Homework 5 Solution

1 Problem 8.1

(a) For a single-factor model with the market portfolio serving as the factor, the return of the stock i is

$$r_i = r_f + \beta_i(r_m - r_f) + \varepsilon_i, \quad i = 1, \dots, 3 \tag{1}$$

where $E\varepsilon_i = 0$ (the formula was taken from the Part 8.3 of the textbook). Using the data from the table, we can find that

$$\bar{r}_1 = 0.05 + 1.1(0.12 - 0.05) = 12.7\%$$

$$\bar{r}_2 = 0.05 + 0.8(0.12 - 0.05) = 10.6\%$$

$$\bar{r}_3 = 0.05 + 1(0.12 - 0.05) = 12\%$$

Then the expected rate of return of the portfolio is

$$\bar{r}_p = w_1\bar{r}_1 + w_2\bar{r}_2 + w_3\bar{r}_3 = 0.2 \times 12.7 + 0.5 \times 10.6 + 0.3 \times 12 = 11.44\%$$

(b) Now, applying the formula (1) (see above) for the portfolio return we can get

$$r_p = r_f + \sum_{i=1}^{3} w_i\beta_i(r_m - r_f) + \sum_{i=1}^{3} w_i\varepsilon_i =$$

$$= r_f + (r_m - r_f)\sum_{i=1}^{3} w_i\beta_i + \sum_{i=1}^{3} w_i\varepsilon_i$$

Since the error terms and the market return are all uncorrelated, the variance of our portrolio is

$$\sigma_{r_p}^2 = \sigma_{r_m}^2 \left(\sum_{i=1}^{3} w_i\beta_i\right)^2 + \sum_{i=1}^{3} \sigma_{\varepsilon_i}^2 w_i^2 =$$

$= 0.18^2(1.1\times0.2+0.8\times0.5+1\times0.3)^2+(0.2\times0.07)^2+(0.5\times0.023)^2+(0.3\times0.01)^2 = 0.027761$

From which $\sigma_{r_p} = \sqrt{0.027761} \approx 16.6\%$

2 Problem 8.2

Using the formulas

$$\bar{r}_1 = \lambda_0 + \lambda_1 b_{11} + \lambda_2 b_{12}$$

$$\bar{r}_2 = \lambda_0 + \lambda_1 b_{21} + \lambda_2 b_{22}$$

from the problem formulation we can find that $\bar{r}_1 = 0.15, \bar{r}_2 = 0.2, \lambda_0 = 0.1, b_{11} = 2, b_{12} = 1, b_{21} = 3, b_{22} = 4$. Therefore, we need to solve the following system to find λ_1, λ_2

$$\begin{cases} 0.15 = 0.1 + 2\lambda_1 + \lambda_2 \\ 0.2 = 0.1 + 3\lambda_1 + 4\lambda_2 \end{cases}$$

From which $\lambda_1 = 0.02, \lambda_2 = 0.01$

3 Problem 8.4

Having that

$$\hat{\bar{r}} = \frac{1}{n}\sum_{i=1}^{n} r_i$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \hat{\bar{r}})^2$$

let us derive Es^2.

$$Es^2 = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(r_i - \hat{\bar{r}})^2\right] = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(r_i^2 - 2\hat{\bar{r}}r_i + \hat{\bar{r}}^2)\right] =$$

$$= E\left[\frac{1}{n-1}\left(\sum_{i=1}^{n} r_i^2 - 2\hat{\bar{r}}\sum_{i=1}^{n} r_i + n\hat{\bar{r}}^2\right)\right] = E\left[\frac{1}{n-1}\left(\sum_{i=1}^{n} r_i^2 - 2\hat{\bar{r}} \times n\hat{\bar{r}} + n\hat{\bar{r}}^2\right)\right] =$$

$$= E\left[\frac{1}{n-1}\left(\sum_{i=1}^{n} r_i^2 - n\hat{\bar{r}}^2\right)\right] = \left[\frac{1}{n-1}\left(\sum_{i=1}^{n} Er_i^2 - nE\hat{\bar{r}}^2\right)\right]$$

We know that $\sigma^2(r_i) = \sigma^2, \sigma^2(\hat{\bar{r}}) = \frac{\sigma^2}{n}$ and $\sigma^2(r) = Er^2 - (Er)^2$. Then

$$Es^2 = \left[\frac{1}{n-1}\left(\sum_{i=1}^{n}(\sigma^2 + \mu^2) - n(\sigma^2/n + \mu^2)\right)\right] =$$

$$= \left[\frac{1}{n-1}\left(n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2)\right)\right] = \sigma^2$$

4 Problem 8.5

(a) The following sequence proves the statement

$$\sigma(\hat{\bar{r}}) = \sigma(n\hat{\bar{r}}_n) = n\sigma(\hat{\bar{r}}_n) = n\frac{\sigma_n}{\sqrt{n}} = n\frac{\frac{\sigma}{\sqrt{n}}}{\sqrt{n}} = \sigma$$

(b)

$$\sigma(\hat{\sigma}^2) = \sigma(n\hat{\sigma}_n^2) = n\sigma(\hat{\sigma}_n^2)$$

Now, for the norlmally disributed returns we khow that

$$\sigma(\hat{\sigma}_n^2) = \frac{\sqrt{2}\sigma_n^2}{\sqrt{n-1}} = \frac{\sqrt{2}\sigma^2}{n\sqrt{n-1}}$$

Then

$$\sigma(\hat{\sigma}^2) = n\frac{\sqrt{2}\sigma^2}{n\sqrt{n-1}} = \frac{\sqrt{2}\sigma^2}{\sqrt{n-1}}$$

So, the more data we have the more precisely we can estimate variance, but the mean estimation does not depend on n.

5 Problem 8.7

Let

$$\hat{\bar{r}} = \frac{1}{24}\sum_{i=1}^{24} r_i$$

be the estimate of r. Our goal is to find $\sigma^2(\hat{\bar{r}})$ and compare it to the some usual result, i.e $\frac{\sigma^2}{12}$. We can find it by using the formula

$$\sigma^2(\hat{\bar{r}}) = \frac{1}{24^2}\sigma^2(\sum_{i=1}^{24} r_i) = \frac{1}{24^2}(\sum_{i=1}^{24}\sigma^2(r_i) + \sum_{1\le i<j\le 24} 2cov(r_i, r_j)) \tag{1}$$

Let $q_i, i = 1, \ldots, 24$ be the half-monthly returns, we can assume that they are independent. Then $r_i = q_i + q_{i+1}$ and $\sigma^2(q_i) = \sigma^2/2$ so

$$cov(r_i, r_j) = cov(q_i + q_{i+1}, q_j + q_{j+1}) = cov(q_{i+1}, q_j) = \begin{cases} \sigma(q_{i+1}) = \sigma^2/2, & i+1 = j \\ 0, & i+1 \neq j \end{cases}$$

Putting this into the (1) gives

$$\sigma^2(\hat{\bar{r}}) = \frac{1}{24^2}(\sum_{i=1}^{24}\sigma^2 + \sum_{i=1}^{23} 2cov(r_i, r_{i+1})) = \frac{1}{24^2}(24\sigma^2 + 2\times 23\sigma^2/2)$$

Ignoring the missing half-month term we can obtain that $\sigma^2(\hat{\bar{r}}) = \sigma^2/12$. This is the same result as we can have by usual estimation.