Homework 4 Solution

1 Problem 7.1

$$\bar{r}_M = 0.23, \quad r_f = 0.07, \quad \sigma_M = 0.32$$

1. $\bar{r} = r_f + \frac{\bar{r}_M - r_f}{\sigma_M}\sigma = 0.07 + \frac{1}{2}\sigma$

2.
   - $\sigma = 2 * (0.39 - 0.07) = 0.64$
   - Two equations need to be solved: $A + B = 1000$, $(1 + r_f)A + (1 + \bar{r}_M)B = 1000(1 + 39\%)$ or $A + B = 1000, 0.07A + 0.23B = 390$. From which we can find that $B = 2000, A = -1000$. So, we should borrow 1000 and invest 2000 in the market portfolio.

3. By the end of the year we should expect to have: $300 * 1.07 + 700 * 1.23 = 1182$.

2 Problem 7.3

(a) Since the market portfolio lies on the minimum-variance set, it can be expressed as a weighted sum of the two portfolios from the problem formulation as

$$\alpha w + (1 - \alpha)v$$

Where $w = (0.6, 0.2, 0.2), v = (0.8, -0.2, 0.4), \alpha$ is unknown coefficient. Obviously, the expected return of the market portfolio is

$$\bar{r}_M = \alpha\bar{r}_w + (1 - \alpha)\bar{r}_v =$$

$$= \alpha(0.6 * 0.1 + 0.2 * 0.2 + 0.2 * 0.1) + (1 - \alpha)(0.8 * 0.1 - 0.2 * 0.2 + 0.4 * 0.1) =$$
$$= 0.12\alpha + 0.08(1 - \alpha) = 0.08 + 0.04\alpha$$

Now, using the fact that the market portfolio cannot have short positions in the assets, we should impose the following system of constraints on the α

$$\begin{cases} 0.6\alpha + 0.8(1 - \alpha) \geq 0 - & \text{position of the market portfotio in the first security} \\ 0.2\alpha - 0.2(1 - \alpha) \geq 0 - & \text{position of the market portfotio in the second security} \\ 0.2\alpha + 0.4(1 - \alpha) \geq 0 - & \text{position of the market portfotio in the third security} \end{cases}$$

From which, we can find that

$$\begin{cases} \alpha \leq 4 \\ \alpha \geq \frac{1}{2} \\ \alpha \leq 2 \end{cases} \Rightarrow \alpha \in [0.5, 2]$$

Hence,

$$\bar{r}_M = 0.08 + 0.04\alpha \in [0.1, 0.16]$$

(b) If we know that w represents the minimum-variance portfolio, then the market portfolio could not lie below this point on the minimum-variance set, i.e it could not have an expected return less then the expected return of the portfolio w. Therefore

$$\bar{r}_M \in [0.12, 0.16]$$

3 Problem 7.4

As soon as the market portfolio is efficient, we can apply Equation (6.9) for this (also the hint to this problem is used)

$$\sum_{i=1}^{n} \sigma_{iM} \lambda w_i = \bar{r}_M - r_f = \lambda cov(r_M, r_M) = \lambda \sigma_M^2$$

And the same derivation can be done for asset k

$$\sum_{i=1}^{n} \sigma_{ik} \lambda w_i = \bar{r}_k - r_f = \lambda cov(r_k, r_M) = \lambda \sigma_{kM}$$

Combining these two equations we can immediately conclude that

$$\frac{\bar{r}_k - r_f}{\bar{r}_M - r_f} = \frac{\sigma_{kM}}{\sigma_M^2} \Rightarrow \bar{r}_k = r_f + \frac{\sigma_{kM}}{\sigma_M^2} (\bar{r}_M - r_f)$$

So, the CAPM formula is derived.

