AFOSR Cooperatve Control Theme

Mixed Integer/LMI Programs for Low-Level Path Planning

Presentation by

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Trajectory Generation Problem

- **Given the system**

$$\dot{x} = f(x) + g(x)u$$
$$h(x,u) \geq 0$$

initial state x_0, final state x_f, final time T

- **Find admissible u to go from x_0 to x_f in T**
- **Assume system is differentially flat. Then,**

$$\begin{aligned} x &= x(z, z^{(1)}, \ldots, z^{(l-1)}) \\ u &= u(z, z^{(1)}, \ldots, z^{(l)}) \end{aligned}$$

- **Let**

$$w = \left[z^T, \frac{dz}{dt}^T, \cdots, \frac{d^{(l-1)}z}{dt^{(l-1)}}^T \right]^T$$

- **Problem reduces to:**

Find w with the above structure satisfying the boundary conditions

$$\phi_1(w(0)) \leq 0, \quad \phi_2(w(T))) \leq 0$$

and the path constraints

$$h(w(t)) \geq 0 \;\; \textbf{\textit{for all}} \;\; t \in [0, T]$$

Standard Solution Method

- **Inner-approximate feasible set by LMI set:**

$$\{v : \mathcal{L}v \geq 0\} \subset \{v : h(v) \geq 0\}$$

- **Restrict to polynomial functions:**

$$w \in \left\{ \sum_{k=0}^{K} w_k t^k : w_k \textbf{ structured} \right\}$$

- **Impose constraints at a finite number of points:**

$$\mathcal{L}w(j\delta t) \geq 0, \qquad j = 0, 1, \cdots, J$$

- **Find** $w_0, w_1, \cdots, w_K$**:**

$$\sum_{k=0}^{K} \mathcal{L}(w_k)\,(j\delta t)^k \geq 0, \qquad j = 0, 1, \cdots, J$$

- **Disadvantages**
  - **Neither necessary nor sufficient**
  - **No bound on number of grid points**
  - **Number of inequalities independent of** K
- **We give necessary and sufficient LMI whose size depends only on** K

Problem Statement

- **Given: $\mathcal{C}$ polytope in $\mathbb{R}^{n_c}$, $\mathcal{D}$ a finite subset of $\mathbb{R}^{n_d}$, $b \in \mathbb{R}^m$, and linear operators**

$$\mathcal{L}_d : \mathbb{R}^{n_d} \to \mathbb{R}^m$$

and

$$\mathcal{L}_k : \mathbb{R}^{n_c} \to \mathbb{R}^m \quad k = 0, 1, 2, \cdots, K$$

- **Define:**

$$\mathcal{S} = \left\{ \begin{array}{r} (x,\ \mu) : b + \Sigma_{k=0}^{K} \mathcal{L}_k\,(x)\,t^k + \mathcal{L}_d\,(\mu) \geq 0 \\ \textbf{for all}\ \ t \in [-1, 1] \end{array} \right\}$$

- **Problem:** ***check if $\mathcal{S}$ is nonempty and, if so, find an element in it***

- **$\mathcal{S}$ defined in terms of polynomials that are positive semi-definite in $[-1, 1]$**

- **Derive a finite dimensional characterization of positive semi-definite polynomials**

Sum of Squares

- **Notation:**

$$e_K(t) = [\,1 \quad t \quad t^2 \quad \cdots \quad t^K\,]'$$
$$c_f = [\,f_0 \quad f_1 \quad \cdots \quad f_K\,]'$$

- **For a scalar polynomial f, we have:**

$$f = \sum_{k=0}^{K} t^k f_k = {c_f}' e_K$$

and

$$f^2 = e'_K c_f {c_f}' e_K = e'_K F e_K$$

where $F \geq 0$

- **Similarly,**

$$f^2 + g^2 = e'_K \widehat{F} e_K$$

for some $\widehat{F} \geq 0$

- ***Sum of squares (sos) of polynomials can be written as a positive semi-definite quadratic form***

- **What are the symmetric matrices F such that $e'_K F e_K$ is a sos of polynomials ?**

Sum of Squares

- **Define:**

$$\mathcal{I}_{Kk} = \left\{ \begin{array}{r} (i,j) : i \in \{0,1,\cdot\cdot,K\},\ j \in \{0,1,\cdot\cdot,K\}, \\ i+j=k \end{array} \right\}$$

