Hence, by the time of delivery, there will be $(1 - q)^M$ assets in our portfolio.

Now, consider two different portfolios. The first one has 1 long position in forward contract on delivery 1 unit of the asset after M periods with a price of delivery F, and the second portfolio borrowed $S/(1 - q)^M$ and bought $1/(1 - q)^M$ units of the asset to have exactly 1 unit after M periods. These both portfolios will have 1 unit of asset after M periods. To have no arbitrage opportunity they should have the same payoffs after M periods. The payoff of the first portfolio is F, and the payoff of the second portfolio is $S/(d(0, M)(1 - q)^M)$ (returning the borrowed money). Therefore,

$$F = \frac{S}{d(0, M)(1 - q)^M}$$

5 Problem 10.7

$$f_t = (F_t - F_0)d(t, T)$$

We know that $F_0 = 940$, let us compute the current forward price F_t. The settings are similar to the example 10.5, p.271.

$$920 = \frac{80}{(1 + 7\%/2)} + \frac{80 + F_t}{(1 + 8\%/2)^2}$$

$$F_t = 831.47$$

Then,

$$f_t = (831.47 - 940)/(1 + 8\%/2)^2 = -100.34$$