4 Problem 7.5

For the standard deviation of market portfolio we have

$$\sigma_M^2 = \sum_{i=1}^{n} x_i^2 \sigma_i^2$$

and, obviously, due to uncorrellation property

$$\sigma_{jM} = cov(r_j, r_M) = cov(r_j, \sum_{i=1}^{n} x_i r_i) = \sum_{i=1}^{n} x_i cov(r_j, r_i) = x_j \sigma_j^2$$

Therefore,

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^2} = \frac{x_j \sigma_j^2}{\sum\limits_{i=1}^{n} x_i^2 \sigma_i^2}$$

5 Problem 7.8

(a) Due to independence of p and c the expected rate of return can be found as

$$\text{Expected rate of return } = E\left[\frac{p}{c}\right] - 1 = EpE\left[\frac{1}{c}\right] - 1 =$$

$$= 24 \times (0.5 \times \frac{1}{20} + 0.5 \times \frac{1}{16}) - 1 = \frac{24 \times 18}{16 \times 20} = 35\%$$

(b) Using the suggested Hint and formula for β we can derive that

$$\beta = \frac{\sigma_{project,M}}{\sigma_M^2} = \frac{E\left[(\frac{p-c}{c} - E\left[\frac{p-c}{c}\right])(r_M - \bar{r}_M)\right]}{\sigma_M^2} = \frac{E\left[(\frac{p}{c} - \bar{p}E\frac{1}{c})(r_M - \bar{r}_M)\right]}{\sigma_M^2} =$$

$$= \frac{E\left[(\frac{p}{c} - \frac{\bar{p}}{c} + \frac{\bar{p}}{c} - \bar{p}E\frac{1}{c})(r_M - \bar{r}_M)\right]}{\sigma_M^2} = \frac{E\left[(\frac{p-\bar{p}}{c} + \bar{p}(\frac{1}{c} - E\frac{1}{c}))(r_M - \bar{r}_M)\right]}{\sigma_M^2} =$$

$$= \frac{E\left[(\frac{p-\bar{p}}{c})(r_M - \bar{r}_M)\right]}{\sigma_M^2} + \underbrace{\bar{p}\frac{E\left[(\frac{1}{c} - E\frac{1}{c}))(r_M - \bar{r}_M)\right]}{\sigma_M^2}}_{=0 \text{ due to uncorrelation of c and } r_M} =$$

$$= E(\frac{1}{c})\frac{E\left[(p - \bar{p})(r_M - \bar{r}_M)\right]}{\sigma_M^2} =$$

$$= (0.5 * \frac{1}{20} + 0.5 * \frac{1}{16}) * 20 = 1.125$$

(c) The excess rate of return, predicted by CAPM should be

$$\bar{r} - r_f = \beta(\bar{r_m} - r_f) = 1.125 * (33 - 9) = 27\%$$

But the expected excess rate of return is 33% -9% = 24%, so the project is not acceptable, based on the CAPM criterion.

6 Problem 7.9

1. Use the first pricing formula

$$P = \frac{\bar{Q}}{1 + r_f + \beta(\bar{r}_M - r_f)}$$

In our case, $\bar{Q} = 100(1 + \alpha r_f + (1-\alpha)\bar{r}_M)$, $cov(r_Q, r_M) = cov(1 + \alpha r_f + (1-\alpha)r_M, r_M) = (1-\alpha)\sigma_M^2$, then $\beta = 1 - \alpha$. Putting all these values to the first pricing formula, we will get

$$P = \frac{100(1 + \alpha r_f + (1-\alpha)\bar{r}_M)}{1 + r_f + (1-\alpha)(\bar{r}_M - r_f)} = \frac{100(1 + \alpha r_f + (1-\alpha)\bar{r}_M)}{1 + \alpha r_f + (1-\alpha)\bar{r}_M} = 100$$

2. Use the second pricing formula

$$P = \frac{1}{1+r_f}\left[\bar{Q} - \frac{cov(Q, r_M)(\bar{r}_M - r_f))}{\sigma_M^2}\right]$$

Using the same value for $\bar{Q}$ and $cov(Q, r_M) = cov(100(1 + \alpha r_f + (1 - \alpha)r_M), r_M) = 100(1 - \alpha)\sigma_M^2$ we can get

$$P = \frac{1}{1+r_f}\left[100(1 + \alpha r_f + (1-\alpha)\bar{r}_M) - \frac{100(1-\alpha)\sigma_M^2(\bar{r}_M - r_f))}{\sigma_M^2}\right] =$$

$$= \frac{1}{1+r_f}\left[100(1 + \alpha r_f + (1-\alpha)\bar{r}_M) - (100(1-\alpha)(\bar{r}_M - r_f))\right] =$$

$$= \frac{1}{1+r_f}\left[100 + 100\alpha r_f + 100(1-\alpha)r_f\right] = 100$$