- **Fix a polynomial f of degree at most K**

- **Define the subset of $\mathbb{R}^{(K+1)\times(K+1)}$:**

$$\mathcal{F}_f = \left\{ \begin{array}{r} F = F' : \Sigma_{(i,j)\in\mathcal{I}_{Kk}} F_{(i+1)(j+1)} = \Sigma_{(i,j)\in\mathcal{I}_{Kk}} f_i f_j \\ \forall\ k = 0,\cdots,2K \end{array} \right\}$$

Lemma: ***The following are true.***

1. ***Let $F \in \mathbb{R}^{(K+1)\times(K+1)}$ be symmetric. Then, $f^2 = e_K' F e_K$ if and only if $F \in \mathcal{F}_f$.***
2. ***There exists a positive semi-definite matrix in $\mathcal{F}_f$.***

- **A polynomial f: $f(t) \geq 0,\ \forall t \in [-1,1]$ is called positive semi-definite (psd)**

- **f sos $\Rightarrow$ f is positive semi-definite (psd)**

- **The converse is not true**

Positive Semi-Definite Polynomials

Lemma [Fekete]: ***Let p be a scalar polynomial of degree K. The following statements are equivalent.***

1. ***$p(t) \geq 0$ for all $t \in [-1, 1]$***
2. ***There exist polynomials f of degree K and g of degree $K-1$ such that $p = f^2 + (1-t^2)g^2$.***

- **Note that**

  $$t^2g^2 = (tg)^2$$

  and

  $$\begin{aligned} p &= f^2 + g^2 - (tg)^2 \\ &= e_K' F e_K + e_{K-1}' G e_{K-1} - e_K' \widetilde{G} e_K \end{aligned} \qquad (1)$$

  by the sos lemma

- **But $e_{K-1}' G e_{K-1} = e_K' \widetilde{G} e_K$ for some $\widetilde{G}$**

- **We use these to obtain a characterization**

Positive Semi-Definite Polynomials

<u>Theorem:</u> ***Let*** $p_0, p_1, \cdots, p_K$ ***be real numbers. The following statements are equivalent.***

1. ***The polynomial***

$$p = \sum_{k=0}^{K} p_k t^k$$

is psd, i.e., $p(t) \geq 0$ ***for all*** $t \in [-1, 1]$***.***

2. ***There exist*** $P \in \mathbb{R}^{(K+1)\times(K+1)}$ ***and*** $G \in \mathbb{R}^{K\times K}$ ***that are symmetric and satisfying***

$$\begin{gathered}
\sum_{(i,j)\in\mathcal{I}_{Kl}} P_{(i+1)(j+1)} = p_l \quad l = 0, 1, \cdots, K \\
\sum_{(i,j)\in\mathcal{I}_{Kl}} P_{(i+1)(j+1)} = 0 \quad l = K+1, \cdots, 2K \\
G \geq 0 \\
P - \begin{bmatrix} I_K \\ 0_{1\times K} \end{bmatrix} G \begin{bmatrix} I_K & 0_{K\times 1} \end{bmatrix} + \begin{bmatrix} 0_{1\times K} \\ I_K \end{bmatrix} G \begin{bmatrix} 0_{K\times 1} & I_K \end{bmatrix} \geq 0
\end{gathered}$$

- **LMIs in P, G and coefficients $\{p_k\}$**
- **Problem of checking if a polynomial is psd is a LMI feasibility problem**
- **Size depends only on polynomial degree**

Problem Solution

- **Decompose $\mathbb{R}^m$ in to m 1-D subspaces**
- **Let b_i, $\mathcal{L}_{di}$ and $\mathcal{L}_{ki}$ be projections of b, $\mathcal{L}_d$ and $\mathcal{L}_k$ onto ith subspace**
- **Define**

$$\mathcal{S}_i = \left\{ \begin{array}{r} (x,\ \mu) : b_i + \mathcal{L}_{di}\left(\mu\right) + \Sigma_{k=0}^{K} \mathcal{L}_{ki}\left(x\right) t^k \geq 0 \\ \textbf{for all} \ \ t \in [-1, 1] \end{array} \right\}$$

- **$\mathcal{S}$ not empty iff $\mathcal{S}_i$ not empty for $1 \leq i \leq m$**

<u>**Corollary:**</u> ***$\mathcal{S}_k$ is not empty iff there exist $(x, \mu) \in \mathcal{C} \times \mathcal{D}$, $P \in \mathbb{R}^{(K+1)\times(K+1)}$ and $G \in \mathbb{R}^{K\times K}$ that satisfy***

$$P_{11} = b_k + \mathcal{L}_{dk}(\mu) + \mathcal{L}_{0k}(x)$$

$$\sum_{(i,j)\in\mathcal{I}_{Kl}} P_{(i+1)(j+1)} = \mathcal{L}_{lk}(x) \quad \textit{\textbf{for}} \ \ l = 1, \cdots, K$$

and the last three LMIs of previous theorem

- **This is a mixed integer/LMI problem**
- **When discrete variables are fixed, we get an LMI problem**

