Homework 3 Solution

1 Problem 6.1

$$\begin{array}{c} \text{amount invested} = X_0 \text{ (deposit)} \\ \text{amount received} = X_0 - X_1 + X_0 \\ \text{total return} = \frac{2X_0 - X_1}{X_0} = 2 - R \end{array}$$

2 Problem 6.2

Let X_1, X_2 be the two random variables which are equal to resulting the values of two rolling dices. Obviously, X_1, X_2 are i.i.d with

$$EX_1 = EX_2 = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6}$$

$$EX_1^2 = EX_2^2 = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$$

Therefore,

$$Ez = EX_1X_2 = EX_1EX_2 = \frac{21}{6} \times \frac{21}{6} = 12.25$$

$$Var(z) = E[(z-Ez)^2] = E(z^2)-(Ez)^2 = EX_1X_2X_1X_2-(Ez)^2 = E[X_1^2]E[X_2^2]-(Ez)^2 =$$

$$= \frac{91}{6} \times \frac{91}{6} - (12.25)^2 \approx 79.96$$

3 Problem 6.4

Let w be the portion of investment in the first stock, then $1-w$ is the portion of investment in the second stock, $\bar{r}_p, \sigma_p$ are the mean and variance of such portfolio. Then,

$$\sigma_p^2 = w^2\sigma_1^2 + 2w(1-w)\sigma_{12} + (1-w)^2\sigma_2^2$$

We wish to minimize this value with respect to w. The first order condition gives:

$$2w\sigma_1^2 + 2(1-2w)\sigma_{12} + 2(w-1)\sigma_2^2 = 0$$

Thus,

$$w = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 - 2\sigma_{12} + \sigma_2^2}$$

$$1 - w = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 - 2\sigma_{12} + \sigma_2^2}$$

And

$$\bar{r}_p = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 - 2\sigma_{12} + \sigma_2^2} \times \bar{r}_1 + \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 - 2\sigma_{12} + \sigma_2^2} \times \bar{r}_2$$

4 Problem 6.5

(a)

$$\text{amount invested} = 10^6 + 0.5u$$
$$\text{expected amount received} = 0.5 \times 3 * 10^6 + 0.5 \times u$$

Then the expected rate of return will be

$$r = \frac{0.5 \times 3 \times 10^6 + 0.5 \times u}{10^6 + 0.5u} - 1 = \frac{10^6}{2 \times 10^6 + u}$$

(b)
We can reduce the variance of this return to zero if the insurance gives us 3×10^6 dollars in case of rain. Then we will receive 3×10^6 dollars if it is rain or not, so there is no uncertainty here. To do this, we need to buy 3×10^6 units of insurance, which cost 1.5×10^6 dollars. In this case we will have a fixed rate of return

$$r = \frac{10^6}{2 \times 10^6 + u} = \frac{10^6}{2 \times 10^6 + 3 \times 10^6} = 20\%$$

5 Problem 6.7

(a) Form a portfolio $(w_1, w_2, w_3)^t$ with these three assets. The variance of this portfolio is

$$\sigma_p = w^t V w = 2w_1^2 + 2w_2^2 + 2w_3^2 + 2w_1 w_2 + 2w_2 w_3$$

Using the hint (symmetry between w_1 and w_3), define $w_1 = w_3 = w, w_2 = (1 - 2w)$. Therefore,

$$\sigma_p = 2w^2 + 2(1 - 2w)^2 + 2w^2 + 4w(1 - 2w) =$$
$$= 2(w^2 + 1 - 4w + 4w^2 + w^2 + 2w - 4w^2) = 2(2w^2 - 2w + 1)$$

The first order condition gives,

$$4w - 2 = 0$$

The minimum variance portfolio will be attained with the following weights: $w_1 = w_3 = 1/2, w_2 = 0$.
(b)
Using the formula (6.5a) (page 159 of the Textbook) for efficient portfolio we can write the following system to be solved (also the fact $\lambda = 1, \mu = 0$ is used).

$$\begin{cases} 2w_1 + w_2 - .4 = 0 \\ w_1 + 2w_2 + w_3 - .8 = 0 \\ w_2 + 2w_3 - .8 = 0 \end{cases}$$

Hence, $w_1 = .1, w_2 = .2, w_3 = .3$. As soon as $\lambda = 1, \mu = 0$ were fixed originally then the solution vector w has to be scaled. Then the following weights represent the solution to this problem

$$w_1 = 1/6, w_2 = 1/3, w_3 = 1/2$$

(c)
To do this part the following system has to be solved (it comes from (6.10) p. 168)

$$\begin{cases} 2v_1 + v_2 = .2 \\ v_1 + 2v_2 + v_3 = .6 \\ v_2 + 2v_3 = .6 \end{cases}$$

The solution to this problem is the following: $v_1 = 0, v_2 = .2, v_3 = .2$. Scaling these values by their sum, we can immediately obtain the weights of the risky assets in the portfolio: $w_1 = 0, w_2 = 1/2, w_3 = 1/2$

6 Problem 6.9

In any outcome of the wheel you will get $B_i A_i = \frac{1}{A_i} A_i = 1$ dollar, so you will receive \$1 independenlty of any outcome. The total bet on the wheel is

$$\sum_{i=1}^{n} \frac{1}{A_i}$$

Therefore, the risk-free rate for this wheel is

$$r_f = \frac{1}{\sum\limits_{i=1}^{n} \frac{1}{A_i}} - 1$$

For the wheel in the Example 6.7 we have $A_1 = 3, A_2 = 6, A_3 = 2$, then the risk-free rate is

$$r_f = \frac{1}{\frac{1}{2} + \frac{1}{3} + \frac{1}{6}} - 1 = 0$$