Generalization to Nonlinear Case

- $\mathcal{L}_d$ **and** $\mathcal{L}_k$ **are nonlinear**

- **Differentially flat systems without approximating feasible set**

- **Define**

$$\mathcal{S} = \left\{ \begin{array}{r} (x,\ \mu) : b + \Sigma_{k=0}^{K} \mathcal{L}_k\,(x)\, t^k + \mathcal{L}_d\,(\mu) \geq 0 \\ \textbf{for all} \ \ t \in [-1, 1] \end{array} \right\}$$

- **Problem:** ***check if*** $\mathcal{S}$ ***is nonempty and, if so, find an element in it***

- **Check existence of** (x, μ)**,** P **and** G **that satisfy**

$$P_{11} = b_k + \mathcal{L}_{dk}(\mu) + \mathcal{L}_{0k}(x)$$

$$\sum_{(i,j) \in \mathcal{I}_{Kl}} P_{(i+1)(j+1)} = \mathcal{L}_{lk}(x) \quad \textbf{for} \ \ l = 1, \cdots, K$$

and the last three LMIs of main theorem

- **Only approximation is in use of polynomial basis**

New Method for Flat Systems

- **Find polynomial w with structure satisfying**

$$h(w(t)) \geq \epsilon \ \ \textbf{for all} \ \ t \in [0, T]$$

where $\epsilon > 0$

- **w belongs to**

$$\left\{ \sum_{k=0}^{n} \mathcal{M}_k(a) t^k : a \ \textbf{arbitrary} \right\}$$

where $\mathcal{M}_k$ are linear

- **Assume h is rational. Substitute for w**

$$h\left(w(t)\right) = \frac{\Sigma_{k=0}^{K} p_k(a) t^k}{\Sigma_{k=0}^{K} q_k(a) t^k} \triangleq h(a, t)$$

for some polynomials (in a) p_k and q_k

- **Define:**

$$\mathcal{W} = \{a : h(a, t) \geq \epsilon \quad \textbf{for all} \ \ t \in [-1, 1]\}$$

$$\mathcal{W}_{\geq}(\delta) = \left\{ \begin{array}{l} a : R\delta + \Sigma_{k=0}^{K} z_k(a) t^k \geq 0 \ \textbf{and} \\ R\delta + \Sigma_{k=0}^{K} q_k(a) t^k > 0 \ \forall \ t \in [-1, 1] \end{array} \right\}$$

$$\mathcal{W}_{\leq}(\delta) = \left\{ \begin{array}{l} a : R\delta + \Sigma_{k=0}^{K} z_k(a) t^k \leq 0 \quad \textbf{and} \\ R\delta + \Sigma_{k=0}^{K} q_k(a) t^k < 0 \ \ \forall \ t \in [-1, 1] \end{array} \right\}$$

where $z_k(a) = p_k(a) - \epsilon q_k(a)$, $R >> 0$

New Method for Flat Systems

<u>**Lemma:**</u> ***The following hold for very large*** R.

1. ***$\mathcal{W}$ is the disjunction of $\mathcal{W}_{\geq}(0)$ and $\mathcal{W}_{\leq}(0)$***
2. ***Let a be given. Then, $a \in \mathcal{W}$ if and only if there exist $\delta \in \{0,1\}$ such that a is in the conjunction of $\mathcal{W}_{\geq}(\delta)$ and $\mathcal{W}_{\leq}(\delta - 1)$.***

- **Disjunctions ⇒ many feasibility problems**
- **Conjunctions ⇒ single feasibility problem with discrete-variables**
- **Common theme in trajectory generation and more generally hybrid systems**
- **$\mathcal{L}_k$ for this case are "polynomial":**

$$\mathcal{L}_k(a) = \begin{bmatrix} p_k(a) - \epsilon q_k(a) & q_k(a) & -p_k(a) + \epsilon q_k(a) \\ & & -q_k(a) \end{bmatrix}'$$

- **Approximate $\mathcal{L}_k$**

Conclusions

- Many approaches to path planning problems grid time and impose constraints at grid points
- We gave an exact condition without gridding
  - leads to mixed integer/LMI problem whose size depends only on the polynomial basis dimension
  - more generally to mixed integer/nonlinear programs involving “polynomial” nonlinearities
  - discrete-variables arise when converting disjunctions to conjunctions
- Possible to solve exactly using nullstellensatz and positivstellensatz